ON ACYCLIC CURVES.
A SURVEY OF RESULTS AND PROBLEMS

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1. Introduction

We start with necessary notation and definitions. All considered spaces are assumed to be metric and all mappings are continuous. We denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the real line, by $I$ the closed unit interval $[0, 1]$ of reals, and by $S^1$ the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Given a subset $A$ of a space $X$, we denote its cardinality by $\text{card} A$, its dimension by $\dim A$, its diameter by $\text{diam} A$, its closure by $\text{cl} A$, its interior by $\text{int} A$, its boundary by $\text{bd} A$.

A. Spaces

An arc is defined as a homeomorphic image of the interval $I$, and a simple closed curve means a homeomorphic image of the unit circle $S^1$. A continuum means a compact connected space. A curve means a one-dimensional continuum. A space is said to be locally connected provided that each of its points has an arbitrarily small connected neighborhood. A subset of a space is said to be arcwise connected provided that every two of its points can be joined by an arc lying in this set. An arc with end points $a$ and $b$ will be denoted by $ab$.

A property of a continuum is said to be hereditary provided each subcontinuum of the continuum has the property. A continuum $X$ is defined to be

1991 Mathematics Subject Classification. 54F50.

Key words and phrases: acyclic, arc, confluent, connected, continuum, contractible, curve, dendrite, dendroid, fan, fixed point, locally connected, monotone, open, order, planable, property of Kelley, selectable, smooth, tree-like.
unicoherent if for every two subcontinua $A$ and $B$ of $X$ such that $X = A \cup B$ the intersection $A \cap B$ is connected. A continuum $X$ is said to be irreducible (between points $a$ and $b$ of $X$) provided that no proper subcontinuum of $X$ contains both $a$ and $b$. E. g., an arc is irreducible between its end points; the \text{sin}(1/x)-curve \\
\[ S = \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\} \cup \{(x, \text{sin}(1/x)) \in \mathbb{R}^2 : x \in (0, 1)\} \]
is irreducible between any point $a \in \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\}$ and $b = (1, \text{sin} 1)$; and no simple closed curve is irreducible. It is well-known (see e.g. [112], (11.2), p. 17) that each continuum $C$ containing some two points $a$ and $b$ contains at least one continuum $I(a, b)$ irreducible from $a$ to $b$. H. C. Miller [89] gave the following characterization of hereditarily unicoherent continua in terms of irreducible continua.

**Theorem (1.1).** A continuum $C$ is hereditarily unicoherent if and only if for every two points $a$ and $b$ of $C$ there exists exactly one continuum $I(a, b)$ irreducible between $a$ and $b$ and contained in $C$.

A continuum $X$ is said to be decomposable provided that there are two proper subcontinua of $X$ whose union is $X$. Otherwise it is said to be indecomposable. Obviously, if a continuum is hereditarily decomposable, then all of its indecomposable subcontinua must be degenerate. See [60], [68] and [94] for further information. In particular, the following results are well-known (see [68]).

**Theorem (1.2).** The following conditions are equivalent for a continuum $X$:

(a) $X$ is indecomposable;
(b) each proper subcontinuum of $X$ has empty interior;
(c) there are three distinct points in $X$ such that $X$ is irreducible between any two of them.

**Theorem (1.3).** Every compact space of dimension greater than one contains an indecomposable (nondegenerate) continuum.

As an immediate consequence of the above result we conclude that:

**Theorem (1.4).** Each hereditarily decomposable continuum is a curve (i.e., it is one-dimensional).

A dendrite means a locally connected continuum containing no simple closed curve. A dendroid is defined as an arcwise connected and hereditarily unicoherent continuum. A $\lambda$-dendroid is defined as a hereditarily decomposable and hereditarily unicoherent continuum. By a tree we understand a one-dimensional acyclic connected polyhedron, i.e., a dendrite with finitely many end points. A continuum is said to be tree-like provided that for each $\varepsilon > 0$ there is a tree $T$ and a surjective mapping $f : X \to T$ such that $f$ is an $\varepsilon$-mapping (i.e., $\text{diam} f^{-1}(y) < \varepsilon$ for each $y \in T$). In particular, if $T$ is an arc, then $X$ is said to be arc-like (or chainable).
It is known that each dendrite is arcwise connected (being a locally connected continuum) and hereditarily unicoherent ([112], (1.1) (v), p. 88) so it is a dendroid. Further, each dendroid is hereditarily decomposable ([10], (47), p. 239), so it is a \( \lambda \)-dendroid (cf. [14], p. 15). H. Cook proved [40] that each \( \lambda \)-dendroid (thus each dendroid) is tree-like. Case and Chamberlin [9] characterized tree-like continua as such continua \( X \) for which every mapping \( f: X \to G \) (where \( G \) is any linear graph) is inessential (i.e., is homotopic to a constant mapping). This property implies one-dimensionality and acyclicity. Namely, a continuum \( X \) is said to be acyclic provided that each mapping from \( X \) into \( S^1 \) is homotopic to a constant mapping, i.e., for all mappings \( f: X \to S^1 \) and \( c: X \to \{p\} \subset S^1 \) there exists a mapping \( h: X \times I \to S^1 \) such that for each point \( x \in X \) we have \( h(x, 0) = f(x) \) and \( h(x, 1) = c(x) \). Finally, it is known that every acyclic curve is hereditarily unicoherent (see property (b) in [112], p. 226). So we have the following seven classes of curves, and each of them is essentially larger than the previous one (see [21], where some mapping properties of these classes are discussed).

\[
\{\text{an arc}\} \subset \{\text{dendrites}\} \subset \{\text{dendroids}\} \subset \{\lambda\text{-dendroids}\} \subset \{\text{tree-like continua}\} \subset \{\text{acyclic curves}\} \subset \{\text{hereditarily unicoherent curves}\}.
\]

Recall that all seven classes above are hereditary.

**B. Mappings**

A surjective mapping \( f: X \to Y \) between continua \( X \) and \( Y \) is said to be:

- **monotone** provided for each point \( y \in Y \) the set \( f^{-1}(y) \) is connected;
- **open** if images of open sets under \( f \) are open;
- **confluent** provided for each subcontinuum \( Q \) in \( Y \) each component of \( f^{-1}(Q) \) is mapped onto \( Q \) under \( f \) ([12], [58]);
- **semi-confluent** provided for each subcontinuum \( Q \) in \( Y \) and for every two components \( C_1 \) and \( C_2 \) of \( f^{-1}(Q) \) either \( f(C_1) \subset f(C_2) \) or \( f(C_2) \subset f(C_1) \) ([82]);
- **weakly confluent** provided for each subcontinuum \( Q \) in \( Y \) some component of \( f^{-1}(Q) \) is mapped onto \( Q \) under \( f \) ([73]);
- **light** provided for each point \( y \in Y \) the set \( f^{-1}(y) \) has one-point components (note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

Obviously monotone mappings are confluent, and open mappings (of compact spaces) are confluent (see [112], Theorem 7.5, p. 148). Evidently, each confluent mapping is semi-confluent; also, each semi-confluent mapping is weakly confluent (see [82], (3.8), p. 13).

The following results on the preservation of the above discussed classes of curves under the considered mappings are known:
(1.5). Monotone, as well as open mappings preserve arcs and dendrites
(see [112], (1.1), p. 165; (1.3), p. 184; (2.41), p. 140 and (6.21), p. 145; (7.7), p. 148 and p. 185).

(1.6). Monotone, as well as open mappings preserve arc-like continua
(see [4], [106]).

(1.7). Semi-confluent (in particular confluent) mappings preserve arcs, dendrites, dendroids, \( \lambda \)-dendroids, tree-like continua and acyclic curves
(see [12], [87], [71]) for confluent mappings, and [55], [74] for semi-confluent ones).

(1.8). Weakly confluent mappings do not preserve any class of the above discussed curves
(because \( f: \mathbb{I} \to \mathbb{S}^1 \) defined by \( f(t) = \exp(4\pi it) \) is weakly confluent).

Recall that it is still not known whether the property of being an arc-like continuum is preserved under confluent ([72] p. 94) or under semi-confluent mappings ([74], p. 263).

2. Dendrites

The following global properties of dendrites are known (see e.g. [68], [94], [112]).

THEOREM (2.1). Every dendrite has the following properties:

(a) it is (uniquely) arcwise connected;
(b) it is hereditarily unicoherent;
(c) it is hereditarily decomposable;
(d) it is a curve;
(e) every subcontinuum of a dendrite is a dendrite.

To formulate local properties of a dendrite we need a concept of the order of a point \( p \) in a continuum \( X \) (in the sense of Menger-Urysohn, written \( \text{ord}(p, X) \)). Let \( n \) stand for a cardinal number. We write

\[ \text{ord}(p, X) \leq n \] provided that for every \( \varepsilon > 0 \) there is an open neighborhood \( U \) of \( p \) such that \( \text{diam}U \leq \varepsilon \) and \( \text{card bd } U \leq n \);

\[ \text{ord}(p, X) = n \] provided that \( \text{ord}(p, X) \leq n \) and for each cardinal number \( m < n \) the condition \( \text{ord}(p, X) \leq m \) does not hold;

\[ \text{ord}(p, X) = \omega \] provided that the point \( p \) has arbitrarily small open neighborhoods \( U \) with finite boundaries \( \text{bd } U \) and \( \text{card bd } U \) is not bounded by any \( n \in \mathbb{N} \).

