ON IRREDUCIBLE SMOOTH CONTINUA

The notion of an irreducible continuum is well known. The theory of irreducible continua has been investigated and developed by a large number of authors, see e.g. [6], [7], [11] and [1]. Recently the notion of smoothness of continua has been introduced [2] and studied in several papers [3], [9]. The aim of this paper is to prove some necessary and sufficient conditions under which an irreducible continuum is smooth.

A continuum means a compact connected metric space. It is well known that for every irreducible continuum $X$ there exists an upper semi-continuous decomposition of $X$ into continua (called layers of $X$, see e.g. [8], §48, IV, p. 199) with the property that the decomposition of $X$ into layers is the finest of all linear upper semi-continuous decompositions of $X$ into continua ([8], §48, IV, Theorem 3, p. 200; [7], Fundamental theorem, p. 259). If the irreducible continuum $X$ reduces to one layer, i.e. if the decomposition space is degenerate, then $X$ is called monostratic, see [8], p. 199. If $X$ is not monostratic, i.e. if $X$ has more than one layer, then the decomposition space is a non-degenerate arc (we can assume that it is the unit interval), and $X$ is called to be of type $A$ (see [11], p. 9). It follows that for each irreducible continuum $X$ there is a continuous monotone mapping $\varphi : X \to [0, 1]$ from $X$ into the unit interval $[0, 1]$, (namely the quotient mapping of the decomposition of $X$ into layers, called the canonical mapping) which is minimal in the sense that, given any continuous monotone mapping $f$ from $X$ into $[0, 1]$, each point-inverse of $f$ is the union of some point-inverses of $\varphi$, i.e., the union of layers of $X$.

If each layer of $X$ has void interior, then $X$ is called to be of type $A$ (see [8], §48, III, p. 197, the footnote, and also [11], Definition 4, p. 13, where these continua are called to be of type $A'$). It is well known (see [8], §48, VII, Theorem 3, p. 216; [11] Theorem 10, p. 15; [1], Theorem 2.7, p. 650) that an irreducible continuum $X$ is of type $A$ if and only if each indecomposable subcontinuum of $X$ has void interior.

A continuum is said to be hereditarily unicoherent at a point $p$ (see [2], p. 52; cf. also [3]) if the intersection of any two subcontinua, each of which contains $p$, is connected. It is easily verified that a continuum $X$ is hereditarily unicoherent at $p$ if and only if given any point $x \in X$, there exists a unique subcontinuum which is irreducible between $p$ and $x$ (see [2], Theorem 1.3, p. 52). If a continuum $X$ is hereditarily unicoherent at a point $p$, and if $q \in X \setminus \{p\}$, then the symbol $pq$ will denote the unique subcontinuum of $X$ which is irreducible between $p$ and $q$. 
A continuum $X$ is said to be smooth at the point $p$ (see [2], p. 52; cf. also [3]) if $X$ is hereditarily unicoherent at $p$ and if for each convergent sequence of points $\{a_n\}$ the condition

$$\lim_{n \to \infty} a_n = a$$

implies that

(2) the sequence $\{pa_n\}$ is convergent

and

$$\lim_{n \to \infty} pa_n = pa.$$ 

It can be proved that if a continuum is smooth at $p$, then it is locally connected at $p$ ([2], Corollary 3.1, p. 54). A continuum $X$ is said to be smooth if there is a point $p \in X$ such that $X$ is smooth at $p$ (see [2], p. 53; cf. also [3] and [9]). It is known ([2], Corollary 3.3, p. 55) that if a continuum $X$ is smooth, then each indecomposable subcontinuum of $X$ has void interior. This implies by the definition of an irreducible continuum of type $\lambda$ the following

**Proposition 1.** If an irreducible continuum is smooth, then it is of type $\lambda$.

Easy examples show that the inverse is not true.

