SOME PROBLEMS CONCERNING MONOTONE DECOMPOSITIONS OF CONTINUA

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The bibliography on upper semi-continuous decompositions of continua is rather large. Especially a great number of papers concerns monotone decompositions of irreducible continua. E.g. Z. Janiszewski in [11], B. Knaster in [14] and [15], K. Kuratowski in [16] and [17] and also W.A. Wilson in [26] and [27] investigated such decompositions for metric continua irreducible between two points. A continuation of this topic can be found in a sequence of papers. In the last two decades several papers in this field have appeared, but from some other point of view. E.g. H.C. Miller in [21] has combined the investigations of irreducible with these of unicoherent spaces. E.S. Thomas, Jr. in his extensive study on irreducible continua [25] applied, among other things, the method of inverse limits as well as the notion of aposyndesis introduced by F. B. Jones (see [12] and [13]) to obtain some results on the structure of such continua.

Besides studies of decompositions of irreducible continua investigations of decompositions of arbitrary continua into closed sets or of mono-
tone decompositions (i.e. into closed connected sets) were developed by a number of authors (see K. Kuratowski [18], M.E. Shanks [24], R.W. Fitzgerald and P.M. Swingle [8]). Some of Miller's results of [21] concerning decompositions of continua irreducible between two points were generalized by M.J. Russell in [23] to decompositions of continua irreducible about a finite set. Recently G.R. Gordh, Jr. considered upper semi-continuous monotone decompositions of continua with some special properties, namely of smooth [9] and nearly smooth [10] continua. Such decompositions were studied by the present author for \( \lambda \)-dendroids in [2] and [3], and for arbitrary continua in [6] (see also [7] which is an abstract of [6]).

A continuum means a compact connected metric space. It is well known that for every irreducible continuum \( I \) there exists an upper semi-continuous decomposition of \( I \) into continua (called layers of \( I \), see e.g. [20], §48, IV, p. 199) with the property that the decomposition of \( I \) into layers is the finest of all linear upper semi-continuous decompositions of \( I \) into continua ([20], §48, IV, Theorem 3, p. 200; [17], Fundamental theorem, p. 259).

Let \( X \) be a continuum. A decomposition \( \mathcal{D} \) of \( X \) is said to be admissible (see [6]) if 1° \( \mathcal{D} \) is upper semi-continuous, 2° \( \mathcal{D} \) is monotone, and 3° for every irreducible continuum \( I \) in \( X \) every layer \( T_t, 0 \leq t \leq \leq 1, \) of \( I \) is contained in some element of \( \mathcal{D} \). Of course every continuum \( X \) has an admissible decomposition, namely the trivial one, i.e. such that the whole \( X \) is the only element of the decomposition.

Problem 1. (see [6]). Characterize continua \( X \) which have a non-trivial admissible decomposition.

It is known ([6], Theorem 1) that if a decomposition \( \mathcal{D} \) of a continuum \( X \) is admissible, then the induced quotient space \( X/\mathcal{D} \) is a here-ditarily arcwise connected.

If \( \mathcal{D} \) and \( \mathcal{E} \) are upper semi-continuous monotone decompositions of a continuum \( X \), then \( \mathcal{D} \preceq \mathcal{E} \) means that every element of \( \mathcal{D} \) is contained in some element of \( \mathcal{E} \), i.e. \( \mathcal{D} \) refines \( \mathcal{E} \). Thus \( \preceq \) defines a partial ordering on the family of upper semi-continuous monotone decompo-
sitions of $X$ (see [25], p. 8; cf. [24], p. 100). It is known (see [6], theorems 2 and 3) that for every continuum $X$ there exists an admissible decomposition of $X$ which is minimal with respect to $\leq$, and that this minimal admissible decomposition is unique. The structure of elements of the minimal admissible decomposition of a continuum $X$ can be seen from the following construction (see [6]; cf. also [2], p. 18-24).