Thus, for any continuum \( X \) we have

\[ \text{ord}(p, X) \in \{1, 2, \ldots, n, \ldots, \omega, \aleph_0, 2^{\aleph_0}\} \]

(convention: \( \omega < \aleph_0 \)).
Definitions. If \( \text{ord}(p, X) = 1 \), then \( p \) is called an end point of \( X \); if \( \text{ord}(p, X) = 2 \), then \( p \) is called an ordinary point of \( X \); and if \( \text{ord}(p, X) \geq 3 \), then \( p \) is called a ramification point of \( X \). The sets of end points, of ordinary points and of ramification points of a continuum \( X \) are denoted by \( E(X) \), \( O(X) \) and \( R(X) \) respectively.

The following \( n \)-arc theorem is due to K. Menger (see e.g. [5], [68]).

**Theorem (2.2).** If a continuum \( X \) is locally connected and \( \text{ord}(p, X) = n \in \mathbb{N} \), then there are \( n \) points \( a_1, \ldots, a_n \) in \( X \) and \( n \) arcs \( pa_i \) for \( i \in \{1, \ldots, n\} \) such that \( pa_i \cap pa_j = \{p\} \) for every two distinct indices \( i \) and \( j \).

The following local properties of dendrites are known (see e.g. [68], [94], [112]).

**Theorem (2.3).** Every dendrite \( X \) has the following properties:

(a) for every point \( p \in X \) the cardinality of the set of components of \( X \setminus \{p\} \) equals \( \text{ord}(p, X) \) whenever either of these is finite;
(b) \( \dim E(X) = 0 \), (hence \( E(X) \neq \emptyset \));
(c) \( E(X) \in G_6 \);
(d) \( O(X) \) is dense in \( X \);
(e) \( \text{card} R(X) \leq \aleph_0 \) (i.e., every dendrite has at most countably many ramification points);
(f) \( E(X) \) is dense in \( X \) if and only if \( R(X) \) is dense in \( X \).

While the set \( R(X) \) is at most countable in every dendrite \( X \), the set \( E(X) \) can be uncountable. As an example recall the Gehman dendrite [51], pictured in Figure 1.

![Figure 1](image-url)

Figure 1.
Now let us pass to mapping properties of dendrites. Some of them were mentioned previously, viz. invariance of the property of being a dendrite with respect to various classes of mappings (monotone, open, confluent, semi-confluent). Here we recall other ones.

In 1921 Stefan Mazurkiewicz asked in Fund. Math.: Is there a locally connected nonplanable continuum without any simple closed curve? In 1923 Tadeusz Ważewski [111] answered the question in the negative, by constructing a universal dendrite $D_\omega$ (lying in the plane).

Given a class $\mathcal{K}$ of topological spaces, an element $U$ of $\mathcal{K}$ is said to be universal for $\mathcal{K}$ provided for each $X \in \mathcal{K}$ there is a homeomorphism $h : X \to h(X) \subset U$. Examples: a) The Hilbert cube $I^{\aleph_0}$ is universal for the following classes of spaces: separable metric, separable connected metric, compact metric, metric continua. b) The Sierpiński curve is universal for the class of all plane curves. c) The Menger curve (see e.g. [5] for the picture) is universal for the class of all curves. d) The dendrite of Figure 2 is universal for the class of all dendrites $D$ satisfying $\text{card } R(D) \leq 1$.

\[ F_\omega = \]

Figure 2.

Moreover, for each integer $n \geq 3$ Ważewski constructed (also in the plane) a dendrite $D_n$ which is universal for the class of all dendrites with order of points not greater than $n$ (called the standard universal dendrite of order $n$). They are characterized as follows.

**Theorem (2.4).** Let $n \in \{3, 4, \ldots, \omega \}$. A dendrite $X$ is homeomorphic to the standard universal dendrite $D_n$ if and only if

\[
\text{ord}(p, X) = n \quad \text{for each point} \quad p \in R(X),
\]

and for each arc $A \subset X$ we have $\text{cl } (A \cap R(X)) = A$.

Sam B. Nadler Jr. has decided to put a picture of the universal dendrite $D_\omega$ on the front cover of this book [94].

The standard universal dendrites $D_n$ and $D_\omega$ have many interesting mapping properties. For example, $D_3$ and $D_\omega$ have the property of an arc that all their open images are homeomorphic to the domain (the fan $F_\omega$ has this property, too). Moreover (see [18], p. 493).

(2.5). Among all standard universal dendrites $D_n$ for $n \in \{3, 4, \ldots, \omega \}$ only $D_3$ and $D_\omega$ are homeomorphic with all their open images.
A wide spectrum of mapping properties of dendrites is presented in [30]. As an immediate consequence of Ważewski’s result we have a corollary.

**Corollary (2.6).** Every dendrite is planable.

A continuum is called a *local dendrite* provided that each of its points has a closed neighborhood which is a dendrite. Local dendrites are characterized as locally connected continua containing at most finitely many simple closed curves. In 1930 K. Kuratowski proved [67] that a local dendrite is nonplanable if and only if it contains one of the two primitive skew graphs: $K_{3,3}$ which is the union of all six edges of a tetrahedron and of a segment joining two midpoints of a pair of disjoint edges, and $K_5$ which is the union of all six edges of a tetrahedron and of four segments joining the center of the tetrahedron with its four vertices. The same characterization is true if the class of local dendrites is replaced by that of graphs. In 1937 S. Claytor obtained [39] a complete characterization of nonplanable locally connected continua. A locally connected continuum distinct from 2-sphere is nonplanable if and only if it contains either one of the Kuratowski primitive skew graphs $K_{3,3}$ and $K_5$ or one of the two curves $C_1$ and $C_2$ pictured in Figure 3.

Finally let us recall one more mapping property of dendrites. In 1932 K. Borsuk, using Whyburn’s cyclic element theory (see [68] and [112]), proved the following important result (compare [94] for another proof).

**Theorem (2.7).** Every dendrite has the fixed point property.

3. **Dendroids-global properties. Fixed point property.**

In 1957 B. Knaster in Wrocław initiated a further study of acyclic curves. His idea was to delete local connectedness from the definition of a dendrite $X$ keeping the following property:

\[(3.1) \quad \forall p, q \in X \exists! \quad I(p, q) \subset X \text{ and } I(p, q) = pq.\]

Note that it is not the same if we demand that

\[\forall p, q \in X \quad \exists! \quad pq \subset X\]

(sin(1/x)-circle has this property).

Knaster named the new class of continua *dendroids*. Recall that according to Miller’s result (Theorem (1.1) above) condition (3.1) is equivalent to hereditary unicoherence of $X$. Thus the two concepts coincide, i.e., the dendroid as defined by Knaster does agree with the dendroid as defined previously, in Part 1.

According to the definition, each dendroid has all (global) properties of dendrites, listed in Theorem (2.1) (obviously with “dendroid” in place of “dendrite” in (e)). Further, it can be proved ([11], T26, p. 197) that
(3.2). A continuum is a dendroid if and only if it is uniquely arcwise connected and hereditarily arcwise connected.

Let us observe further that

(3.3). Every locally connected dendroid is a dendrite.

Another important structural property of dendroids was shown by K. Borsuk in 1954 [6].

**Proposition (3.4).** In each dendroid one has:

(3.5). The closure of the union of an increasing sequence of arcs is an arc.

As a consequence it follows that:

(3.6). In each dendroid each arc is contained in a maximal arc.
Recall that the property (3.5) was later taken as the definition of a new class of spaces called \( B \)-spaces (\( B \) is for Borsuk) which was studied by W. Holsztyński [61]. This property was used by K. Borsuk in [6] to show:

**THEOREM (3.7).** Every dendroid has the fixed point property.

This result was generalized by L. E. Ward, Jr., to continuous multivalued functions and to upper semicontinuous continuum-valued functions.

Recall that a multivalued function \( F \) from a space \( X \) to a space \( Y \) means a correspondence which assigns to each point \( x \) of \( X \) a nonempty closed subset \( F(x) \) of \( Y \). If \( Y = X \), then a point \( x \in X \) is called a fixed point under \( F \) provided \( x \in F(x) \). If the equality \( \lim x_n = x \) implies the inclusion \( \text{Ls } F(x_n) \subset F(x) \) (the inclusion \( F(x) \subset \text{Li } F(x_n) \)), then \( F \) is said to be upper (lower) semicontinuous. \( F \) is defined to be continuous if both conditions hold. Recall:

\[
\begin{align*}
p \in \text{Ls } A_n & \quad \text{provided that each neighborhood of } p \text{ intersects} \\
p \in \text{Li } A_n & \quad \text{provided that each neighborhood of } p \text{ intersects} \\
\text{Li } A_n = \text{Ls } A_n, & \quad \text{then } A_n \text{ is convergent to } \text{Lim } A_n.
\end{align*}
\]

In the case when \( F(x) \) is a singleton for each \( x \in X \), upper semicontinuity becomes continuity. If \( F(x) \) is a continuum for each \( x \in X \), then \( F \) is said to be continuum-valued.