Now let a continuum $X$ be smooth at a point $p$. The equivalence relation $\rho_p$ defined by

$$x \rho_p y \text{ if and only if } px = py$$

is studied in [2], p. 57 and in [9]. Let $\psi: X \to X/\rho_p$ denote the quotient mapping.

**Proposition 2.** Let an irreducible continuum $X$ be hereditarily unicoherent at a point $p$. Then $X$ is smooth at $p$ if and only if the decomposition of $X$ into sets $\psi^{-1}(t)$, where $t \in X/\rho_p$, coincides with the canonical decomposition of $X$ into layers.

**Proof.** If $X$ is smooth, then the result is known (see [2], Lemma 5.1, p. 57). If both the decompositions mentioned above coincide, then the decomposition space $X/\rho_p$ is an arc, thus a smooth dendroid, and we see that $X/\rho_p$ smooth at $\psi(p)$. These conditions are sufficient to prove the smoothness of $X$ at $p$ (see [9], Theorem 3.1).

Let a continuum $X$ be irreducible but not monostratic (i.e. of type $A$) and let $T_t$, $t \in [0, 1]$, denote a layer of $X$. Thus $X = \bigcup \{T_t: 0 < t < 1\}$. Put

$$L_t = \bigcup \{T_u: 0 < u < t\} \text{ and } R_t = \bigcup \{T_v: t < v < 1\}.$$ 

Thus we have $X = L_t \cup T_t \cup R_t$ for each $t \in [0, 1]$, and since

$$L_t = \varphi^{-1}([0, t)) \text{ and } R_t = \varphi^{-1}((t, 1]),$$

where $\varphi$ is the canonical (thus monotone) mapping from $X$ to the unit interval $[0, 1]$, we see that both $L_t$ and $R_t$ are connected. (Here the capital letters $L$ and $R$ stand for left and right, respectively.)

Adopt the following definitions. A layer $T_t$ is called to be a layer of left cohesion if either $t = 0$ or $T_t \subset L_t$. A layer $T_t$ is called to be a layer of right cohesion if either $t = 1$ or $T_t \subset R_t$. In other words $T_t$ is a layer of left cohesion
if either \( t = 0 \) or \( T_t = \overline{L_t} \setminus L_t \); and \( T_t \) is a layer of right cohesion if either \( t = 1 \) or \( T_t = R_t \setminus R_t \). One can see that \( T_1 \) is a layer of left cohesion and that \( T_0 \) is a layer of right cohesion provided the interior of \( T_0 \) and of \( T_1 \) is empty.

A layer \( T_t \) is called to be a layer of cohesion if it is a layer of both left and right cohesions. Thus if the interior of \( T_0 \) (or \( T_1 \)) is empty, then \( T_0 \) (or \( T_1 \)) is a layer of cohesion. For \( t \in (0,1) \) the layer \( T_t \) is a layer of cohesion if \( T_t \subseteq \overline{L_t} \cap R_t \) (see [7], p. 260). Observe that the above inclusion can be replaced by the equality (see [7], p. 260; cf. also [8], § 48, IV, p. 201). It is known that the family of all layers of \( X \) which are not layers of cohesion is at most countable ([7], Theorem XIV, p. 261, and [8], p. 201).

**Proposition 3.** Let a continuum \( X \) be irreducible between points \( a \) and \( b \). Then the continuum \( X \) is smooth at \( a \) if and only if

(4) \( X \) is locally connected at \( a \)

and

(5) each layer \( T_t \) of \( X \) is of left cohesion.

