1. \[ T = \frac{m}{2} x^2 + \frac{1}{2} I \dot{\theta}^2 \]
   \[ x = R \theta \]
   \[ \dot{x} = R \ddot{\theta} \]
   \[ \text{so} \quad T = \frac{m}{2} \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \]
   \[ U = -mgx \]
   \[ \lambda = \frac{1}{2} (m + \frac{I}{R^2}) \dot{x}^2 + mgx \]

   \[ \frac{\lambda}{x} = mg \quad \frac{\ddot{x}}{x} = (m + \frac{I}{R^2}) \dot{x} \]

   \[ \text{so} \quad (m + \frac{I}{R^2}) \ddot{x} = mg \]
   \[ \ddot{x} = \frac{mg}{m + \frac{I}{R^2}} \]

2. \[ T = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) \]
   \[ U = mgsz \]
   \[ \lambda = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - mgsz \]
   \[ \rho = R \quad \text{and} \quad z = \lambda \phi \]

   \[ \lambda = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \frac{\dot{z}^2}{\lambda^2} + \dot{z}^2) - mgsz \]

   Since \( \rho = R \), \( \dot{\rho} = 0 \)

   and

   \[ \lambda = \frac{m}{2} (\rho^2 + 1) \dot{z}^2 - mgsz \]

   \[ \frac{\lambda}{\dot{z}} = -mg \quad \frac{\ddot{z}}{\lambda} = m \left( \frac{\rho^2}{\lambda^2} + 1 \right) \frac{\dot{z}}{\dot{z}} \]

   So \( \dot{\rho} \left( \frac{\dot{z}^2}{\lambda^2} + 1 \right) \frac{\dot{z}}{\dot{z}} = -mg \)

   \[ \ddot{z} = \frac{-s}{1 + \left( \frac{\rho^2}{\lambda^2} \right)^2} \quad \rho \to 0 \quad -s \]
3. a) \[ T_m = \frac{m}{2} \dot{x}^2 \]

\[ x = x + L \sin \phi \]  
\[ \dot{x} = \dot{x} + L \dot{\phi} \cos \phi \]

\[ \ddot{x} = \frac{M}{2} \left( \dot{x}^2 + L^2 \dot{\phi}^2 + 2 \dot{L} \dot{\phi} \cos \phi \right) \]

\[ T = \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left( \dot{x}^2 + L^2 \dot{\phi}^2 + 2 \dot{L} \dot{\phi} \cos \phi \right) \]

\[ U = \frac{1}{2} k x^2 - M g L \cos \phi \]

\[ L = \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left( \dot{x}^2 + L^2 \dot{\phi}^2 + 2 \dot{L} \dot{\phi} \cos \phi \right) + M g L \cos \phi - \frac{1}{2} k x^2 \]

\[ \frac{\partial L}{\partial \dot{x}} = -k x \]
\[ \frac{\partial L}{\partial \dot{\phi}} = M \ddot{x} + M \left( \dot{x} + L \dot{\phi} \cos \phi \right) \]
\[ \frac{\partial L}{\partial \ddot{x}} = M \dddot{x} + M \left( \ddot{x} + L \ddot{\phi} \cos \phi - L \dot{\phi}^2 \sin \phi \right) \]

\[ (m + M) \dddot{x} + M \left( \dddot{x} + L \dddot{\phi} \cos \phi - L \dot{\phi}^2 \sin \phi \right) + k x = 0 \]

\[ \phi \]
\[ \frac{\partial^2 L}{\partial \phi \partial \dot{\phi}} = -M g L \sin \phi - M L \ddot{x} \sin \phi \]
\[ \frac{\partial^2 L}{\partial \phi^2} = M L^2 \dddot{\phi} + M L \dddot{x} \cos \phi \]
\[ \frac{\partial^2 L}{\partial \phi \partial \dddot{x}} = M L^2 \dddot{\phi} + M L \dddot{x} \cos \phi - M L \dddot{x} \phi \sin \phi \]

\[ L^2 \phi + ML \dddot{x} \cos \phi - ML \dddot{x} \phi \sin \phi + M \dddot{x} \phi \sin \phi + M g \phi \sin \phi = 0 \]

\[ L^2 \phi + L \dddot{x} \cos \phi + g L \sin \phi = 0 \]
6) For $x$ and $\phi$ small we get

\[
(m+M)\dddot{x} + ML\dddot{\phi} + kx = 0
\]

and $L^2\dddot{\phi} + L\dddot{x} + 8\phi = 0$

or $L^2\dddot{\phi} + x + 8\phi = 0$

\[
T = \frac{M}{2}(\dot{\rho}^2 + \rho^2\phi^2 + \dot{\phi}^2)
\]

\[
U = mgz = mgk\rho^2
\]

\[
K = \frac{m}{2}[L(1+4k^2\rho^2)\dot{\rho}^2 + \rho^2\omega^2] - mgk\rho^2
\]

\[
\frac{\partial T}{\partial \rho} = 4mk^2\rho\dot{\rho}^2 + m\rho\omega^2 - 2mgk\rho
\]

\[
\frac{\partial K}{\partial \rho} = m(L^2\rho^2)\dot{\rho}
\]

\[
\frac{\partial T}{\partial \rho} = m(L^2\rho^2)\dddot{\rho} + 8mk^2\rho\dot{\rho}^2
\]

\[
(1 + 4k^2\rho^2)\dddot{\rho} + 8k^2\rho\dot{\rho}^2 = 4k^2\rho\dot{\rho}^2 + \rho\omega^2 - 2gk\rho
\]

\[
(1 + 4k^2\rho^2)\dddot{\rho} + 4k^2\rho\dot{\rho}^2 + 2gk\rho + \rho\omega^2 = 0
\]

4th eq. $\dddot{\rho}$ and $\dddot{\phi} = 0$

So

\[
\rho(2gk - \omega^2) = 0
\]

\[
\Rightarrow \rho = 0 \quad \text{or} \quad \omega^2 = 2gk
\]
For $p = 0$ we get

$$\ddot{p} + (2gk - w^2)p = 0$$

or

$$\ddot{p} = (w^2 - 2gk)p$$

so motion will be stable about $p = 0$ provided that

$$2gk > w^2$$

If $w^2 > 2gk$ the bead accelerates away from the bottom.

