Quantum phase transitions on percolating lattices

Thomas Vojta
Department of Physics, University of Missouri-Rolla

- Geometric percolation
- Classical magnet on a percolating lattice
- Percolation quantum phase transitions
- Quantum Ising magnet and activated scaling
- Percolation and dissipation

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Geometric percolation

- regular (square or cubic) lattice
- sites are occupied at random
  site **empty** (vacancy) with probability p
  site **occupied** with probability $1 - p$

Question: Do the occupied sites form a connected infinite spanning cluster?

- sharp **percolation threshold** at p_c

$p > p_c$: only disconnected finite-size clusters
length scale: connectedness length ξ_c

$p = p_c$: ξ_c diverges, clusters on all scales, clusters are **fractals** with dimension $D_f < d$

$p < p_c$: infinite cluster covers finite fraction P_∞ of sites

$p > p_c$

$p = p_c$

$p < p_c$

Percolation as a critical phenomenon

- percolation can be understood as continuous phase transition
- **geometric fluctuations** take the role of usual thermal or quantum fluctuations
- concepts of **scaling** and **critical exponents** apply

cluster size distribution:
(number of clusters with s sites):

$$n_s(p) = s^{-\tau_c} f\left[(p - p_c)\, s^{\sigma_c}\right]$$

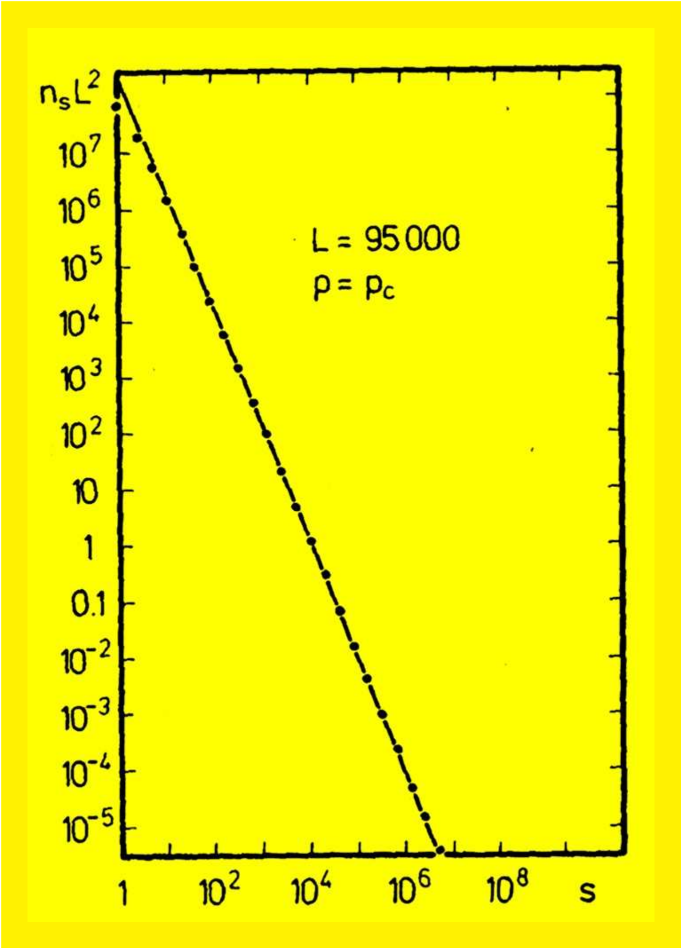
scaling function $f(x)$

$$\begin{array}{ll} f(x) \sim \exp(-B_1 x^{1/\sigma_c}) & (p > p_c) \\ f(x) = \text{const} & (p = p_c) \\ f(x) \sim \exp[-(B_2 x^{1/\sigma_c})^{1-1/d}] & (p < p_c) \end{array} .$$

infinite cluster: $P_\infty \sim |p - p_c|^{\beta_c}$ $\quad (\beta_c = (\tau_c - 2)/\sigma_c)$

length scale : $\xi_c \sim |p - p_c|^{-\nu_c}$ $\quad (\nu_c = (\tau_c - 1)/d\sigma_c)$