The above mentioned results of Ward are as follows ([107], Theorem 2, p. 926; [108], Theorems 1 and 2, p. 162 and 163):

**THEOREM (3.8).** Every dendroid has the fixed point property for continuous multivalued functions.

**THEOREM (3.9).** An arcwise connected continuum has the fixed point property for upper semicontinuous continuum-valued functions if and only if it is hereditarily unicoherent.

A further step forward was to prove the fixed point property for some kinds of multivalued functions of \( \lambda \)-dendroids. An essential difference (and trouble) was the lack of arcwise connectivity in \( \lambda \)-dendroids. Important progress was made by R. Manfré in 1976, who replaced natural linear ordering of an arc by linear ordering of tranches in hereditarily decomposable irreducible continua, using Kuratowski's theory on the structure of these continua. He proved ([86], p. 120) the following result (which has been generalized to the non-metric case by T. Maćkowiak [81]).

\[ (3.10) \text{ THEOREM.} \quad \text{Every } \lambda \text{-dendroid has the fixed point property for upper semi-continuous continuum-valued functions.} \]

Several further generalizations of this result to various classes of multivalued functions of \( \lambda \)-dendroids were obtained by T. Maćkowiak in [83].
The next class of considered curves consists of tree-like continua. The problem concerning the fixed point property of tree-like continua (for mappings) was posed in 1951 by R. H. Bing [4] and has been solved in the negative after 29 years by D. P. Bellamy [1] who constructed a suitable counterexample in 3-space (compare also [50] for a more geometrical description of a very similar example of a tree-like continuum without the fixed point property). The problem of the possibility of constructing a planar example is still open, as well as the problem of a structural characterization of those tree-like continua which do have the fixed point property.

4. Dendroids-local properties

Local properties of dendrites were connected with the concept of the order of a point in the sense of Menger-Urysohn. However, it can be immediately observed that this concept is not good enough for studying local properties of dendroids. To illustrate this, note that, according to the definition of ord (in the sense of Menger-Urysohn) each point of the Cantor fan $F_C$ (i.e., the cone over the Cantor set) is of order $2^{\aleph_0}$, while it would be better to consider points of the Cantor set as end points, interior points of the straight line segments as ordinary points, and the top of the fan as the only ramification point of order $2^{\aleph_0}$.

Given a dendroid $X$ and a point $p \in X$, let $r(p, X)$ denote the number of arc components of $X \setminus \{p\}$, that is, the number of arcs $pa_i$ for $a_i \in X \setminus \{p\}$ such that $pa_i \cap pa_j = \{p\}$ for $i \neq j$. We also keep $r(p, X) = \omega$ in the previous sense (of ord). This is just the definition of order of a point in the classical sense, due to W.C. Young (about 1905) and studied later (about 1912) by Z. Janiszewski. Since every locally connected dendroid is a dendrite, we see by Theorem (2.3) (a) that the two definitions do agree. As previously, we set, for a dendroid $X$:

\begin{align*}
E(X) &= \{p \in X \mid r(p, X) = 1\}, \\
O(X) &= \{p \in X \mid r(p, X) = 2\}, \\
R(X) &= \{p \in X \mid r(p, X) \geq 3\}.
\end{align*}

If $\text{card } R(X) = 1$, the the dendroid $X$ is called a fan. It is said to be finite (countable) if $E(X)$ is finite (resp. countable). The mapping cylinder for the Cantor-Lebesgue mapping of the Cantor set $C$ onto $\mathbb{I}$, i.e., the example of Figure 4 shows that (e) of Theorem (2.3) (i.e., $\text{card } R(X) \leq \aleph_0$) is not true if $X$ stands for a dendroid. Moreover, it is known (see [10], Corollary 7, p. 241 and p. 245) that

\begin{equation}
(4.1). \text{ For every dendroid } X \text{ there exists a dendroid } B(X) \supset X \text{ such that } R(B(X)) = X. \text{ Furthermore, there is an estimate }
\end{equation}

\[ r(p, B(X)) \leq r(p, X) + 4 \quad \text{for each point} \quad p \in X. \]
(4.2). There exists a dendroid $X$ such that $R(X)$ is homeomorphic to $X$ and $r(p, X) \leq 4$ for each point $p \in X$.

In the light of these results, any dendroid can be considered as the set of ramification points of a dendroid. The methods of construction of (4.1) and (4.2) rely on an embedding the dendroid $X$ into the Hilbert cube $I^8$, next mapping the Cantor set $C$ onto $X$ in a special way and taking the mapping cylinder. Using similar methods and repeating this construction countably many times, a smooth dendroid $X$ was constructed in [20] without ordinary points (i.e., such that $O(X) = \emptyset$, and consequently $X = E(X) \cup R(X)$). Such a phenomenon is not possible for plane dendroids. A subset of a space is said to be planable at a point provided the point has a planable neighborhood.

(4.3). If a dendroid $X$ satisfies $O(X) = \emptyset$ then $X$ is not planable (moreover, it is planable at no one of its points).

Consequently, Corollary (2.6), when true for dendrites, is no longer true if dendroids are under consideration.

Recall that a dendroid $X$ is said to be smooth at a point $p \in X$ provided that for every sequence of points $x_n$ of $X$ converging to a point $x$ the sequence of arcs $px_n$ converges to the arc $px$. A dendroid $X$ is said to be smooth provided that there exists a point $p \in X$ at which $X$ is smooth.

Now let us fix our attention on the set of end points of a dendroid. Recall that we already know (see Theorem (2.3) (b) and (c)) that for each dendrite $X$ the set $E(X)$ is a 0-dimensional $G_\delta$ set. One of the first questions asked for dendroids was if the same is true for this larger class of curves (when $r(p, X)$ is considered in place of ord $(p, X)$). In 1961 A. Lelek showed [70] that this is not the case. First of all observe that:

(4.4). For every dendroid $X$ the set $E(X)$ of end points of $X$ does not contain any nondegenerate continuum.
In fact, if there is a nondegenerate continuum $K \subset E(X)$, then
\[
\forall p, q \in K \quad \exists I(p, q) \subset K \text{ and } I(p, q) = pq
\]
so $pq \subset K \subset E(X)$, a contradiction.

Concerning the dimension of $E(X)$, A. Lelek constructed in [70] a dendroid (even a fan, i.e., a dendroid with only one ramification point) having 1-dimensional set of end points. This fan is denoted by $F_L$, is called the Lelek fan, and it is located in the Cantor fan: $F_L \subset F_C$. Its construction is the following.

Set all dyadic rationals $d_n$ of the open unit interval $(0, 1)$ in a sequence, e.g., $d_n \in \{1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, \ldots \}$. Let the Cantor set $C$ be understood as $E(F_C)$ and let $C_n$ stand for the $n$-th "portion" of $C$ with diameter $1/3^n$ so that $C_n = C \cap [2/3^n, 1/3^{n-1}]$. Thus
\[
C = E(F_C) = \{0\} \cup \bigcup \{C_n : n \in \{1, 2, 3, \ldots \}\}
\]

Let $F_n \subset F_C$ be the upper part of the cone over $C_n$ cut on the level $d_n$ (thus $F_n$ is similar to $F_C$ with the ratio equal to $d_n$). Define inductively
\[
\Delta^1(F_C) = \text{cl} \left( \bigcup \{F_n : n \in \{1, 2, 3, \ldots \}\} \right) \subset F_C,
\]
\[
\Delta^2(F_C) = \text{cl} \left( \bigcup \{\Delta^1(F_n) : n \in \{1, 2, 3, \ldots \}\} \right),
\]

and, for each integer $i > 0$,
\[
\Delta^{i+1}(F_C) = \text{cl} \left( \bigcup \{\Delta^i(F_n) : n \in \{1, 2, 3, \ldots \}\} \right).
\]

Then $F_L$ is defined by
\[
F_L = \bigcap \left\{ \Delta^i(F_C) : i \in \{1, 2, 3, \ldots \} \right\}.
\]

Another construction: let $v$ denote the top of $F_C$. For each point $c_n = 2/3^n \in C = E(F_C)$ let $S_n$ be the straight line segment of length $d_n$ contained in the straight line segment $vc_n \subset F_C$. Put
\[
D_1 = \text{cl} \left( \bigcup \{S_n : n \in \{1, 2, 3, \ldots \}\} \right).
\]

Continuing in this way with $F_n$ in place of $F_C$ we define an increasing sequence of fans $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ and we put
\[
F_L = \text{cl} \left( \bigcup \{D_n : n \in \{1, 2, 3, \ldots \}\} \right).
\]
Properties: \( \{v\} \cup E(F_L) \) is connected, hence it is of dimension 1, and consequently \( E(F_L) \) is one-dimensional. Further, \( E(F_L) \) is dense in \( F_L \). In 1989 W.J. Charatonik [38] and in 1990 W.D. Bula and L.G. Oversteegen [8] proved that the Lelek fan is unique, in the sense that

\[(4.5)\] Every smooth fan \( X \) having dense set \( E(X) \) is homeomorphic to \( F_L \).