For $w^2 = 2gk$ we get $\dot{p} = 0$ at all values of $p$.

If we move the bead slightly we see from

$$\left(1 + 4k^2p^2\right)\ddot{p} + 8k^2p\dot{p}^2 = (w^2 - 2gk)p$$

becomes

$$\left(1 + 4k^2p^2\right)\ddot{p} + 8k^2p\dot{p}^2 = 0$$

$$\ddot{p} = -\frac{8k^2\dot{p}^2}{1 + 4k^2p^2}$$

and $\dot{p}$ is always negative for any $p > 0$. Thus nudging it outward the bead's acceleration will slow its motion but not reverse it. If nudged inward it speeds up. In both cases it never returns to its original position and thus eq. is unstable.

5.

Since $p$ is const,

$$T = \frac{m}{2}(R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{w}^2)$$

$$V = -mgR \cos \theta (\text{wrt hoop center})$$

$$K = \frac{m}{2}(R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{w}^2) + mgR \cos \theta$$
\[
\frac{d^2 \theta}{dt^2} = mR^2 \sin \theta \omega_0 \omega_0 \omega^2 - mgR \sin \theta
\]

\[
\frac{d \theta}{dt} = mR^2 \omega
\]

So \( mR^2 \theta'' = mR^2 \sin \theta \omega_0 \omega_0 \omega^2 - mgR \sin \theta \)

\[
\theta'' = \sin \theta \omega_0 \omega_0 \omega^2 - \frac{g}{R} \sin \theta
\]

At eq.,

\[
(w^2 \omega_0^2 - \frac{g}{R}) \sin \theta = 0
\]

For \( \sin \theta = 0 \Rightarrow \theta = 0 \) or \( \pi \)

For \( w^2 \omega_0^2 = \frac{g}{R} \)

\[
\omega_0 = \sqrt{\frac{g}{R}}
\]

This only occurs when \( w^2 > \frac{g}{R} \)

and then

\[
\theta_{eq} = \pm \cos^{-1} \left( \frac{g}{w^2 R} \right)
\]

If \( w^2 < \frac{g}{R} \), the only eq. pts are \( \theta = 0 \) and \( \pi \)

let \( \theta = \theta_0 + \epsilon \Rightarrow \theta = \epsilon \)

Then

\[
\epsilon'' = (w^2 \omega_0^2 (\theta_0 + \epsilon) - \frac{g}{R}) \sin (\theta_0 + \epsilon)
\]

\[
\epsilon' = \left[ w^2 \omega_0^2 \cos \theta_0 \cos \epsilon - \sin \theta_0 \sin \epsilon \right] \sin \theta_0 \cos \epsilon
\]

\[
\epsilon' = \left[ w^2 \omega_0^2 \epsilon_0 - 3 \sin \theta_0 \sin \epsilon \right] - \frac{8 g}{R} \left[ \sin \theta_0 \cos \epsilon \right] + \cos \theta_0 \epsilon_0, \epsilon
\]

For \( \theta_0 = \pi \)

\[
\epsilon' \sim \left[ w^2 (-1) - \frac{g}{R} \right] (-\epsilon) = \left[ w^2 + \frac{g}{R} \right] \epsilon
\]
This is always unstable.

For $\Theta_0 = 0$

$$\varepsilon \sim [\omega^2 - \frac{8}{R}] \varepsilon$$

This is stable for $\frac{8}{R} > \omega^2$

and unstable for $\frac{8}{R} < \omega^2$

When stable, the frequency of small oscillations is

$$\frac{\omega^2}{R^2} \approx \frac{8}{R}$$

For $\Theta_0 = \pm 1 \cos^{-1} \left( \frac{8}{R\omega^2} \right)$ we get

$$\varepsilon \sim \frac{\omega^2}{R} \left( \frac{8}{R\omega^2} \right) - \varepsilon \frac{\omega^2}{R^2}$$

$$\varepsilon \sim -\frac{\omega^2}{R} + \varepsilon \left( \frac{8}{R\omega^2} \right)$$

$$\varepsilon \sim -\left( \frac{\omega^2}{R^2} \right) \varepsilon$$

$$\varepsilon \sim -\left( \omega^2 - \left( \frac{8}{R} \right)^2 \right) \varepsilon$$

This is stable when $\omega^2 \geq \left( \frac{8}{R} \right)^2$

$\omega^4 \geq \left( \frac{8}{R} \right)^2$

KTN $\omega^2 \geq \frac{8}{R}$

with freq. of small osc. $\sqrt{\omega^2 - \left( \frac{8}{R} \right)^2}$

heat $\omega^2 = \frac{8}{R}$

Then we have stable eq. at $\Theta_0 = 0$ when $\omega < \omega_c$

and stable eq. at $\Theta_0 = \cos^{-1} \left( \frac{\omega^2}{R} \right)$ when $\omega > \omega_c$

The frequencies of small osc. are $\sqrt{\omega^2 - \omega_c^2}$

and $\omega \left( 1 - \left( \frac{\omega}{\omega_c} \right)^4 \right)$ respectively.