fractal dimension: $D_f = d/(\tau_c - 1)$



from Stauffer/Aharony

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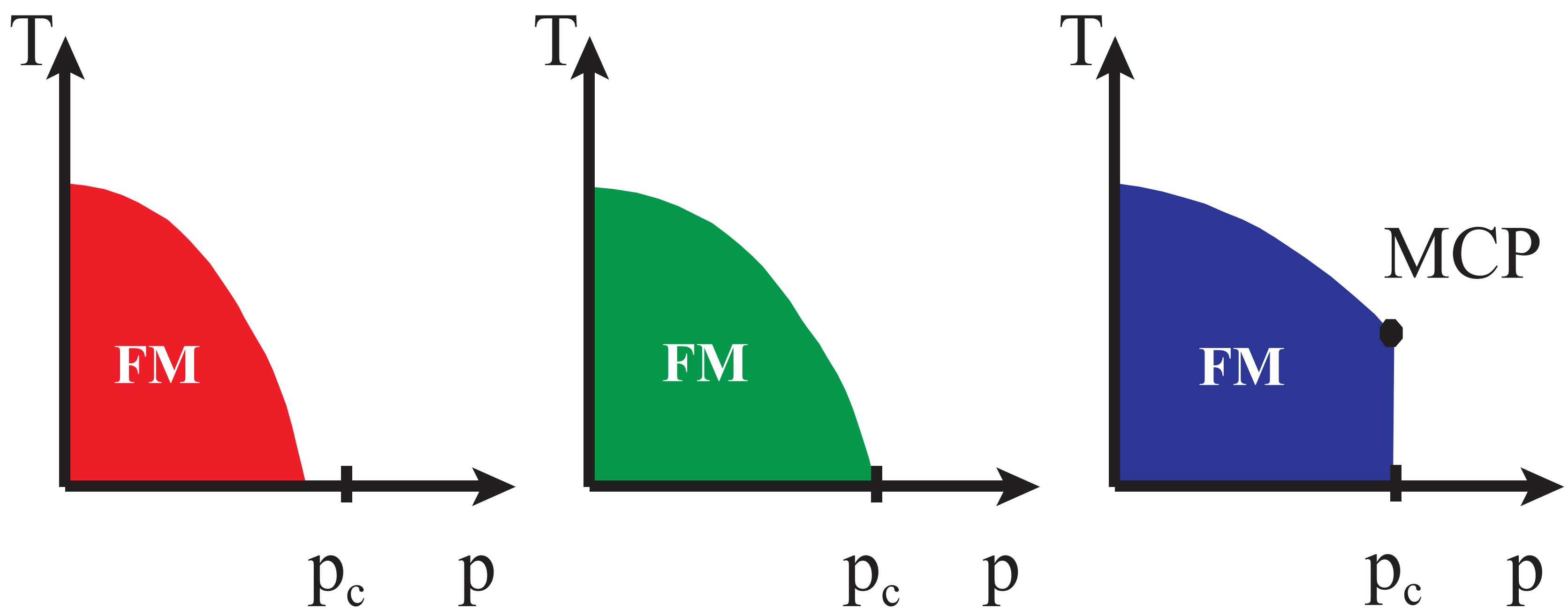
Classical diluted magnet

$$H = -J \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j \, S_i S_j - h \sum_i \epsilon_i S_i$$

- S_i $-$ classical Ising or Heisenberg spin
- ϵ_i random variable, 0 with probability p, 1 with probability $1 - p$

Question:
Phase diagram as function of temperature T and impurity concentration p?



Is a classical magnet on the critical percolation cluster ordered?

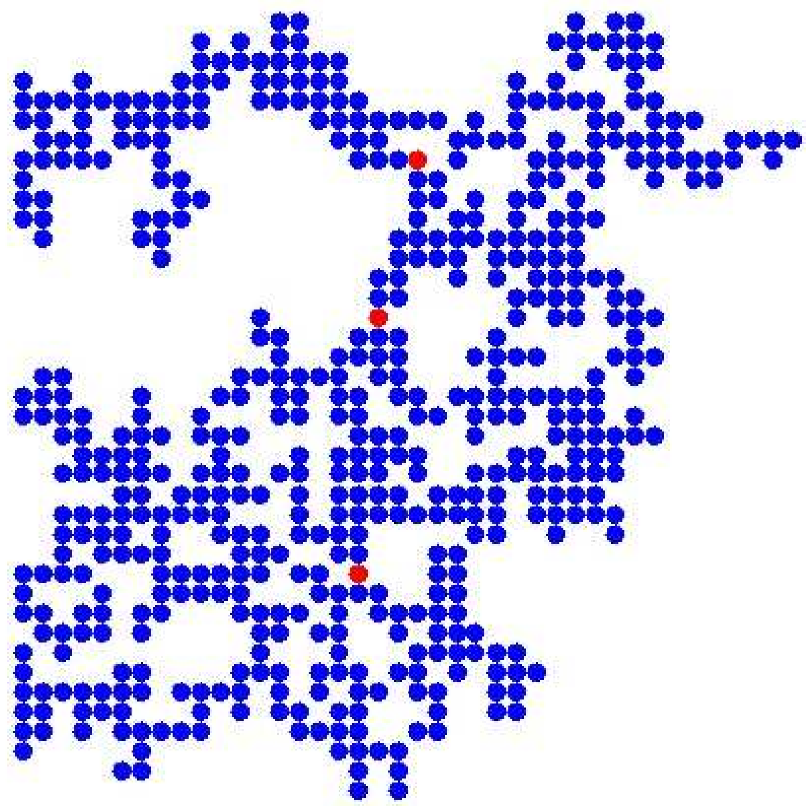
naive argument: fractal dimension $D_f > 1 \quad \Rightarrow \quad$ Ising magnet orders at low T