The Lelek fan has a very strong mapping property, namely [38]

\[(4.6)\] Any confluent image of the Lelek fan \( F_L \) is homeomorphic to \( F_L \).

Other properties of the Lelek fan will be recalled in Theorem (5.26).

Another plane dendroid was constructed in 1980 by David P. Bellamy [2]. Using universe limits and geometric techniques, he constructed a dendroid \( M \subset \mathbb{R}^2 \) with \( E(M) \) connected and such that the end points and only the end points of \( M \) are arcwise accessible from the complement of \( M \). By a modification of this example he constructed (in the same paper) another example of a plane dendroid \( K \) whose end points and accessible points coincide, but with \( E(K) \) not connected.

A metric \( d \) on a dendroid \( X \) is said to be radially convex with respect to a point \( p \in X \) provided that for every two points \( x \) and \( y \) of \( X \) if \( y \in px \) and \( y \neq x \), then \( d(p, y) < d(p, x) \). In other words, for each point \( x \in X \) the metric \( d \) considered on the arc \( px \) is an isometry. It is known ([31], Theorem 10, p. 310) that

\[(4.7)\] If a dendroid \( X \) is smooth at a point \( p \), then there is an equivalent metric \( d \) on \( X \) that is radially convex with respect to \( p \).

Using the radially convex metric it can easily be shown that

\[(4.8)\] Every smooth fan can be embedded into the Cantor fan \( F_C \).

Let a dendroid \( X \) be smooth at a point \( p \in X \) and let a metric \( d \) be radially convex with respect to \( p \). For each \( n \in \mathbb{N} \) put

\[F_n = \{ y \in X : (\exists x \in X) \ (y \in px \text{ and } d(x, y) \geq 1/n) \} ,\]

and note that each \( F_n \) is closed (by smoothness of \( X \) at \( p \)). Obviously,

\[E(X) = X \setminus \bigcup \{ F_n : n \in \mathbb{N} \} ,\]

whence we conclude that

\[(4.9)\] If a dendroid \( X \) is smooth, then \( E(X) \in G_{\delta} \).

Lelek has shown in [70] the following results on \( E(X) \).

\[(4.10)\] If a dendroid \( X \) is planable, then \( E(X) \in G_{\delta0\delta} \).
(4.11). For every dendroid $X$ if $E(X) \cap \text{cl} R(X) \in G_\delta$, then $E(X) \in G_\delta$.

Thus,

(4.12). If the set $R(X)$ is a dendroid $X$ is closed, then $E(X) \in G_\delta$,

whence

(4.13). For every fan $X$ we have $E(X) \in G_\delta$.

In particular, $E(F_L) \in G_\delta$.

In the same paper [70] Lelek constructed a plane dendroid $X$ such that $E(X)$ is dense in $X$, and $E(X) \in F_\sigma \setminus G_\delta$. Namely $X$ is obtained from the Cantor fan $F_C$ by replacing each rational segment $pc$ in $F_C$ with a homeomorphic copy of a dendrite $D$ pictured in Figure 5.

![Figure 5.]

The one-point union of the constructed dendroid $X$ and of $F_L$ with the “tops” identified gives an example of a dendroid $Y$ for which $E(Y)$ is not of the first Borel class (it is neither $F_\sigma$ nor $G_\delta$).

Further progress was made in a sequence of papers by Jacek Nikkel during 1983–1991 (see [96], [97], [98], [99], [100]). A basic step in this direction was the investigation of the structure of $E(X)$ and $R(X)$ of a dendroid $X$. His results can be summarized as follows.

Let $G$ be the Gehman dendrite (see Part II). It can be characterized as a dendrite $G$ satisfying the following two conditions: 1) $E(G)$ is homeomorphic to the Cantor ternary set, and 2) $\text{ord}(p, G) \leq 3$ for each $p \in G$.

A dendroid $X$ is called a Gehman dendroid provided that there is a one-to-one mapping from $G \setminus E(G)$ onto some dense subset of $X$ (i.e., $X$ is a compactification of $G \setminus E(G)$). For an example see Figure 6.

**Theorem** (4.14). Each Gehman dendroid contains (topologically) the Gehman dendrite $G$. 
Theorem (4.15). The following conditions are equivalent for each dendroid $X$:

(I) $X$ has uncountably many end points;

(II) $X$ contains (topologically) either some Gehman dendroid or an arc $J$ (may be degenerate) from which uncountably many distinct branches begin (i.e., $X \setminus J$ has uncountably many arc-components);

(III) $X$ contains some Gehman dendroid or there exists an uncountable family of pairwise disjoint nondegenerate arcs in $X$;

(IV) $X$ contains (topologically) either the Gehman dendrite $G$, or the Cantor fan $F_C$, or the Cantor comb.

The Cantor comb is $I \times \{0\} \cup C \times I$ (where $C$ is the Cantor set).

A property that plays an important role in the study of the set $R(X)$ in a dendroid $X$ is the possibility of covering $R(X)$ by countably many arcs.

Theorem (4.16). Let $X$ be a dendroid such that $R(X)$ cannot be covered by countably many arcs. Then either $X$ contains (topologically) the Gehman dendrite $G$, or there is an uncountable family of pairwise disjoint triods in $X$.

Recall that, by a theorem of R.L. Moore [92], every uncountable collection of triods in the plane contains an uncountable subcollection every two elements of which do intersect. So, there is no uncountable collection of pairwise disjoint
trioids in the plane. Therefore, for planable dendroids $X$ the structure of $R(X)$ can be different from that for nonplanable ones.

**Theorem (4.17).** For each planable dendroid $X$ the set $R(X)$ can be covered by countably many arcs.

Consider the dendroid of the Figure 7 $X$ which is sometimes called the Gehman dendrite with ski.

![Figure 7.](image)

It has the following properties:

(i) $R(X)$ cannot be covered by countably many arcs (so $X$ is nonplanable), and

(ii) $X$ does not contain any uncountable family of pairwise disjoint trioids.

For each cardinal number $\alpha \leq 2^{\aleph_0}$ and for a dendroid $X$ put

$$R_\alpha(X) = \{ p \in X : r(p,X) = \alpha \} \text{ and } S_\alpha(X) = \{ p \in X : r(p,X) \geq \alpha \} .$$

Thus $R_1(X) = E(X), R_2(X) = O(X), S_3(X) = R(X)$.

**Theorem (4.18).** If the dendroid $X$ is planable, and $A \subset X$ is an arc, then $A \cap S_3(X) \in G_{\delta\sigma}$ and $A \cap S_4(X) \in G_{5\sigma}$. Consequently, $S_3(X) \in G_{5\sigma}$ and $S_4(X) \in G_{5\sigma}$.
THEOREM (4.19). Let the dendroid $X$ be planable. Then

1. $R_1(X) \in G_{\delta\sigma}$;  
2. $R_2(X) \in F_{\sigma\delta\sigma}$ and $S_2(X) \in F_{\sigma\delta\sigma}$;  
3. $R_3(X) \in F_{\sigma\delta\sigma} \cap G_{\delta\sigma}$ and $S_3(X) \in G_{\delta\sigma}$;  
4. $R_4(X) \in G_{\delta\sigma}$ and $S_4(X) \in G_{\delta\sigma}$;  
5. $S_5(X)$ is at most countable.

Recall that a continuous image of a Borel set is called an analytic set. A set is defined to be co-analytic if its complement is analytic.

THEOREM (4.20). Let an arbitrary dendroid $X$ be given. Then

6. $S_\alpha(X)$ is analytic for each $\alpha \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$;  
7. $R_{2^{\aleph_0}}(X)$ is analytic;  
8. $R_\alpha(X)$ is co-analytic for each $\alpha \leq \aleph_0$.

In particular, $R(X)$ is analytic and $O(X)$ is co-analytic.

Nikiel and Tymchatyn in [100] showed that the results presented in Theorem (4.20) are the best possible in the sense that the following five examples were constructed.

Example (4.21). There exists a dendroid $X$ such that $S_3(X)$, $R_{2^{\aleph_0}}(X)$ and $R_2(X)$ are not Borel. (By Theorem (4.20), $S_3(X)$ and $R_{2^{\aleph_0}}(X)$ are analytic, and $R_2(X)$ is co-analytic).

Example (4.22). For each cardinal number $\alpha \in \{3, 4, \ldots, \aleph_0\}$ there is a smooth dendroid $X_\alpha$ such that $R_\alpha(X_\alpha)$ is not Borel.

Example (4.23). There exists a smooth dendroid $Y$ (being a wedge of dendroids of Examples (4.21) and (4.22)) such that $R_2(Y)$, $R_\beta(Y)$ and $S_\beta(Y)$ for $\beta \in \{3, 4, \ldots, \aleph_0, 2^{\aleph_0}\}$ are not Borel.

