A Theorem on Non-planar Dendroids

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Presented by K. KURATOWSKI on July 19, 1975

Summary. It is shown that there is no finite family of dendroids with the property that a dendroid is non-planar if and only if it contains a member of this family. The question is asked if there exists a countable family having the same property.

Kuratowski has proved [4] that a (metric) locally connected continuum which contains only a finite number of simple closed curves (i.e. a local dendrite) is imbeddable in the 2-sphere is and only if it contains neither of the two "primitive skew graphs" $K_1$ and $K_2$, defined as follows. $K_1$ consists of the points $p_1, p_2, \ldots, p_5$ and the arcs $p_i p_j$, where $1 \leq i < j \leq 5$ and different arcs intersect only at their end points; $K_2$ consists of the points $p_1, p_2, p_3$ and $q_1, q_2, q_3$ and the arcs $p_i q_i$, where $i = 1, 2, 3$ and different arcs intersect only at their end points.

Claytor has shown [2] that a locally connected continuum which has no cut point is imbeddable in the 2-sphere if and only if it contains neither $K_1$ nor $K_2$. He has also shown [3] that a locally connected continuum can be imbedded in the 2-sphere if and only if it contains no homeomorphic image of the primitive skew graphs $K_1$ and $K_2$ or of some two skew curves $C_1$ and $C_2$ (see [3], p. 631, where these curves are pictured).

The aim of this note is to show that no theorem of this kind can be proved in case when the curves under consideration are not locally connected. This will be shown for dendroids, i.e. hereditarily unicoherent and arcwise-connected continua. To state this more precisely, let $\mathcal{A}$ be a class of spaces and let $\mathcal{P}$ be a property. Recall that $\mathcal{P}$ is said to be finite (or countable) in the class $\mathcal{A}$ provided there is a finite (or countable) set $\mathcal{F}$ of members of $\mathcal{A}$ such that a member $X$ of $\mathcal{A}$ has property $\mathcal{P}$ if and only if $X$ contains a homeomorphic image of some member of $\mathcal{F}$ (cf. [1], p. 304). For example, the result of Kuratowski mentioned at the beginning can be restated as follows: the property of not being imbeddable in the 2-sphere is finite in the class of local dendrites.

The main result is the following

**Theorem.** The property of not being imbeddable in the 2-sphere (or in the Euclidean plane) is not finite in the class of dendroids.
Auxiliary constructions. Let \((x, y)\) denote a point in the plane \(R^2\) having \(x\) and \(y\) as its rectangular coordinates. Put \(p_0=(0, 0)\), \(p_i=(1/i, 0)\) for \(i=1, 2, 3, \ldots\) and \(I_0=\{(x, 0): 0 \leq x \leq 2\}\). Let \(I_i\) for \(i=1, 2, 3, \ldots\) denote the union of \(i\) straight segments which have only the point \(p_0\) in common, lie in the upper half-plane \((y \geq 0)\) and arc of length \(1/i\) each. Put \(P_0=\{(0, 0): 0 \leq x \leq 2\}\) and \(I_0=\{I_i: i=1, 2, 3, \ldots\}\) and define \(D_0\) as the union of \(I_0\) and of a sequence of arcs which have only the point \(p_0\) in common, lie in the upper half-plane and approximate the continuum \(I\). Thus \(D_0\) is a dendroid. Fig. 1 of [1], p. 305 is a picture of a homeomorphic image of \(D_0\). In general, let \(D_n\) denote a dendroid which lies in the upper half-plane and looks like \(D_0\), except that \(p_i\) has \(n+i\) segments erected above it instead of \(i\) arcs (i.e. we substitute \(I_{n+i}\) for \(I_i\), where \(i=1, 2, 3, \ldots\)). Now let \(C_n\) be the union of \(D_n\) and of a sequence of straight segments which join \(p_0\) with \(q_i=(2, -1/i)\) for \(i=1, 2, 3, \ldots\). In other words, \(C_n\) is the union of \(D_n\) and of the harmonic fan lying in the lower half-plane, having \(p_0\) as its top and \(I_0\) as its limit segment. Let us call a point \(p\) of a set \(P\) contained in the plane \(R^2\) to be strongly inaccessible in \(P\) provided that there is no homeomorphism \(h: P \to h(P) \subset R^2\) such that \(h(p)\) is accessible from the complement \(R^2 \setminus h(P)\), i.e. that there is no arc \(A\) with \(h(p)\) as its end point and lying in \([R^2 \setminus h(P)] \cup \{h(p)\}\). Observe that each point \(p_i\) for \(i=1, 2, 3, \ldots\) is strongly inaccessible from \(R^2 \setminus C_n\), i.e. for no homeomorphism \(h: C_n \to h(C_n) \subset R^2\) is the image \(h(p_i)\) accessible from \(R^2 \setminus h(C_n)\). Therefore, if we add to \(C_n\) a straight segment lying in the space and having only the point \(p_i\) in common with \(C_n\), we obtain a dendroid which is non-planar, i.e. which cannot be imbedded in the plane. Let \(\sigma\) denote an arbitrary infinite sequence \(\sigma=\{\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots\}\), the terms \(\sigma_i\) of which are equal to 0 or 1.