Assign to each point $x \in X$ an increasing sequence of continua $A_\alpha(x)$ (where $\alpha$ is a countable ordinal) defined by the transfinite induction. Firstly consider in $X$ all irreducible continua $I$ with $x \in I$, take in each of them the layer $T(x)$ to which $x$ belongs and put $A_0(x) = \bigcup T(x)$, where the union in the right side of the equality runs over all irreducible continua $I$ such that $x \in I \subset X$. Secondly suppose that $A_\beta(x)$ are defined for $\beta < \alpha$, and put

$$A_\alpha(X) = \begin{cases} \bigcup_{n \to \infty} \{ \text{Ls } A_\beta(x_n) : \lim_{n \to \infty} x_n \in A_\beta(x) \}, & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta < \alpha} A_\beta(x), & \text{if } \alpha = \lim_{\beta < \alpha} \beta, \end{cases}$$

where, in the case $\alpha = \beta + 1$, the union is taken over all convergent sequences of points $x_n \in X$ with $\lim_{n \to \infty} x_n \in A_\beta(x)$. Since $X$, as a metric continuum, is separable, there exists a countable ordinal $\xi$ such that if $\xi < \eta$, then $A_\xi(x) = A_\eta(x)$. Call the continuum $A_\xi(x)$ a stratum of the point $x$ in the continuum $X$. For various points $x$ strata of these points are either disjoint or identical. Thus the relation on $X$ to belong to the same stratum is an equivalence. The decomposition of $X$ into its strata is called canonical. It is known ([6], Theorem 4) that for every continuum $X$ the canonical decomposition of $X$ coincides with its minimal admissible decomposition. If the continuum $X$ is irreducible, then its canonical decomposition coincides with the decomposition into layers, and the induced quotient space is an arc. It is well known that there are examples of irreducible continua $X$ with the property that each layer of $X$ is a non-trivial continuum – see e.g. Knaster's Example 5 in [20], §48, I, p. 191. In the above context of ideas one can ask the following problem, which is due to J. Krasinkiewicz.
Problem 2. Let \( Y \) be a given hereditarily arcwise connected continuum. Does there exist a continuum \( X \) every stratum of which is non-trivial and such that \( Y \) is the decomposition space of the canonical decomposition of \( X \)?

Recall that a decomposition \( \mathcal{D} \) of a metric space \( X \) is said to be continuous at its element \( D \in \mathcal{D} \) (in other words, \( D \) is called an element of continuity of \( \mathcal{D} \)) provided that, if \( \{D_n\} \) is a sequence of elements in \( \mathcal{D} \) and there exists a point \( x_n \in D_n \) for \( n = 1, 2, \ldots \), such that the sequence \( \{x_n\} \) converges to a point of \( D \), then \( \lim_{n \to \infty} D_n = D \).

The decomposition \( \mathcal{D} \) is said to be continuous provided it is continuous at each of its elements (see [19], §19, II, p. 185 and [20], §43, V, p. 67; cf. also [25], p. 59). It is known that the family of all elements of continuity of \( \mathcal{D} \) is a dense \( G_\delta \)-set in the quotient space \( X/\mathcal{D} \) (see [20], §43, VII, p. 73; cf. also [25], Theorem 1, p. 60). B. Knaster has proved ([15], section 5, p. 574-577) that there exists an irreducible continuum \( I \) (of type \( \lambda \)) such that each layer of \( I \) is a non-trivial layer of continuity. In view of these it is natural to ask the following

Problem 3. Given a hereditarily arcwise connected continuum \( Y \), does there exist a continuum \( X \) every stratum of which is a non-trivial stratum of continuity and such that \( Y \) is the decomposition space of the canonical decomposition of \( X \)?

Of course the positive answer to Problem 3 yields one to Problem 2.

A hereditarily decomposable and hereditarily unicoherent continuum is called a \( \lambda \)-dendroid. If a \( \lambda \)-dendroid is arcwise connected, then it is a dendroid. It is known ([2], Corollary 1, p. 27) that the decomposition space of a canonical decomposition of a \( \lambda \)-dendroid is a dendroid. Therefore it is natural to ask the following modification of Problem 3.

Problem 4. Given a dendroid \( Y \), does there exist a \( \lambda \)-dendroid \( X \) which has properties formulated in Problem 3?