Example (4.24). There exists a smooth dendroid $X$ such that

(a) $S_3(X)$ is contained in an arc in $X$;  
(b) $R_1(X)$ is closed;  
(c) $S_3(X) = R_{2^{\aleph_0}}(X)$ (i.e., each ramification point is of order $2^{\aleph_0}$);  
(d) $R_{2^{\aleph_0}}(X)$ is analytic and not Borel;  
(e) $R_2(X)$ is co-analytic and not Borel.

Example (4.25) There exists a dendroid $Z$ such that $R_1(Z)$ is co-analytic and not Borel.
5. Dendroids with some special properties

Consider the two examples of the Figure 8 of fans $X$ and $Y$.

Lengths of arcs in $X$ are bounded, while in $Y$ they are not. But length is not a topological concept, it does not belong even to metric topology. To express this idea in a suitable way let us recall the following definition.

A continuum $S$ is said to be uniformly arcwise connected (u.a.c.) [11] provided that it is arcwise connected and that for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that for every arc $A \subset X$ there are points $a_0, a_1, \ldots, a_k \in A$ such that

$$A = \bigcup \{a_i a_{i+1} : i \in \{0, 1, \ldots, k-1\}\}$$

and $\text{diam} \ a_i a_{i+1} < \varepsilon$ for each $i \in \{0, 1, \ldots, k-1\}$ (i.e., if each arc in $X$ contains $k+1$ points which cut the arc into subarcs of diameter less than $\varepsilon$).

Note that to be u.a.c. is a hereditary property for dendroids.

**Theorem (5.1).** Uniform arcwise connectedness is an invariant under continuous mappings onto one-arcwise connected continua.

One-arcwise connectedness of the range space is essential (Fig. 9):

A continuum $X$ is said to be uniformly pathwise connected (u.p.c.) [66] provided that there is a family $\mathcal{P} = \{p : I \to X\}$ of paths satisfying

1. for every two points $x$ and $y$ in $X$ there is a path $p \in \mathcal{P}$ such that $p(0) = x$ and $p(1) = y$ (i.e., a path joining $x$ with $y$);

2. for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that for each path $p \in \mathcal{P}$ there are numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for each $i \in \{1, \ldots, k\}$ we have $\text{diam} \ p([t_{i-1}, t_i]) \leq \varepsilon$.

Obviously, each u.a.c. continuum is u.p.c., but not conversely: $I^2$ is u.p.c. while not u.a.c. For one-arcwise connected continua these two concepts coincide. W. Kuperberg proved [66] the following results.
THEOREM (5.2). Uniform pathwise connectedness is an invariant under continuous mappings.

THEOREM (5.3). A continuum $X$ is u.p.c. if and only if there exists a mapping of the Cantor fan $F_C$ onto $X$.

COROLLARY (5.4). The Cantor fan is a common model in the class of all u.a.c. fans and in the class of all u.a.c. dendroids.

Recall that a dendroid $X$ is said to be smooth at a point $p \in X$ provided that

$$\forall a \in X \forall \{a_n \in X : n \in \mathbb{N}\} \quad (a_n \rightarrow a) \implies (pa_n \rightarrow pa).$$

A dendroid $X$ is said to be smooth provided that there exists a point $p \in X$ such that $X$ is smooth at $p$. The point $p$ is called the initial point of $X$. The concept of smoothness was defined first for fans (with the top of the fan as an initial point, [13]), next for dendroids [31]. It was later extended to continua which are hereditarily unicoherent at a point (i.e., such that the intersection of any two subcontinua each containing the given point is connected, [52]) and finally extended to arbitrary continua [76] in the following way. A continuum $X$ is said to be smooth at a point $p \in X$ provided that for each point $x \in X$, for each continuum $K$ containing the points $p$ and $x$ and each sequence of points $x_n$ in $X$ converging to $x$ there exists a sequence of continua $K_n$ each of which contains the points $p$ and $x_n$ such that $K_n$ converge to $K$. A continuum $X$ is said to be smooth provided that there exists a point $p$ at which it is smooth. If the considered continuum is a dendroid, then the two definitions do agree. Note that smoothness of dendroids (not of continua) is a hereditary property. It is known that

(5.5). If a continuum is smooth at $p$, then it is locally connected at $p$.

Irreducible smooth continua were studied in [15] and [76].

T. Maćkowiak obtained the following result ([75] and [76]).
THEOREM (5.6). Smoothness of continua (of dendroids, in particular) is an invariant under confluent mappings.

The result was extended to weakly monotone mappings. A mapping \( f: X \to Y \) between continua \( X \) and \( Y \) is said to be \textit{weakly monotone} provided that for each subcontinuum \( Q \) in \( Y \) with \( \text{int} Q \neq \emptyset \) each component of \( f^{-1}(Q) \) is mapped onto \( Q \) under \( f \). Obviously, each confluent mappings is weakly monotone. T. Mackowiak proved in [77] that

(5.7). A \textit{weakly monotone image of a dendroid is a dendroid}.

Exploiting their characterization of non-smooth dendroids E. E. Grace and E. J. Vought proved [53] the following result.

THEOREM (5.8). \textit{Weakly monotone mappings preserve smoothness of dendroids}.

The mentioned characterization is the following. A dendroid \( X \) is of \textit{type 1} provided that there are in \( X \): a point \( p \) and a sequence of points \( a_n \) converging to a point \( a \) such that \( X = \text{cl} \bigcup \{pa_n: n \in \mathbb{N}\} \), the sequence of arcs \( pa_n \) converges to a continuum \( L \), and there are a point \( s \in E(L) \setminus \{a\} \) and an open neighborhood \( U \) of \( s \) such that if \( C \) is a component of \( U \cap L \) containing \( s \), then \( C \cap \bigcup \{pa_n: n \in \mathbb{N}\} = \emptyset \).

A dendroid \( X \) is of \textit{type 2} provided that there are in \( X \): two points \( s \) and \( t \), two sequences of points \( a_n \) and \( b_n \) which converge to some points \( a \) and \( b \) respectively, such that

\[
X = \left( \bigcup \{sa_n: n \in \mathbb{N}\} \right) \cup \text{sabt} \cup \left( \bigcup \{tb_n: n \in \mathbb{N}\} \right),
\]

where \text{sabt} stands for the arc from \( s \) to \( t \) passing thru \( a \) and \( b \), the sequences of arcs \( sa_n \) and \( tb_n \) converge to the arcs \( sa \) and \( tb \) respectively, and \( \text{diam} (sa_n \cap st) \) as well as \( \text{diam} (tb_n \cap st) \) tend to \( 0 \).
THEOREM (5.9). A dendroid is non-smooth if and only if it contains either a dendroid of type 1 or a dendroid of type 2.

Fitzgerald Burton Jones defined in [62] the following set-valued function on a given continuum $X$. Let $x \in X$. Put

$$T(x) = \{y \in X: \text{if } K \text{ is a subcontinuum of } X \text{ and if } y \in \text{int } K \subset K \subset X, \text{ then } x \in K\}.$$ 

Example in Figure 12.

The following results are shown in [31].

THEOREM (5.10). A dendroid $X$ is smooth at $p \in X$ if and only if

$$px \cap T(x) = \{x\} \text{ for each point } x \in X.$$
THEOREM (5.11). A dendroid $X$ is smooth if and only if for every two of its points $x$ and $y$ we have either $xy \cap T(x) = \{x\}$ or $xy \cap T(y) = \{y\}$.

A partial order $\leq$ on a set $X$ is a reflexive, antisymmetric and transitive relation. A partial order $\leq$ is continuous on $X$ if $\leq$ is a closed subset of $X \times X$. Let $X$ be a dendroid and let $p \in X$. For $x, y \in X$ we write $x \leq_p y$ if and only if $x \in py$. Note that $\leq_p$ is a partial order on $X$. It is shown in [64] that

THEOREM (5.12). A dendroid $X$ is smooth if and only if there is in $X$ a point $p$ such that the partial order $\leq_p$ is continuous on $X$.

We say that a mapping $f : X \to Y$ between dendroids $X$ and $Y$ is order preserving with respect to a point $p \in X$ provided that for every two points $x$ and $y$ in $X$ the condition $x \leq_p y$ implies the condition $f(x) \leq_{f(p)} f(y)$. Note that a mapping $f$ is $\leq_p$-preserving if and only if for each point $x \in X$ the partial mapping $f \mid px$ is monotone.

The following theorem is proved in [31].

THEOREM (5.13). Let $X$ be a dendroid and let $p \in X$. The following conditions are equivalent:

1. $X$ is smooth at $p$;
2. $X$ admits a radially convex metric with respect to $p$;
3. $\leq_p$ is closed;
4. there exists a surjective mapping $f : F_C \to X$ from the Cantor fan $F_C$ onto $X$ which is $\leq_v$-preserving, where $v$ is the top of $F_C$.

COROLLARY (5.14). Every smooth dendroid is u.a.c.