Wrong !!!

- critical percolation cluster contains **red sites**
- parts on both sides of red site can be flipped with **finite energy cost**

$\Rightarrow$ no long-range order at any finite T,
$T_c(p)$ **vanishes at percolation threshold**

- fractal (mass) dimension D_f not sufficient to characterize magnetic order



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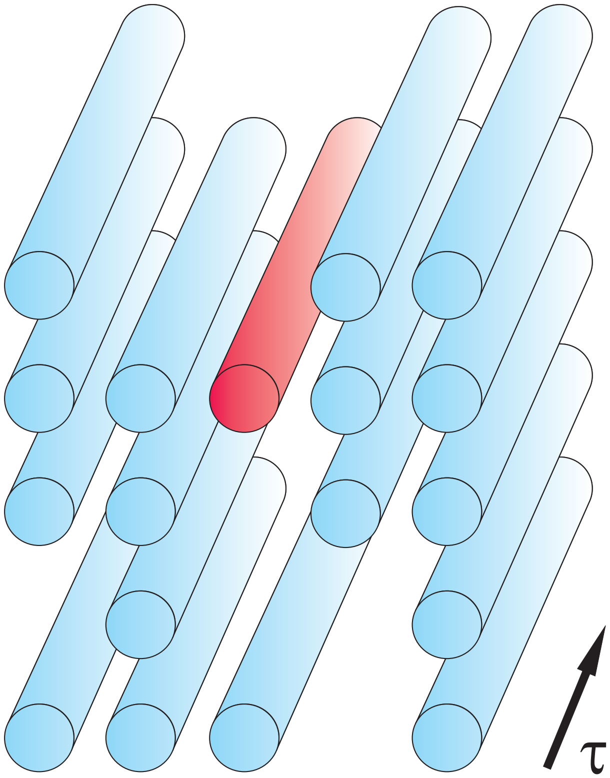
Percolation and quantum fluctuations

- quantum fluctuations are less effective in destroying long-range order
- red sites ⇒ **red lines, infinite at $T = 0$**
- flipping cluster parts on both sides of red line requires **infinite energy**

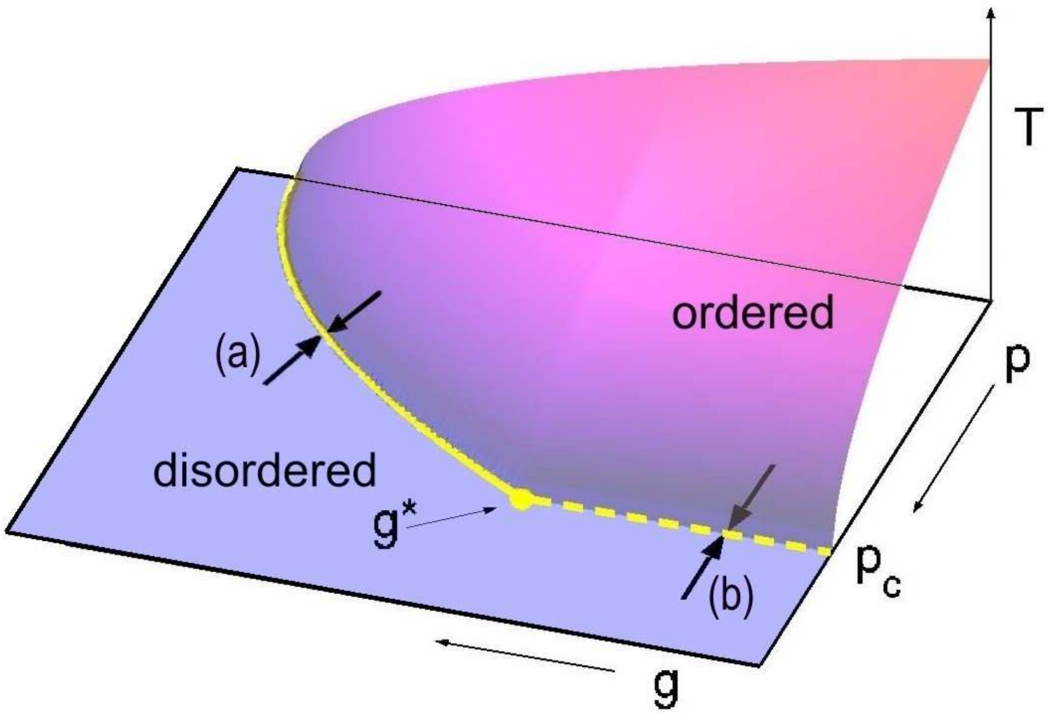
Long-range order survives on the critical percolation cluster