If a sequence \(\sigma\) is fixed, we define \(L_{i, \sigma} = \{p_i\}\) if \(\sigma_i=0\); if \(\sigma_i=1\), let \(L_{i, \sigma}\) be the straight line segment of length \(1/i\) lying in the space (e.g. perpendicular to the plane \(R^2\) in which \(C_n\) is contained) and having \(p_i\) as its end point. Putting

\[A_{n, \sigma}=C_n \cup \bigcup \{L_i: i=1, 2, 3, \ldots\}\]

we see that \(A_{n, \sigma}\) is a non-planar dendroid. Taking various sequences \(\sigma\) and various \(n\) we obtain an uncountable family of skew dendroids \(A_{n, \sigma}\).

Proof of the theorem. Suppose there exists a finite set \(\mathcal{F}=\{F_1, F_2, \ldots, F_j, \ldots, F_k\}\) of dendroids such that a dendroid \(X\) is non-planar if and only if \(X\) contains a homeomorphic image of some member of \(\mathcal{F}\). Fix a sequence \(\sigma\) putting \(\sigma_i=1\) for every \(i=1, 2, \ldots\), i.e. let \(\sigma=\{1, 1, \ldots\}\), and take the countable family of non-planar dendroids \(\{A_{n, \sigma}: n=0, 1, 2, \ldots\}\). Note that for a fixed \(n\) and for each \(i=1, 2, 3, \ldots\) the point \(p_i\) of \(A_{n, \sigma}\) is the common end point of \(n+i+3\) straight segments disjoint out of it. Namely, it is the common end point of \(n+i+3\) segments erected above \(p_i\) in \(D_n\), of two segments contained in \(I_0\) and of the segment \(L_{i, \sigma}\). In other words, \(A_{n, \sigma}\) is of ramification order (in the classical sense) \(n+i+3\) at each point \(p_i\). Since \(A_{n, \sigma}\) is non-planar, it contains a homeomorphic image of some member of \(\mathcal{F}\), say \(F_j\). Neglecting the homeomorphism we can write \(F_j \subset A_{n, \sigma}\). Let \(\mathcal{F}'\) be a subset
of $\mathcal{F}$ such that $F_j \in \mathcal{F}'$ if and only if there exists a natural $n$ such that $F_j \subset A_{n, \sigma}$. As a subset of $\mathcal{F}$, $\mathcal{F}'$ is a finite set.