Another characterization of smooth dendroids, in terms of a homotopy, was given by Mohler in [90]. Recall that a mapping $f : X \to Y \subset X$ is called a retraction provided that the partial mapping $f \mid Y$ is the identity (or, equivalently, if $f \circ f = f$). A homotopy $h : X \times I \to X$ means a mapping such that for every point $x \in X$ we have $h(x, 0) = x$ and $h(x, 1) = c$ for some $c \in X$. (Then we say that $h$ contracts $X$ to the point $c$). A homotopy $h : X \times I \to X$ is called a retracting homotopy if for each $t \in I$ the mapping $h_t : X \to X$ given by $h_t(x) = h(x, t)$ is a retraction.

THEOREM (5.15). A dendroid $X$ is smooth if and only if it admits a retracting homotopy $h : X \times I \to X$ that contracts $X$ to a point $c$ such that $h(c, t) = c$ for each $t \in I$.

As we already know (see (4.8)) the Cantor fan is a universal element for the class of all smooth fans. An analogous question (on the existence of a universal element) for the class of smooth dendroids [31] has been answered in the affirmative.

THEOREM (5.16). There exists a universal smooth dendroid.

At least three constructions of a universal smooth dendroid are known. The first was by Grisolakis and Tymchatyn in [56]. Their proof used embeddings
of smooth dendroids into the hyperspace of the pseudo-arc [57], and it was — for this reason — rather ineffective and hard to understand in the sense that it was hard to see any embedding of a concrete dendroid into the universal one. The next two constructions used inverse limits. One given in [37] was better that the previous one, but still it was not constructive. Finally Mohler and Nikkel [91] constructed a universal smooth dendroid $X$ as the inverse limit of an inverse sequence of trees with open bonding mappings and moreover such that $X$ consists of end points and of ramification points only, i.e., $O(x) = \emptyset$, and that $E(X)$ is closed.

Krasinkiewicz and Minc showed [65] that

(5.17). There is no universal element in the class of all dendroids.

An important structural property of continua is the property of Kelley [63]. A local version of this property is due to Wardle [110] and runs as follows. A continuum $X$ is said to have the property of Kelley at a point $x \in X$ provided that for each sequence of points $x_n$ converging to $x$ and for each continuum $K$ in $X$ containing the point $x$ there is a sequence of continua $K_n$ in $X$ with $x_n \in K_n$ for each $n \in \mathbb{N}$ and converging to $K$. R.W. Wardle proved in [110] that

(5.18). For each continuum $X$ the set \{ $x \in X$: $X$ has the property of Kelley at $x$ \} is a dense $G_\delta$-set.

A continuum $X$ is said to have the property of Kelley if it has the property at each point $x \in X$. In other words, $X$ has the property of Kelley if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every two points $a$ and $b$ in $X$ and for each subcontinuum $A$ of $X$ containing the point $a$ there exists a subcontinuum $B$ of $X$ containing the point $b$ such that the condition $d(a, b) < \delta$ implies the condition $H(A, B) < \varepsilon$, where $d$ is a metric on $X$ and the Hausdorff distance $H$ is defined by

$$H(A, B) = \max \{ \sup \{d(p, B): p \in A \}, \sup \{d(q, A): q \in B \} \}.$$  

(It is well-known that $H$ is a metric on the hyperspace $C(X)$ of all subcontinua of a continuum $X$.)

S.T. Czuba [49] proved the following implications.

**Theorem (5.19).** Let $X$ be a dendroid. Then:

[X has the property of Kelley]  
\[ \implies [\forall x, y \in X \text{ if } xy \cap T(x) \neq \{x\}, \text{ then } y \in T(x)] \]
\[ \implies [X \text{ is smooth}] \]
\[ \implies [X \text{ is locally connected at some point}] . \]

The assumption that $X$ is a dendroid is essential: a $\lambda$-dendroid obtained as a compactification of $F_C \setminus \{\text{top}\}$ such that the remainder is an arc has the property of Kelley and is not smooth (it is not locally connected at any point).
Question (5.20) For what continua $X$ does the property of Kelley imply local connectedness of $X$ at some point?

Fans having the property of Kelley have been characterized in [26] and [27]. To formulate the characterization two auxiliary symbols are needed. $F_{HP}$ is a harmonic fan with the limit segment prolonged out of the end point. $F_C^\omega$ is obtained from the fan $F_\omega$ (see Example d before Theorem (2.4)) replacing each straight line segment of $F_\omega$ from the top to the end point by a copy of the Cantor fan $F_C$ of the same diameter as the diameter of the replaced segment (see Figure 13).

![Figure 13.](image-url)

**Theorem (5.21).** Let a fan $X$ with the top $v$ be given. Then the following conditions are equivalent:

1. $X$ has the property of Kelley;
2. $X$ is smooth and for no countable subset of end points
   \[ \{e_n \in E(X) : n \in \mathbb{N}\} \]
   the union $\bigcup \{ve_n : n \in \mathbb{N}\}$ is homeomorphic to $F_{HP}$;
3. $X$ is smooth and the set $\{v\} \cup E(X)$ is closed;
4. $X$ is embeddable into $F_C^\omega$ in such a way that end points of $X$ are mapped to end points of $F_C^\omega$, i.e., there is a homeomorphism $h : X \to h(X) \subset F_C^\omega$ with $E(h(X)) \subset E(F_C^\omega)$;
5. $X = \lim(X_n, f_n)$, where $X_n$ are locally connected fans and the bonding mappings $f_n$ are open;
6. $X = \lim(X_n, f_n)$, where $X_n$ are finite fans (i.e. having finitely many end points), and $f_n$ are confluent.

Characterizations of dendroids having the property of Kelley are not known.
Images of the Cantor fan under confluent and related mappings are characterized in [28]. Sample results are the following.

**Theorem (5.22).** Let the continuum $Y$ be given. Then the following conditions are equivalent:

(i) $Y$ is the image of $F_C$ under a confluent mapping;
(ii) $Y$ is a fan with the property of Kelley or $Y$ is an arc;
(iii) $Y$ is a smooth fan with $R(Y) \cup E(Y)$ closed or $Y$ is an arc.

**Theorem (5.23).** Let the continuum $Y$ be given. Then the following conditions are equivalent:

(a) $Y$ is the image of $F_C$ under an open mapping;
(b) $Y$ is the image of $F_C$ under an open and light mapping;
(c) $Y$ is the image of $F_C$ under a confluent and light mapping;
(d) $Y$ is a smooth fan with $E(Y)$ closed or $Y$ is an arc.

**Theorem (5.24).** Let the continuum $Y$ be given. The following conditions are equivalent:

1. $Y$ is the image of $F_C$ under a monotone mapping;
2. $Y$ is homeomorphic either to $F_C$ or to $F_C^c$.

**Theorem (5.25).** Let the continuum $Y$ be given. Then the following conditions are equivalent:

A. $Y$ is the image of $F_C$ under a mapping;
B. $Y$ is the image of $F_C$ under a light mapping;
C. $Y$ is uniformly pathwise connected.

Recall the following result concerning fans that was proved in [38].

**Theorem (5.26).** Let the smooth fan $X$ with the top $v$ be given. Then the following conditions are equivalent:

1. $X$ is homeomorphic to the Lelek fan $F_L$;
2. $E(X) \cup \{v\}$ is connected;
3. $E(X)$ is dense in $X$;
4. every confluent image of $X$ is homeomorphic to $X$.

Confluent mappings of fans were studied in [29].

6. **Planability of dendroids**

It is easy to give an example of a dendroid that is nonplanable, i.e., that has no homeomorphic image contained in the plane. To this aim one can take a plane dendroid $X$ containing a point $p$ which is strongly arcwise nonaccessible from the complement $\mathbb{R}^2 \setminus X$ and add to $X$ an arc $pq$ situated in space so that $pq \cap X = \{p\}$ (see [36]). Another reason of nonplanability is connected with Moore's theorem on triods [92] from which it follows that there is no uncountable collection of pairwise disjoint triods in the plane. For example,
let $C \subset \mathbb{I}$ be the Cantor set and let $T$ be a simple triod with its center $v$. Then $(C \times T) \cup (\mathbb{I} \times \{v\})$ is a nonplanable dendroid.

In 1961 (in vol. 8 of Colloq. Math.) Bronislaw Knaster posed the problem of finding an intrinsic characterization of dendroids which can be embedded in the plane. The problem is still open, and seems to be far from solved. Only some partial solutions are known. To present them, let us introduce the following concept.