(if quantum fluctuations are not too strong)

(confirmed by explicit results for quantum Ising and Heisenberg magnets and for quantum rotors)



Phase diagram of a diluted quantum magnet



Schematic phase diagram

p = impurity concentration
g = quantum fluctuation strength
T = temperature

(long-range order at $T > 0$ requires $d \geq 2$ for Ising and $d \geq 3$ for Heisenberg symmetry)

Two zero-temperature quantum phase transitions:

(a) generic quantum phase transition, driven by quantum fluctuations
(b) percolation quantum phase transition, driven by geometry of the lattice

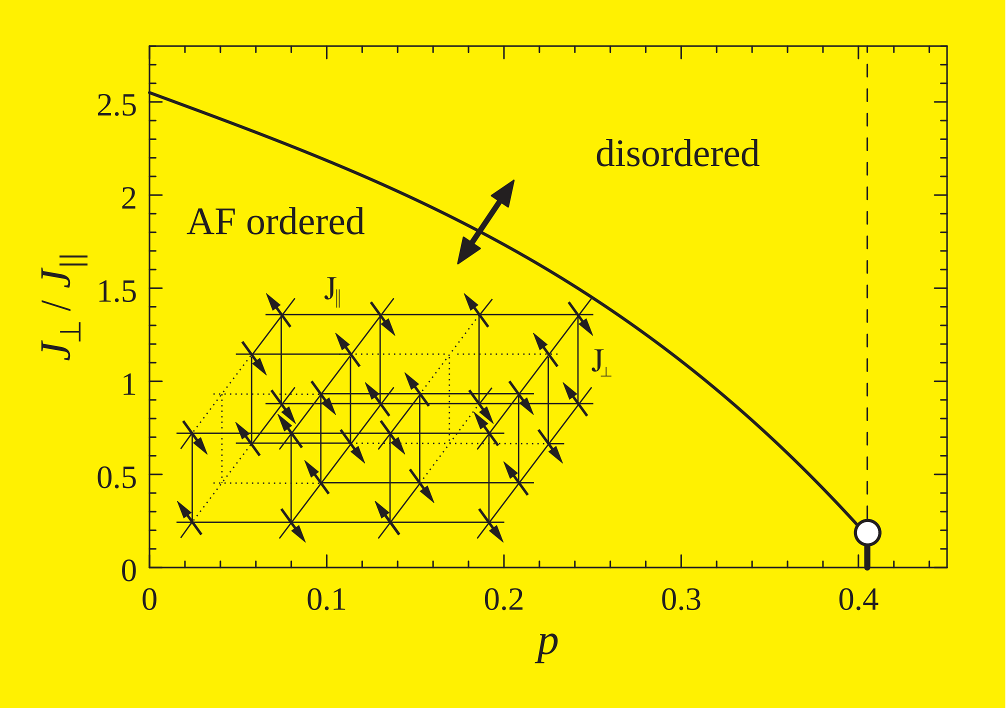
transitions separated by **multicritical point** at $(g^*, p_c, T = 0)$

Example: diluted bilayer Heisenberg antiferromagnet

$$H = J_{\|} \sum_{\substack{\langle i,j \rangle \\ a=1,2}} \epsilon_i \epsilon_j \hat{\mathbf{S}}_{i,a} \cdot \hat{\mathbf{S}}_{j,a} + J_{\perp} \sum_i \epsilon_i \hat{\mathbf{S}}_{i,1} \cdot \hat{\mathbf{S}}_{i,2},$$

- $\hat{\mathbf{S}}_{j,a}$: quantum spin operator $(S = 1/2)$ at site j, layer a
- ratio $J_{\perp}/J_{\|}$ controls strength of **quantum** fluctuations
- **dilution**: random variable ϵ_i=0,1 with probabilities p, $1-p$.