Let a mapping \( f \) of a continuum \( X \) be monotone. The mapping \( f \) is said to belong to the class \( \Phi \) if for any point \( y \in f(X) \), for any point \( x \in X \) and for any irreducible continuum \( I \) in \( X \) it is true that if \( x \in f^{-1}(y) \cap I \), then the layer \( T(x) \) of \( x \) in \( I \) is contained in
In other words \( f \in \Phi \) if it takes each layer of each irreducible continuum into a point. It follows that a monotone mapping \( f \) of a continuum \( X \) is in \( \Phi \) if and only if the induced decomposition of \( X \) into continua \( f^{-1}(y) \), \( y \in f(X) \), is admissible. For some properties of mappings belonging to \( \Phi \) see [6], section 5. A continuum \( X \) is said to belong to the class \( \mathscr{H} \) if every monotone mapping of \( X \) onto a hereditarily arcwise connected continuum is in \( \Phi \). It can be seen that the class \( \mathscr{H} \) contains e.g. all \( \lambda \)-dendroids, all irreducible (hence all indecomposable) continua and also all hereditarily arcwise connected continua ([6], corollary 8). The union of a disk and of an arc which has its end point as the only common point with the disk is an example of a continuum having a non-trivial admissible decomposition but not being in \( \mathscr{H} \). The known characterizations of continua belonging to \( \mathscr{H} \) (see [6], Theorem 5 and Corollary 9; cf. also [7]) are rather external, expressed by transformations into another space, and in fact not too far from the definition. For example one of these characterizations says that a continuum \( X \) is in \( \mathscr{H} \) if and only if for every mapping \( f \) of \( X \) onto a hereditarily arcwise connected continuum there exists one and only one mapping \( g \) of \( \varphi(X) \) onto \( f(X) \) such that \( f(x) = g(\varphi(x)) \) for each \( x \in X \), where \( \varphi \) denotes the canonical mapping, i.e., the quotient mapping induced by the canonical decomposition of \( X \). Since the class \( \mathscr{H} \) of continua has some nice properties and it seems to be interesting enough for further investigations, the following question is very natural.

**Problem 5.** Give an internal characterization of continua belonging to the class \( \mathscr{H} \).

A continuum \( X \) is said to be *monostratic* if it consists of only one stratum, i.e. if the canonical mapping is the trivial one of \( X \) into a point (see [6]). Each indecomposable continuum is monostratic. A monostratiform \( \lambda \)-dendroid (see [3]) is an example of a hereditarily decomposable monostratic continuum. An \( n \)-dimensional cube, where \( n > 1 \), is an example of a monostratic continuum which does not belong to \( \mathscr{H} \). It is known (see [6], Proposition 11) that a continuum \( X \in \mathscr{H} \) is monostratic if and only if every monotone mapping of \( X \) onto a hereditarily arcwise connected continuum is trivial.
Relatively little information concerning the inner structure of monostratic continua, in particular of monostratic \( \lambda \)-dendroids, has appeared in the literature. Thus the following modification of Problem 1 is open.

**Problem 6.** Give an internal characterization of monostratic continua.

**Problem 7.** Give an internal characterization of monostratic continua belonging to the class \( \mathcal{K} \).

**Problem 8.** Give an internal characterization of monostratic \( \lambda \)-dendroids.

A point \( p \) of a continuum \( X \) is said to be a *terminal point* of \( X \) if every irreducible continuum in \( X \) which contains \( p \) is irreducible from \( p \) to some point ([21], p. 190). It is known that if a \( \lambda \)-dendroid \( X \) is monostratic, then every irreducible subcontinuum in \( X \) has empty interior ([5], p. 365); this implies that every such \( X \) has uncountably many terminal points ([5], p. 367). The known examples show that the set of all terminal points is dense in such \( X \). So we have (see [5], p. 367).

**Problem 9.** Let a \( \lambda \)-dendroid \( X \) be monostratic. Does it follow that the set of all terminal points of \( X \) is dense in \( X \)?

It is known (see [6], Proposition 19) that monostraticity of continua belonging to \( \mathcal{K} \) is an invariant under monotone mappings. The question is asked in [6] whether it is an invariant under open mappings. The answer is negative, as it can be seen from an example given in [1], p. 216: van Dantzig's solenoid is an indecomposable (thus belonging to \( \mathcal{K} \)) continuum which admits an open mapping onto a circle. But if we assume that \( X \) is a \( \lambda \)-dendroid, the answer is unknown. Hence the following two problems, due to J. B. Fugate (see [4], p. 340) are still unanswered.