Let $\mathcal{D}$ be a class of spaces and let $\mathcal{P}$ be a property. We say that the property $\mathcal{P}$ is finite (countable) in $\mathcal{D}$ provided that there is a finite (countable, resp.) set $\mathcal{F} \subset \mathcal{D}$ such that a member of $\mathcal{F}$ has the property $\mathcal{P}$ if and only if it contains a homeomorphic copy of a member of $\mathcal{F}$. For example, the property of being not embeddable in the 2-sphere $S^2$ is finite in the class of graphs and of local dendrites [67], of locally connected continua [39] (compare a part of this text after Corollary (2.6)). Nothing similar is true for nonplanability of curves that are not locally connected. Namely it was shown in 1977 [34] that

**Theorem (6.1).** The property of not being planable is not countable in the class of smooth dendroids.

To approach this result let us consider in $\mathbb{R}^2$ a countable fan consisting of a sequence of straight line segments $pa_i$, where $p$ is the origin and $a_i$ are points of the unit circle such that $a_1 > a_2 > \cdots > a_0 = \lim a_i$ (here $a_i$ denote also the angle coordinates of points $a_i$). Let $s = (s_0, s_1, s_2, \ldots)$ be a given zero-one sequence. For each angle $a_i pa_{i+1}$, where $i \in \{0, 1, 2, \ldots\}$ we distinguish an arm of this angle in such a way that the distinguished arm is $pa_i$ if $s_i = 0$ and it is $pa_{i+1}$ if $s_i = 1$. We complete the fan in each angle $a_i pa_{i+1}$ by a sequence of disjoint straight line segments, each of which has one end point in the distinguished arm of the angle, the opposite end point in the unit circle, and such that this sequence converges to the other arm of the angle $a_i pa_{i+1}$ (see the Figure 14). Note that the resulting dendroid is smooth, and the point $p$ is not accessible.

![Figure 14](image-url)
Finally we add a straight line segment \( pq \) which is perpendicular to the plane and we denote the obtained dendroid by \( D(s) \). It is nonplanable and smooth. It is proved in [34] that if \( F \) is a nonplanar subcontinuum of \( D(s) \), then there is no embedding of \( F \) into \( D(t) \) for \( s \neq t \).

In the same year T. Maćkowiak [79] constructed an uncountable collection of (nonsmooth) fans with the same properties.

It is known that there is a universal smooth dendroid (Theorem (5.16)). In 1976 T. Maćkowiak [78] showed that if the plane smooth dendroids \( D_1 \) and \( D_2 \) are defined as pictured below, then there is no plane smooth dendroid containing homeomorphic copies of \( D_1 \) and of \( D_2 \). Thus, it follows that

(6.2). There is no universal element in the class of plane smooth dendroids.

Moreover, in 1982 L. Habiniak [59] proved that

(6.3). There is no plane dendroid containing all plane smooth dendroids.

Now let us pass to problems related to mappings and planability of dendroids. The following results are known [16].

THEOREM (6.4). Let \( X \) be a planable continuum and \( f: X \to f(X) \) be a monotone mapping such that for each \( y \in f(X) \) the inverse image \( f^{-1}(y) \) does not separate the plane. Then \( f(X) \) is planable.

COROLLARY (6.5). A monotone image of a planable \( \lambda \)-dendroid (dendroid, fan) is a planable \( \lambda \)-dendroid (dendroid, fan).

It can be shown that monotoneity in this result is essential, and it cannot be replaced by confluence. Maćkowiak asked in 1976 the following question [16], which is still open:
Question (6.6) Is planability of dendroids an invariant property with respect to open mappings? (For graphs the answer is negative, [112], p. 189.)

As we already know, each smooth fan is planable (it is embeddable into the Cantor fan, (4.8)). Oversteegen proved in 1979 [104] that

Theorem (6.7). If a fan is locally connected at its top, then it is planable.

The first example of a nonplanable fan was constructed by Borsuk [7]. Consider the plane countable fan of Figure 16.

![Figure 16.](image)

Identifying every two opposite points on the limit segments (and taking the induced homeomorphism on the rest) we get Borsuk’s nonplanable fan. Note that the discussed quotient mapping is simple (i.e., \( \text{card} \ f^{-1}(y) \leq 2 \) for each \( y \) in the range space) and confluent, while not open. So we have a question:

Question (6.8) Is planability of fans an invariant property with respect to open mappings?

7. Contractibility

A continuum \( X \) is said to be contractible provided that there are a homotopy \( H: X \times I \to X \) and a point \( p \in X \) such that for \( x \in X \) we have \( H(x, 0) = x \) and \( H(x, 1) = p \). For example, a disk is contractible, while a simple closed curve is not. The following results concerning contractibility of curves are well-known (see e.g. [17], [23], [32], [33]).

(7.1). Every contractible curve is a dendroid.

The inverse is not true — and the main problem related to contractibility of curves is to find a structural characterization of contractible dendroids.
(7.2). Every contractible dendroid is u.a.c.

(7.3). Let a curve be locally connected. Then it is contractible if and only if it is a dendrite.

(7.4). If a space \(X\) contains two subsets \(A\) and \(B\) such that

\[\emptyset \neq A \subset B \neq X\]

and for every homotopy \(H : X \times I \to X\) with \(H(x, 0) = x\) for every \(x \in X\) we have \(H(A \times I) \subset B\), then \(X\) is not contractible.

If \(A = B\) in (7.4), then \(A\) is said to be homotopically fixed.

**Question** (7.5) Does every noncontractible dendroid contain a homotopically fixed subset?

Other known conditions that imply noncontractibility of dendroids are the following. A dendroid \(X\) is of type \(N\) between its points \(p\) and \(q\) if there are two sequences of arcs \(p_n p'_n\) and \(q_n q'_n\) in \(X\) and points \(p''_n \in q_n q'_n \setminus \{q, q_n\}\) and \(q''_n \in p_n p'_n \setminus \{p_n, p'_n\}\) with

\[
Pq = \lim p_n p'_n = \lim q_n q'_n; \quad p = \lim p_n = \lim p'_n = \lim p''_n; \quad q = \lim q_n = \lim q'_n = \lim q''_n.
\]

The above concept is due to L. G. Oversteegen [101] and is related to the following condition of B. G. Graham [54]. A dendroid \(X\) is said to contain a zigzag if there exist in \(X\): an arc \(pq\), a sequence of arcs \(p_n q_n\) and two sequences of points \(p'_n\) and \(q'_n\) situated in these arcs in such a manner that \(p_n < q_n < p'_n < q'_n\) (where \(<\) denotes the natural order on \(p_n q_n\) from \(p_n\) to \(q_n\)), for which the following conditions hold:

\[
Pq = \lim p_n q_n; \quad p = \lim p_n = \lim p'_n; \quad q = \lim q_n = \lim q'_n.
\]

It is known that if a dendroid contains a zigzag, then it is of type \(N\) but not conversely, even for fans.

A point \(p\) of a dendroid \(X\) is called a Q-point of \(X\) provided there exists a sequence of points \(p_n\) of \(X\) converging to \(p\) such that \(Ls \, pp_n \neq \{p\}\) and, if for each \(n \in \mathbb{N}\) the arc \(p_n q_n\) is irreducible between \(p_n\) and the continuum \(Ls \, pp_n\), then the sequence of points \(q_n\) converges also to \(p\).

The third concept we recall here is pairwise smoothness [54]. Let two sequences of points \(r_n^1\) and \(r_n^2\) of a dendroid \(X\) be given, both converging to a common limit point \(r\). We say that the former sequence dominates the latter one provided that whenever there is a point \(s\) in \(X\) and a sequence of points \(s_n^1\) of \(X\) converging to \(s\) with the property that the arcs \(r_n^1 s_n^1\) converge to the arc
rs, then it follows that there also exists a sequence of points $s_n^2$ of $X$ converging to $s$ such that the arcs $r_n^2 s_n^2$ converge to $rs$.

If, for each point $s$ of a dendroid $X$ and for each sequence $s_n^1$ tending to $s$ the limit of the sequence of arcs $r_n^1 s_n^1$ is not an arc, then the sequence $r_n^1$ dominates all the sequences $r_n^2$ whatsoever.

A dendroid $X$ is said to be *pairwise smooth* provided that whenever a pair of sequences converge to a common limit point, then one of the pair dominates the other.

The following internal characterization of contractibility of fans is due to L. G. Oversteegen [103].

**Theorem (7.6).** For every fan $X$ the following conditions are equivalent:

1. $X$ is contractible;
2. $X$ is not of type $N$, contains no $Q$-point and is pairwise smooth;
3. $X$ contains no zigzag, contains no $Q$-point and is pairwise smooth.

The above characterization describes three possible reasons for the noncontractibility of a fan:

1. Being of type $N$ (in particular containing a zigzag),
2. Containing a $Q$-point,
and
3. Not being pairwise smooth.

If one considers these conditions for arbitrary continua, the situation is the following (details are presented in [23] and [35]).

1. If a continuum is of type $N$, then it is noncontractible [101].
2. It is an open question if the existence of a $Q$-point in a dendroid implies noncontractibility.
3. Not being pairwise smooth implies noncontractibility for fans only.

Oversteegen showed [105] the following interesting result.

**Theorem (7.13).** Every contractible fan is locally connected at its top, thus it is planable.

For dendroids (even with two ramification points only) contractibility does not imply planability ([23], p. 571).