**Proof.** Assume firstly that \( X \) is smooth at \( a \). Thus condition (4) follows from Corollary 3.1 in [2], p. 54. To prove (5) suppose on the contrary that there is a layer \( T_t \) which is not of left cohesion. Since \( X \) is of type \( \lambda \) by Proposition 1, we have \( t \neq 0 \) and there is a point \( x \in T_t \setminus \overline{L_t} \). Let \( y \in \overline{L_t} \cap \overline{R_t} \). The point \( a \) being in \( \overline{L_t} \), we have \( ay \subseteq \overline{L_t} \), whence \( x \in T_t \setminus ay \). Since \( X \) is of type \( \lambda \) and since \( y \) belongs to \( \overline{R_t} \), there is a decreasing sequence of reals \( v_n \), where \( t < v_n \), such that we can choose a sequence of points \( y_n \in T_t \subseteq R_t \) with \( \lim_{n \to \infty} y_n = y \).

Then \( T_t \subseteq ay_n \subseteq \overline{L_t} \) for every \( n \), whence \( T_t \subseteq \cap_{n=1}^{\infty} ay_n = \lim_{n \to \infty} ay_n \), the sequence of continua \( \overline{L_t} \) being decreasing. Since \( x \in T_t \setminus ay \), we conclude that \( x \notin \lim ay_n \cdot \ay \) contrary to the smoothness of \( X \) at the point \( a \).

Assume secondly that conditions (4) and (5) are both satisfied. Since \( X \) is locally connected at \( a \), the layer \( T_0 \) to which the point \( a \) belongs is degenerate and we have \( T_0 = \{ a \} \). Let \( x \in X \) and let \( \varphi(x) = t \). To prove that there is exactly one irreducible continuum from \( a \) to \( x \) in \( X \) (i.e. to prove the hereditary unicoherence of \( X \) at \( a \)) we consider the case \( t > 0 \) only, because if \( t = 0 \), then \( x = a \) and the irreducible continuum in question is degenerate. Since \( T_t \) is a layer of left cohesion, we have \( T_t = \overline{L_t} \setminus L_t = Fr L_t \). But \( x \in T_t \) by the definition of \( t \), hence the continuum \( \overline{L_t} \) is irreducible from \( a \) to \( x \) (see [8], § 48, II, Theorem 7, p. 194; cf. also [11], Theorem 1 (c), p. 7). Notice that \( \overline{L_t} \) is the only continuum irreducible from \( a \) to \( x \) in \( X \). In fact, suppose another continuum \( Q \) is irreducible between \( a \) and \( x \) in \( X \). Thus \( y \in Q \setminus \overline{L_t} \). By (5) we have \( T_t \subseteq \overline{L_t} \) which implies \( \overline{L_t} = \varphi^{-1} ([0, t]) \), and hence \( t < \varphi(y) = v \). Thus the continuum \( Q \) contains \( a \), and intersects a layer \( T_v \) with \( t < v \). This implies that \( L_t \subseteq \overline{L_t} \subseteq Q \). We see that \( \overline{L_t} \) is a proper subcontinuum of \( Q \) which contains both \( a \) and \( x \), contrary to the irreducibility of \( Q \) between these points. Therefore \( X \) is hereditarily unicoherent at \( a \).
Suppose on the contrary that $X$ is not smooth at $a$. Then (see [2], Theorem 2.3, p. 53) there exist convergent sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} y_n = y$, $y_n \subseteq ax_n$ and $y \subseteq X \setminus ax$. Hence the continuity of the canonical mapping $\varphi : X \to [0,1]$ implies correspondingly

\begin{align*}
(6) & \quad \lim_{n \to \infty} \varphi(x_n) = \varphi(x), \\
(7) & \quad \lim_{n \to \infty} \varphi(y_n) = \varphi(y), \\
(8) & \quad \varphi(y_n) < \varphi(x_n), \\
(9) & \quad \varphi(x) < \varphi(y).
\end{align*}

Taking $t = \frac{1}{2} [\varphi(x) + \varphi(y)]$ we see by (9) that

\begin{align*}
(10) & \quad \varphi(x) < t < \varphi(y),
\end{align*}

and we conclude from (6) that there is a natural $n_0$ such that if $n > n_0$, then $\varphi(x_n) < t$ which gives by (8) that if $n > n_0$, then $\varphi(y_n) < t$, and we have a contradiction with (7) by (10).