**Problem 10.** Is monostraticity of \( \lambda \)-dendroids an invariant under open mappings?

**Problem 11.** Is monostraticity of \( \lambda \)-dendroids an invariant under confluent mappings?

(a mapping of \( X \) onto \( Y \) is *confluent* if for every continuum \( Q \) in \( Y \)

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every component of the inverse image of $Q$ is mapped onto the whole $Q$; see [1], p. 213). Since each open mapping is confluent, the positive answer to Problem 11 implies the positive answer to Problem 10. But even under a more narrow class of mappings the question is open. Namely a mapping from a topological space $X$ to a topological space $Y$ is said to be a local homeomorphism if for every point $x \in X$ there exists a neighbourhood $U$ of $x$ such that $f(U)$ is a neighbourhood of $f(x)$ and such that $f$ restricted to $U$ is a homeomorphism between $U$ and $f(U)$. If $f$ is a local homeomorphism, then it is an open mapping.

**Problem 12.** Is monostraticity of continua an invariant under local homeomorphisms?

A continuum is said to be stratified if it has a non-trivial stratification, i.e. if it consists of more than one stratum. In other words, a continuum is stratified if it is not monostratic. It is immediately seen that a continuum is stratified if and only if it has a non-trivial admissible decomposition. A continuum each subcontinuum of which is stratified is called hereditarily stratified. It is easy to observe that a continuum is hereditarily stratified if and only if it has singletons as the only monostratic subcontinua. An example of a hereditarily stratified $\lambda$-dendroid is given in [4], p. 340, which has a monotone mapping onto a monostratic $\lambda$-dendroid. Thus the hereditary stratification of $\lambda$-dendroids is not an invariant under monotone mappings.

**Problem 13.** Is the hereditary stratification of continua an invariant under local homeomorphisms?

**Problem 14.** Is the hereditary stratification of continua an invariant under open mappings?

A continuum $X$ is said to belong to the class $\mathcal{L}$ if it admits a nontrivial admissible decomposition each of whose elements has void interior. In other words, $X \in \mathcal{L}$ if and only if each stratum of $X$ has void interior. The class $\mathcal{L}$ contains by definition all irreducible continua of type $\lambda$ (see [20], p. 197, the footnote; cf. also [25], p. 13 — continua of type $A'$), all hereditarily arcwise connected continua, and also all smooth continua (see [9], Theorem 5.2, p. 58). There are examples of con-
tinua described in [6] which show that neither $\mathcal{N} \setminus \mathcal{L}$ nor $\mathcal{L} \setminus \mathcal{N}$ is empty. It is known (see [6], Proposition 24) that if a continuum $X$ is in the class $\mathcal{L}$, then every monostratic subcontinuum of $X$ has void interior. This property does not characterize continua of the class $\mathcal{L}$ — there is a continuum $K$ each monostratic subcontinuum of which is a singleton, and which is not in $\mathcal{L}$ (see [6]). So we have

**Problem 15.** Characterize continua belonging to the class $\mathcal{L}$.

The continuum $K$ mentioned above is not a $\lambda$-dendroid. The following problem (see [6]) is open.

**Problem 16.** Let every monostratic subcontinuum of a $\lambda$-dendroid $X$ have void interior. Does it follow that $X$ is in $\mathcal{L}$?

L. Mohler has proved ([22], Theorem 6, p. 73) that if an irreducible continuum $X$ is of type $\lambda$ and if $f$ is a local homeomorphism defined on $X$, then $f(X)$ is an irreducible continuum of type $\lambda$. The question arises if this result can be extended to continua of the class $\mathcal{L}$ (not necessarily irreducible). So we have

**Problem 17.** Let a continuum $X$ belong to $\mathcal{L}$, and let $f$ be a local homeomorphism defined on $X$. Does it follow that $f(X)$ is in the class $\mathcal{L}$?

Although property of being an irreducible continuum of type $\lambda$ is not preserved under open mappings (see [1], p. 216; cf. [22], p. 73), if we neglect the irreducibility, we obtain the following

**Problem 18.** Let a continuum $X$ belong to $\mathcal{L}$ and let $f$ be an open mapping defined on $X$. Does it follow that $f(X)$ is in the class $\mathcal{L}$?

**REFERENCES**