Another set of conditions that imply noncontractibility of dendroids was introduced by S.T. Czuba [41], [42], [44], [48]. Let a nonempty proper subcontinuum $K$ of a dendroid $X$ be given. Then $K$ is called an $R^i$-continuum (where $i = 1, 2, 3$) if there exist an open set $U$ containing $K$ and two sequences $\{C_n^1\}$ and $\{C_n^2\}$ of components of $U$ such that

$$
K = \begin{cases} 
\text{Ls } C_n^1 \cap \text{Ls } C_n^2 & \text{for } i = 1, \\
\text{Lim } C_n^1 \cap \text{Lim } C_n^2 & \text{for } i = 2, \\
\text{Li } C_n^1 & \text{for } i = 3.
\end{cases}
$$
THEOREM (7.14). If a subcontinuum \( K \) of a dendroid \( X \) is an \( \mathbb{R}^i \)-continuum (where \( i = 1, 2, \) or \( 3 \)), then \( K \) is homotopically fixed, and so \( X \) is not contractible.

Contractibility is not a hereditary property, even for fans. However, we have the following result [33].

THEOREM (7.15). Every smooth dendroid is hereditarily contractible (but not conversely).

A dendroid \( X \) is said to be pointwise smooth provided that for each point \( a \in X \) there is a point \( p(a) \in X \) (called an initial point for \( a \) in \( X \)) such that for every convergent sequence of points \( a_n \) of \( X \) the condition \( \lim a_n = a \) implies that the sequence of arcs \( p(a)a_n \) is convergent, and \( \lim p(a)a_n = p(a)a \). This concept plays an important role in study of hereditary contractibility of dendroids ([43], [45], [46], [47]).

THEOREM (7.16). A dendroid \( X \) is pairwise smooth if and only if for every two of its points \( x \) and \( y \) we have

\[
\text{either } \ xy \cap T(x) \neq \{x\}, \ \text{or } \ xy \cap T(y) \neq \{y\}, \ \text{or } \ T(x) \cap T(y) = \emptyset.
\]

(Here \( T \) stands for the Jones function [62], see the paragraph before Theorem (5.10)).

It is known that

(7.17). If a dendroid is hereditarily contractible, then it is pointwise smooth.

Question (7.18) Does pointwise smoothness of dendroids imply their hereditary contractibility?

The answer to (7.18) is known to be affirmative in the case when the dendroid is a fan [47].

THEOREM (7.19). For every fan, smoothness, pointwise smoothness and hereditary contractibility are equivalent.

The above result can be extended to a wider class of dendroids [47]. A nondegenerate collection \( \mathcal{G} \) of continua is called a clump provided that \( \mathcal{G}^* \) (i.e. the union of all elements of \( \mathcal{G} \)) is a continuum and there exists a continuum \( C \) (called the center of \( \mathcal{G} \)) such that \( C \) is a proper subcontinuum of each element of \( \mathcal{G} \) and \( C \) is the intersection of every two elements of \( \mathcal{G} \). A dendroid \( X \) is said to have property (CS) provided that there exists a clump \( \mathcal{G} \) of smooth dendroids, having a center \( C \) such that

(\( \alpha \)) \( \mathcal{G}^* = X; \)

(\( \beta \)) there is a point of \( C \) which is an initial point of each element of \( \mathcal{G}; \)

(\( \gamma \)) \( \text{bd} C \) is zero-dimensional.
Theorem (7.20). If a dendroid $X$ has property (CS) and if $C$ denotes the center of a clump for $X$, then the following conditions are equivalent:

1. $X$ is hereditarily contractible;
2. $X$ is pointwise smooth;
3. $X$ is smooth;
4. $X/C$ is hereditarily contractible;
5. $X/C$ is pointwise smooth;
6. $X/C$ is smooth.

As has been shown by L.G. Oversteegen [102],

(7.21). Noncontractibility is not a countable property in the class of fans.

Finally recall the following result [3].

Theorem (7.22). If $A$ and $B$ are closed subsets of a continuum $X$ such that

$$A \cap T(B) = \emptyset = B \cap T(A) \quad \text{and} \quad T(A) \cap T(B) \neq \emptyset,$$

then $X$ is not contractible.

8. Selectibility

Given a metric space $X$ with a metric $d$, we denote by $2^X$ the space of all nonempty closed subsets of $X$ equipped with the Hausdorff distance $H$ defined by

$$H(A, B) = \max \{ \sup \{d(p, B) : p \in A \}, \sup \{d(q, A) : q \in B \} \}.$$

Further, $C(X)$ means the subspace of $2^X$ composed of all nonempty closed connected subsets of $X$. If $X$ is a continuum, $C(X)$ is called the hyperspace of subcontinua of $X$. A continuous selection for a family $\mathcal{F} \subset 2^X$ is defined as a mapping $\sigma : \mathcal{F} \to X$ such that $\sigma(A) \in A$ for each $A \in \mathcal{F}$. Answering a question of Michael [88] Kuratowski, Nadler and Young characterized [69] locally compact separable metric spaces $X$ for which there exists a continuous selection for $2^X$. In particular, they proved that if a continuum $X$ admits such a selection, then $X$ is an arc. Since each arc admits such a selection (taking $\sigma(A) = \min A$, for example), we infer that

(8.1). A continuum $X$ admits a continuous selection for $2^X$ if and only if $X$ is an arc.

So the problem of finding a structural characterization of continua that admit a continuous selection for a given family $\mathcal{F} \subset 2^X$ is solved in the case when $\mathcal{F} = 2^X$. Another very interesting case is when $\mathcal{F} = C(X)$. A continuum $X$ is said to be selectable provided that it admits a continuous selection for $C(X)$. Important results in this area were obtained by Nadler and Ward [95] who proved that

(8.2). Every selectable continuum is a dendroid;
(8.3). A locally connected continuum is selectable if and only if it is a dendrite;
(8.4). Each selectable dendroid is a continuous image of the Cantor fan, so it is u.a.c.

On the other hand, there are u.a.c. dendroids which are not selectable, as the one pictured below [95].

![Figure 17.](image)

A selection \( \sigma: \mathcal{F} \to X \), where \( \mathcal{F} \subset 2^X \), is said to be rigid provided that if \( A, B \in \mathcal{F} \) and \( \sigma(B) \in A \subset B \), then \( \sigma(A) = \sigma(B) \). Ward [109] showed that

(8.5). A continuum \( X \) is a smooth dendroid if and only if there exists a rigid selection for \( C(X) \).

On the other hand there are not smooth dendroids \( X \) admitting non-rigid selections for \( C(X) \), as e.g. pictured below. Nadler posed in his book [93] a problem (still open):

Problem (8.6) Give an internal characterization of selectable dendroids (of selectable fans).

There are many properties of contractible and of selectable dendroids which are similar to each other. Examples of noncontractible and selectable dendroids are known [19], see Figure 18.

The first example of a contractible and nonselectible dendroid \( D \) was constructed by T. Maćkowiak [84]. The example is nonplanable, is not hereditarily contractible, and has many ramification points. Simpler examples are not known. So, we have a sequence of open questions.
Question (8.7) Does there exist a contractible and nonselectible dendroid which is a) planable, b) hereditarily contractible, c) a fan?

Maćkowiak introduced [80] the following concept. Let a continuum $X$ and two of its subcontinua $B \subset A \subset X$ be given. Then $B$ is called a bend set of $A$ provided that there are two sequences $\{A_n\}$ and $\{A'_n\}$ of subcontinua of $X$ such that $A_n \cap A'_n \neq \emptyset$ for every $n \in \mathbb{N}$, $A = \text{Lim } A_n = \text{Lim } A'_n$ and $B = \text{Lim } (A_n \cap A'_n)$. We say that $X$ has the bend intersection property if for each continuum $A$ of $X$ the intersection of all bend sets of $A$ is nonempty. Maćkowiak proved [80] that

(8.8). Each selectable dendroid has the bend intersection property.

The property of being a continuum of type $N$ (discussed previously, in connection with noncontractibility of dendroids, see Part VII) can be generalized by replacing the arc $pq$ mentioned in the definition of type $N$ by an arbitrary continuum $K$ containing these points; i.e., a continuum $X$ is said to be of generalized type $N$ between points $p$ and $q$ provided that there are two sequences of arcs $p_np'_n$ and $q_nq'_n$ in $X$ and points

$$p''_n \in q_nq'_n \setminus \{q_n,q'_n\} \quad \text{and} \quad q''_n \in p_np'_n \setminus \{p_n,p'_n\}$$

with

$$p = \lim p_n = \lim p'_n = \lim p''_n, \quad q = \lim q_n = \lim q'_n = \lim q''_n$$

and $\text{Lim } p_np'_n = \text{Lim } q_nq'_n (= K)$. Note that if a dendroid $X$ is of generalized type $N$ between some points $p, q \in X$, then the singletons $\{p\}$ and $\{q\}$ are bend sets of $K = \text{Lim } p_np'_n = \text{Lim } q_nq'_n$. Thus
(8.9). If a dendroid is of generalized type N (in particular, if it is of type N), then it does not have the bend intersection property, so it is not selectable.

Interrelations between various conditions related to nonselectibility of dendroids are discussed in [22], [24], [25].

Concerning mapping properties of selectable dendroids, neither monotone nor open mappings preserve selectibility of dendroids [80]. However, the following question is open.

**Question** (8.10) Is it true that an open image of a selectable fan is selectable?

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The Editors wish to thank Alejandro Illanes and Sergio Macías for their valuable help in providing the computer-made figures for this paper.