Phase diagram mapped out by Sandvik (2002) and Vajk and Greven (2002)



Model action, order parameter, and correlation length

$O(N)$ quantum rotor model ($N \geq 2$)

$$\mathcal{A} = \int d\tau \sum_{\langle ij \rangle} J\epsilon_i\epsilon_j \mathbf{S}_i(\tau)\cdot\mathbf{S}_j(\tau) + \frac{T}{g}\sum_i \sum_n \epsilon_i |\omega_n|^{2/z_0} \mathbf{S}_i(\omega_n)\mathbf{S}_i(-\omega_n)$$

$\mathbf{S}_i(\tau)$: N-component unit vector at site i and imaginary time τ
$\epsilon_i = 0, 1$: random variable describing site dilution
z_0: (bare) dynamical exponent of the clean system.

Order parameter (magnetization):

• magnetic long-range order only possible on infinite percolation cluster
• for $g < g^*$, infinite percolation cluster is ordered for all $p < p_c$
$\Rightarrow$ magnetization $m \sim P_\infty \sim |p - p_c|^{\beta_c}$

$\beta = \beta_c$

Spatial correlation length:

• magnetic correlations cannot extend beyond connectedness length of the lattice
$\Rightarrow$ correlation length $\xi \sim \xi_c \sim |p - p_c|^{-\nu_c}$

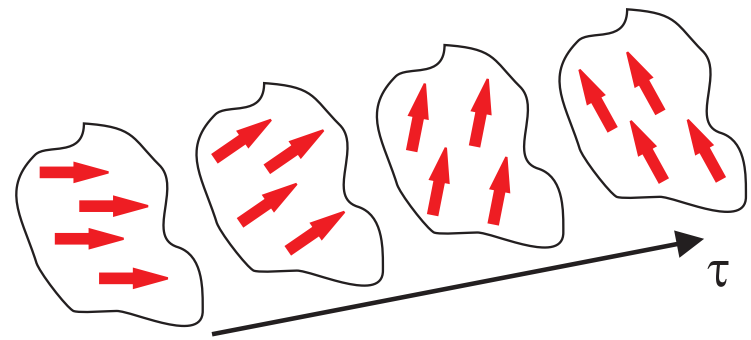
$\nu = \nu_c$

Quantum dynamics of a single percolation cluster

single percolation cluster of s sites

- for $g < g^*$, all rotors on the cluster are correlated but collectively fluctuate in time

$\Rightarrow$ cluster acts as single (0+1) dimensional NLSM model with moment s



$$\mathcal{A}_s = s\frac{T}{g}\sum_i\sum_n |\omega_n|^{2/z_0}\mathbf{S}(\omega_n)\mathbf{S}(-\omega_n) + sh\int d\tau S^{(1)}(\tau)$$

Dimensional analysis or renormalization group calculation:

$$F_s\,(g,h,T) = g^{\varphi}s^{-\varphi}\Phi\left(hs^{1+\varphi}g^{-\varphi}, Ts^{\varphi}g^{-\varphi}\right) \qquad \varphi = z_0/(2-z_0)$$

- free energy of quantum spin cluster more singular than that of classical spin cluster
- **susceptibility**: classically $\chi_s^c \sim s^2$, quantum (at $T=0$): $\chi_s \sim s^{2+\varphi}$

  susceptibility of quantum cluster diverges faster with cluster size s

Scaling theory of the percolation quantum phase transition

- total free energy is sum over contributions of all percolation clusters
- combining **percolation cluster size distribution** + **free energy of single cluster**

$$F\,(p-p_c,h,T) = \sum_s n_s(p-p_c)\,F_s\,(g,h,T)$$

Scaling form of free energy:

- rescaling $s \to s/b^{D_f}$ yields

$$F\,(p-p_c,h,T) = b^{-(d+z)}F\left((p-p_c)b^{1/\nu}, hb^{\left(D_f+z\right)}, Tb^z\right)$$