Since every member $F_j$ of $\mathcal{F}'$ is a non-planar subdendroid of some dendroid $A_{n, \sigma}$, there exists, for this fixed $n$, a natural $i$ such that $F_j$ contains a non-trivial straight segment $S$ contained in $L_{i, \sigma}$ for which the point $p_i$ is an end point. Since the straight segment $S$ as a subset of $F_j$ cannot be imbedded into $R^2$, the dendroid $F_j$ must contain a subarc $B$ of $I_0$ containing $p_i$ as its interior point, together with two sequences of arcs: each arc of the first sequence lies in the upper half-plane, and the first sequence approximates $B$ together with the union of $n+i$ segments erected above $p_i$; each arc of the second sequence lies in the lower half-plane, and the second sequence approximates $B$ only. The dendroid $F_j$ being a continuum, we conclude that $F_j$ contains a homeomorphic image of the dendroid $A_{n+i, \tau}$, where $\tau$ denotes the sequence $\{1, 0, 0, ..., 0, ...\}$, i.e. $\tau_1 = 1$ and $\tau_i = 0$ for $i > 1$. Neglecting the homeomorphism we can write $A_{n+i, \tau} \subset F_j$.

We see that both $F_j$ and $A_{n, \sigma}$ are of the same ramification order $n+i+3$ at $p_i$. Put $i_j = \min \{i \in N : A_{n+i, \tau} \subset F_j\}$. It follows that $F_j$ contains the point $p_{i_j}$ at which it is of ramification order $n+i_j+3$. Let $N$ denote the set of all natural numbers $0, 1, 2, ...$. Consider the set $\{m \in N : F_j \subset A_{m, \sigma}\}$. Since the dendroids $A_{m, \sigma}$ are of ramification order $m+i+3$ at each point, we see by the construction of $A_{m, \sigma}$ that the minimal ramification order of $A_{m, \sigma}$ at points $p_i$ is $m+4$. Since $F_j$ contains a point of order $n+i_j+3$, and since both $F_j$ and $A_{m, \sigma}$ are of the same ramification order at that point (by the argumentation as above), we conclude from $F_j \subset A_{m, \sigma}$ that $m+4 \leq n+i_j+3$, where $n$ and $i_j$ are fixed. Thus the set $\{m \in N : F_j \subset A_{m, \sigma}\}$ must be finite. Therefore, for each member $F_j$ of $\mathcal{F}'$ only a finite number of dendroids $A_{m, \sigma}$ can contain $F_j$. Since $\mathcal{F}'$ is finite, it follows that only for finitely many dendroids $A_{n, \sigma}$ there is a natural $j$ with $F_j \subset A_{n, \sigma}$. Thus there are some dendroids in the sequence $\{A_{n, \sigma} : n \in N\}$ that do not contain any member of $\mathcal{F}'$. The proof is finished.

Problem. Is it true that the property of not being imbeddable in the plane is not countable in the class of dendroids?

Remarks. 1) The theorem answers one of the problems asked in [1], §5, p. 307.

2) On the basis of the previous construction of an uncountable family of skew-dendroids (e.g. of $A_{0, \sigma}$ where $\sigma$ runs over the set of all zero-one sequences) one might be tempted to think that it is easy to show a positive answer to the above problem. However, the above-mentioned family is not good enough to answer the problem affirmatively. Namely, putting $F_j = A_{0, \sigma}(j)$, where $\sigma(j)$ is a sequence which has only the $j$-th term equal to 1, i.e. $\sigma_i(j) = 1$ if and only if $i = j$, we get a countable family of $F_j$, $j = 1, 2, 3, ...$ with the property that each non-planar dendroid $A_{0, \sigma}$ contains some member of this family.
REFERENCES


Я. Е. Харатоник, Теорема о не-плоских дendiroidах

Содержание. Доказывается, что не существует конечное семейство dendroidов, имеющее следующее свойство: dendroid не может быть вложенным в плоскость тогда и только тогда, когда он содержит некоторый элемент этого семейства. Ставится вопрос, существует ли счётное семейство dendroidов, имеющее то же самое свойство.