In the same way one can prove the following

**Proposition 4.** Let a continuum $X$ be irreducible between points $a$ and $b$. Then the continuum $X$ is smooth at $b$ if and only if

\begin{align*}
(11) & \quad X \text{ is locally connected at } b \\
(12) & \quad \text{each layer } T_i \text{ of } X \text{ is of right cohesion.}
\end{align*}

Notice the following

**Proposition 5.** An irreducible continuum $X$ is smooth at a point $p$ if and only if continua $\overline{L_{\varphi(p)}}$ and $\overline{R_{\varphi(p)}}$ both are smooth at $p$.

**Proof.** In fact, if $X$ is smooth at $p$, then $\overline{L_{\varphi(p)}}$ and $\overline{R_{\varphi(p)}}$ both are smooth at $p$ by (2.8) in [9]. To prove the opposite way observe that if $\overline{L_{\varphi(p)}}$ is smooth at $p$, then it is locally connected at $p$ by Corollary 3.1 in [2], p. 54. Similarly if $\overline{R_{\varphi(p)}}$ is smooth at $p$, then it is locally connected at $p$. It implies that if continua in question both are smooth at $p$, then the layer $T_{\varphi(p)}$ to which $p$ belongs reduces to the point $p$ only. Thus $\overline{L_{\varphi(p)}} \cap \overline{R_{\varphi(p)}} = \{p\}$, whence it follows that $X$ is hereditarily unicoherent at $p$ provided both $\overline{L_{\varphi(p)}}$ and $\overline{R_{\varphi(p)}}$ are. Now let $\{a_n\}$ be a convergent sequence of points in $X$. Considering separately subsequences of points $a_n$ which are in $\overline{L_{\varphi(p)}}$ or in $\overline{R_{\varphi(p)}}$ we see that (1) implies (2) and (3).

**Theorem.** An irreducible continuum $X$ is smooth at a point $p$ if and only if all three of the following conditions are satisfied:

\begin{align*}
(13) & \quad X \text{ is locally connected at } p, \\
(14) & \quad \text{for each } t \text{ satisfying } 0 < t < \varphi(p) \text{ the layer } T_t \text{ is of right cohesion,} \\
(15) & \quad \text{for each } t \text{ satisfying } \varphi(p) < t < 1 \text{ the layer } T_t \text{ is of left cohesion.}
\end{align*}
Proof. Let the continuum $X$ be irreducible from $a$ to $b$. Assume firstly that $X$ is smooth at $p$. Then (13) follows from Corollary 3.1 in [2], p. 54. To prove (14) and (15) observe that the continua $L_{\varphi(p)}$ and $R_{\varphi(p)}$ on the one hand both are smooth at $p$ by Proposition 5 and, on the other hand, are irreducible from $a$ to $p$ and from $p$ to $b$ respectively (see [11], Theorem 1 (c), p. 7; cf. also [8], § 48, III, Theorem 1, p. 195). Since the layers of $L_{\varphi(p)}$ and of $R_{\varphi(p)}$ coincide with the layers of $X$, hence putting $L_{\varphi(p)}$ for $X$ and $p$ for $b$ in Proposition 4, and also putting $R_{\varphi(p)}$ for $X$ and $p$ for $a$ in Proposition 3 we see that (14) and (15) hold true.

Assume secondly that all three conditions (13), (14) and (15) are satisfied. As previously we see that continua $L_{\varphi(p)}$ and $R_{\varphi(p)}$ are irreducible from $a$ to $p$ and from $p$ to $b$ respectively, and that each of them is locally connected at $p$ by (13). Further, we conclude from (14) and Proposition 4 (in which we take $L_{\varphi(p)}$ for $X$ and $p$ for $b$) that $L_{\varphi(p)}$ is smooth at $p$. Similarly it follows from (15) and Proposition 3 (with $R_{\varphi(p)}$ for $X$ and $p$ for $a$) that $R_{\varphi(p)}$ is smooth at $p$. Applying Proposition 5 we conclude the proof.