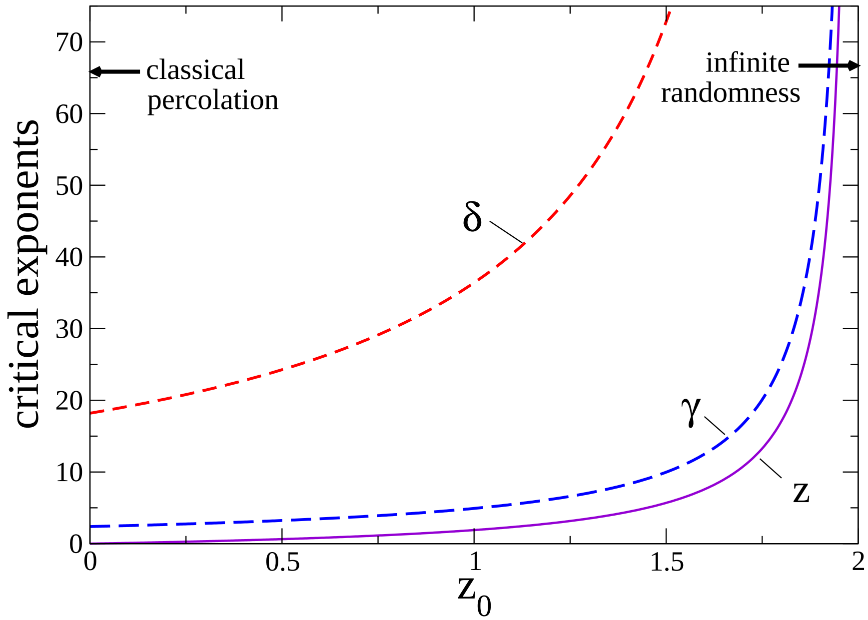
- correlation length exponent identical to the classical value, $\nu = \nu_c$
- $z = \varphi D_f$ plays the role of the dynamic critical exponent.

Critical behavior

- exponents determined by two lattice percolation exponents (e.g., $\nu = \nu_c$, D_f) and the **dynamical exponent** z

$$\begin{aligned}
2-\alpha &= (d+z)\,\nu \\
\beta &= (d-D_f)\,\nu \\
\gamma &= (2D_f - d + z)\,\nu \\
\delta &= (D_f + z)/(d-D_f) \\
2-\eta &= 2D_f - d + z\ .
\end{aligned}$$

- **classical** exponents recovered for $z=0$:
- α, γ, δ, and η are **nonclassical** while β is **unchanged**



Exponents in 2D as function of z_0.

	2d		3d	
	classical	quantum	classical	quantum
α	$-2/3$	$-115/36$	-0.62	-2.83
β	$5/36$	$5/36$	0.417	0.417
γ	$43/18$	$59/12$	1.79	4.02
δ	$91/5$	$182/5$	5.38	10.76
ν	$4/3$	$4/3$	0.875	0.875
η	$5/24$	$-27/16$	-0.06	-2.59
z	-	$91/48$	-	2.53

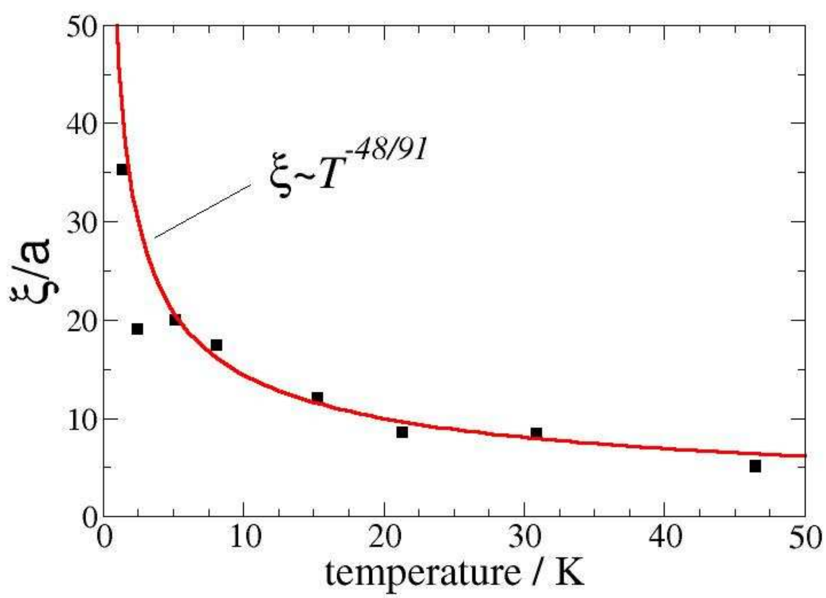
Exponents in 2d and 3d for $z_0 = 1$.