Now we recall a concept which is due to F. B. Jones, see [4] and [5]. Let $x$ and $y$ be distinct points of a continuum $X$. We say that $X$ is aposyndetic at $x$ with respect to $y$ provided there is a subcontinuum of $X$ containing $x$ in its interior and not containing $y$. If this condition fails, i.e., if every subcontinuum of $X$ which contains $x$ in its interior contains $y$, then $X$ is non-aposyndetic at $x$ with respect to $y$.

**Proposition 6.** If a layer $T_t$ of an irreducible continuum $X$ is of left cohesion, then for each two points $x$ and $y$ of $T_t$, the continuum $\varphi^{-1}([0, t])$ is non-aposyndetic at $x$ with respect to $y$.

**Proof.** Assume that a layer $T_t$ is of left cohesion. Let $x$ and $y$ be two arbitrary points of $T_t$, and let a continuum $K \subseteq \varphi^{-1}([0, t])$ contain $x$ in its interior with respect to $\varphi^{-1}([0, t])$. Since the layer $T_t$ is of left cohesion, it follows that $x$, as a point of $T_t$, is in $L_t$. Hence the interior of $K$ with respect to $\varphi^{-1}([0, t])$, as a neighbourhood of $x$, must intersect $L_t$. Thus $K$ intersects a layer $T_{u_0}$, where $0<u_0<t$. Therefore all layers $T_u$ with $u_0<u<t$ are contained in $K$ (see [11], Theorem 5, p. 10; cf. also [10], Remark 2, p. 71). It implies that the closure of the union $\bigcup \{T_u: u_0<u<t\}$ is in $K$. Since the layer $T_t$ is of left cohesion, it is contained in the closure of the union in question, thus in $K$. It follows that $y \subseteq K$, which shows that $\varphi^{-1}([0, t])$ is non-aposyndetic at $x$ with respect to $y$.

Observe further that the inverse to Proposition 6 is not true as the following example shows.

Let $X = \bigcup \{T_t: t \in [0, 1]\}$ be an irreducible continuum (of type $\lambda$) defined in such a way that all layers $T_t$ for $t \neq 1/2$ are points, $T_{1/2}$ is the well-known BROUWER's indecomposable continuum (see e.g. [8], § 48, V, Example 1, p. 204 and 205) the closure of $\bigcup \{T_t: t \in [0, 1]\}$ is an arc, and $\bigcup \{T_t: t \in (1/2, 1]\}$ is a one-to-one image of the real half line which approximates the continuum $T_{1/2}$ (from the right side). We see that $T_{1/2}$ is not a layer of left cohesion although the condition formulated in Proposition 6 is satisfied by the indecomposability of $T_{1/2}$.

In the same way as for Proposition 6 one can prove
Proposition 7. If a layer $T_t$ of an irreducible continuum $X$ is of right cohesion, then for each two points $x$ and $y$ of $T_t$ the continuum $\varphi^{-1}(t, 1]$ is non-aposyndetic at $x$ with respect to $y$.

A similar example to that given after Proposition 6 shows that the inverse to Proposition 7 is not true. Propositions 6 and 7 imply by the Theorem the following

Corollary. If an irreducible continuum $X$ is smooth at a point, $p$, then

(16) for each $t \in [0, \varphi(p))$ and for each two points $x$ and $y$ of the layer $T_t$ the continuum $\varphi^{-1}([t, 1])$ is non-aposyndetic at $x$ with respect to $y$

and

(17) for each $t \in (\varphi(p), 1]$ and for each two points $x$ and $y$ of the layer $T_t$ the continuum $\varphi^{-1}((0, t])$ is non-aposyndetic at $x$ with respect to $y$.

References


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