T.V. + J. Schmalian, PRL **95**, 237206 (2005)

Simulation and experiment

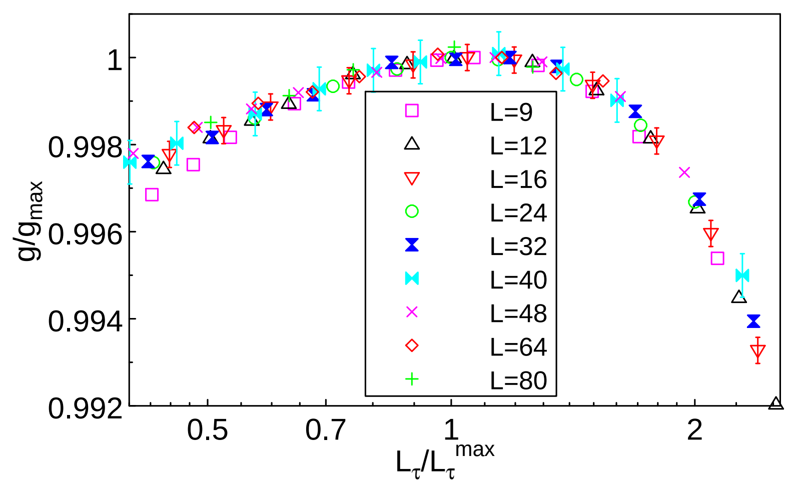
Diluted Heisenberg antiferromagnet $La_2Cu_{1-x}(Zn,Mg)_xO_4$

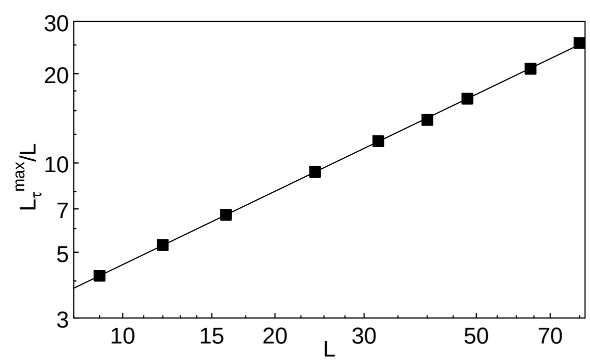
- neutron scattering experiments
  Vajk et al., Science 295, 1691, (2002)
- correlation length at $p = p_c$
  prediction $\xi \sim T^{-1/z}$



Monte-Carlo simulations

- FSS of Binder cumulant at $p = p_c$
  $\Rightarrow z \approx 1.83$





R. Sknepnek, M.V., T.V., PRL **93**, 097201 (2004),
T.V., R. Sknepnek PRB **74**, 094415 (2006)

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Diluted transverse-field Ising model

$$H_I = -J \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j \hat{S}_i^z \hat{S}_j^z - h_x \sum_i \epsilon_i \hat{S}_i^x - h_z \sum_i \epsilon_i \hat{S}_i^z \ ,$$

- $\hat{S}_j^{x,z}$: x and z components of quantum spin operator ($S = 1/2$) at site j
- h_x: transverse magnetic field, controls strength of **quantum** fluctuations
- h_z: ordering (longitudinal) magnetic field, conjugate to order parameter
- **dilution**: random variable ϵ_i=0,1 with probabilities p, $1 - p$.

single percolation cluster of s sites

- for small h_x, all spins on the cluster are correlated but collectively fluctuate in time
- cluster of size s acts as (0+1) dimensional **Ising model** with moment s
- energy gap (inverse susceptibility) of cluster depends **exponentially** on size s

$$\Delta \sim \chi_s^{-1} \sim h_x e^{-Bs} \qquad [B \sim \ln(J/h_x)]$$

Activated scaling

- exponential relation between length and time scales: $\ln \xi_\tau \sim \ln(1/\Delta) \sim s \sim \xi^{D_f}$

Activated scaling: $\ln \xi_\tau \sim \xi^{D_f}$

Scaling form of the magnetization at the percolation transition

(Senthil/Sachdev 96)

- sum over all percolation clusters using size distribution n_s

$$m(p - p_c, h_z) = b^{-\beta_c/\nu_c}\, m\left((p - p_c)b^{1/\nu_c}, \ln(h_z) b^{-D_f}\right)$$

- at the percolation threshold $p = p_c$: $m \sim [\ln(h_z)]^{2-\tau_c}$
- for $p \neq p_c$: power-law quantum Griffiths effects $m \sim h_z^{\zeta}$ with nonuniversal ζ

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Dissipative transverse-field Ising model

- couple each spin to local bath of harmonic oscillators

$$H = H_I + \sum_{i,n} \epsilon_i \left[\nu_{i,n} a^\dagger_{i,n} a_{i,n} + \frac{1}{2} \lambda_{i,n} \hat{S}^z_i (a^\dagger_{i,n} + a_{i,n}) \right]$$

- $a^\dagger_{i,n}, a_{i,n}$: creation and destruction operator of the n-th oscillator coupled to spin i
- $\nu_{i,n}$ frequency of of the n-th oscillator coupled to spin i
- $\lambda_{i,n}$: coupling constant

Ohmic dissipation: spectral function of the baths is **linear** in frequency

$$\mathcal{E}(\omega) = \pi \sum_n \lambda^2_{i,n} \delta(\omega - \nu_{i,n})/\nu_{i,n} = 2\pi\, \alpha\, \omega e^{-\omega/\omega_c}$$

α dimensionless dissipation strength

ω_c cutoff energy

Phase diagram

- percolation cluster of size s equivalent to **dissipative** two-level system with effective dissipation strength $s\alpha$

$\Rightarrow$ **large clusters** with $s\alpha > 1$ **freeze**
small clusters with $s\alpha < 1$ **fluctuate**

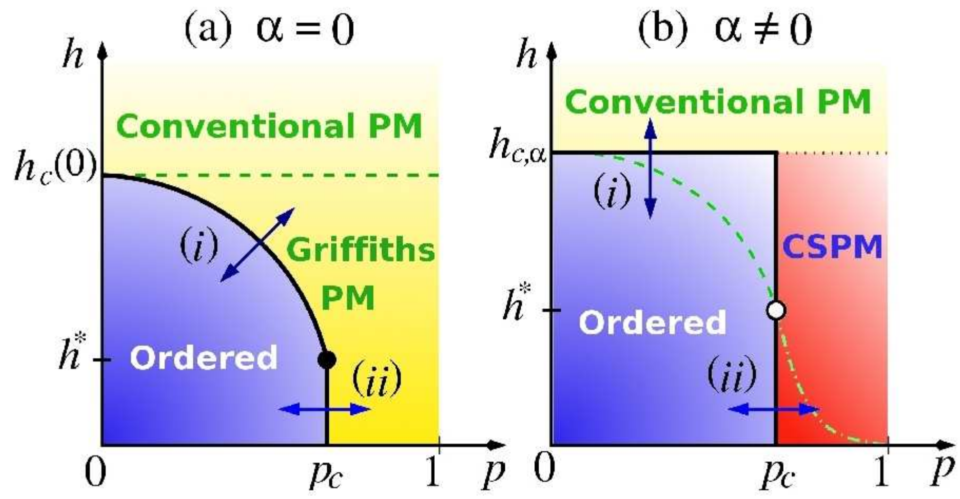
- frozen clusters act as classical superspins, dominate low-temperature susceptibility

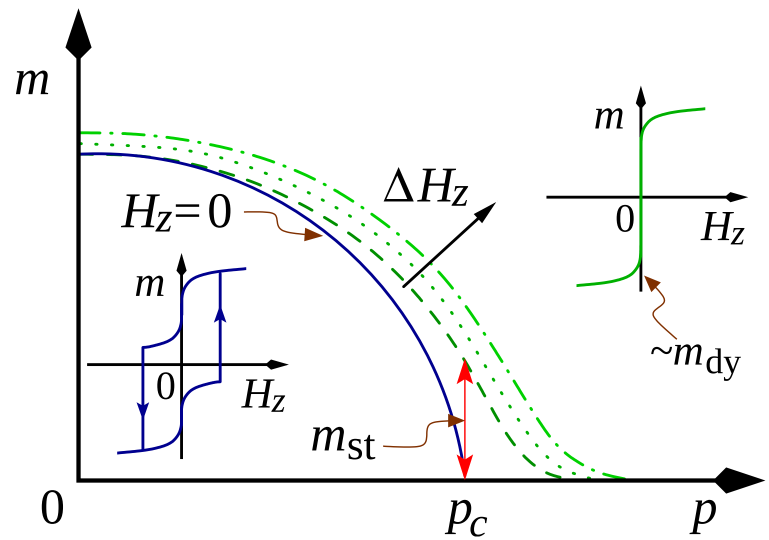
$$\chi \sim |p - p_c|^{-\gamma_c}/T$$

- magnetization of infinite cluster

$$m_\infty \sim P_\infty(p) \sim |p - p_c|^\beta$$

- magnetization of finite-size frozen and fluctuating clusters leads to **unusual hysteresis effects**





J. Hoyos and T.V., PRB **74**, 140401(R) (2006)

Conclusions

- long-range order on critical percolation cluster is destroyed by thermal fluctuations
  long-range order survives a nonzero amount of quantum fluctuations
  $\Rightarrow$ permits **percolation quantum phase transition**

- critical behavior is controlled by lattice percolation exponents but it is
  different from classical percolation

- in diluted quantum Ising magnets $\Rightarrow$ exotic transition, activated scaling

- Ohmic dissipation: large percolation clusters **freeze**, act as **superspins**
  $\Rightarrow$ **classical superparamagnetic cluster phase**

Interplay between geometric criticality and quantum fluctuations leads to novel quantum phase transition universality classes