Mappings on dendrites

Janusz J. Charatonik a,b,*, Alejandro Illanes a

a Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México DF, Mexico
b Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received 7 October 2003; accepted 21 April 2004

Abstract

The existence of a point of order $\omega$ in dendrites with finitely many branch points is characterized in terms of self-mappings on such dendrites. Also in these terms conditions are found under which dendrites without points of order $\omega$ have infinitely many branch points. Structure of dendrites having the $\Omega$EP-property (for each self-mapping $f$ the set of nonwandering points of $f$ is contained in the closure of the set of eventually periodic points of $f$) is characterized by noncontaining of a special dendrite $W$.

© 2004 Elsevier B.V. All rights reserved.

MSC: 54C05; 54F50

Keywords: Continuum; Dendrite; Mapping; Nonwandering point; Periodic point

1. Introduction

Since dendrites have often appeared as Julia sets in complex dynamical systems (see [18], for example), the dynamical behavior of their self-mappings is both important and interesting in the study of dynamical systems (and in continuum theory, too). Therefore questions arise in a very natural way concerning a possibility of extensions of some results in the area proved for trees to dendrites, which form the nearest (in a sense) class of curves containing trees. Such questions were discussed, e.g., in [1,5,10–12] and in certain other papers. In the present paper several results obtained earlier for trees are extended to dendrites or to dendrites satisfying some additional conditions.

* Corresponding author.
E-mail addresses: jjc@math.unam.mx (J.J. Charatonik), illanes@math.unam.mx (A. Illanes).

0166-8641/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
In these studies especially valuable are results that tie properties of mappings $f : X \to X$ with the topological structure of the space $X$. Such results let us to show that some topological objects, in particular some classes of dendrites, can be described using two rather far methods: topological or even geometrical one, that presents their internal structure, and a functional method, that exhibits mapping properties which characterize the studied space.

In [11] the second named author characterized dendrites $X$ that have the PR-property (closures of the sets of periodic and of recurrent points are equal for each self-mapping on $X$) as those ones which do not contain any topological copy of the Gehman dendrite. In this paper two similar results are shown. The first of them says that for mappings $f : X \to X$ on a dendrite $X$ with finitely many branch points the difference between cardinalities of the sets $\Omega(f)$ and $P(f)$ for $f : X \to X$ can be arbitrarily large if and only if $X$ contains a topological copy of the fan $F_\omega$. The second result concerns so called nonwandering-eventually-periodic property (abbreviated as the $\Omega$EP-property), an important property that was previously studied by a number of topologists. Namely we say that a space $X$ has the $\Omega$EP-property provided that for each mapping $f : X \to X$ the set of nonwandering points of $f$ is contained in the closure of the set of eventually periodic points of $f$. The obtained characterization extends earlier results of [1,3,10].

The paper consists of five sections. After Introduction and Preliminaries we discuss dendrites with finitely many branch points in Section 3. The main result in this section is Corollary 3.4 in which the existence of a point of order $\omega$ in such dendrites is characterized in terms of self-mappings on the dendrites. Section 4 is devoted to dendrites with infinitely many branch points but containing no points of order $\omega$. It is shown that there are a dendrite $X$ and a self-mapping $g : X \to X$ such that the numbers $\text{card}(F(g))$ and $\text{card}((\Omega(g) \setminus P(g))$ can be made arbitrarily large. This extends some results of [10]. In Section 5 we prove that a dendrite $X$ has the $\Omega$EP-property if and only if it does not contain a topological copy of the dendrite $W$ defined on the plane by

$$W = [0, 1] \times [0] \cup \bigcup \left\{ \left[ \frac{1}{n}, \frac{1}{n+1} \right] \times \left[ 0, \frac{1}{n} \right] : n \in \mathbb{N} \right\}.$$  \hfill (1.1)

So, any homeomorphic copy of $W$ is a dendrite all branch points of which are of order 3, all lie in an arc so that one of the end points of the arc is the only accumulation point of the set of all branch points of $W$.

A number of open questions related to the subject are also contained in the paper.

2. Preliminaries and auxiliary results

We use the same notation as in [10] or [1]. It is recalled here for the reader convenience.

By a space we mean a metric space, and a mapping means a continuous function. We use $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the positive integers, the spaces of real and of complex numbers, respectively. For $A \subset X$ we denote $\text{cl}_X(A)$ and $\text{bd}_X(A)$ the closure and the boundary of $A$ in $X$, correspondingly. We will omit the subscript $X$ in case when the meaning of the space $X$ is clear. The symbol card$(A)$ stands for the cardinality of $A$, and diam$(A)$ means the diameter of $A$. 
If $p$ and $q$ are points lying in the plane, then $pq$ stands for the straight line segment joining $p$ and $q$.

For a space $X$, a mapping $f : X \to X$ and $n \in \mathbb{N}$ we denote by $f^n$ the $n$th composition of $f$, and by $f^0$ the identity mapping.

Let $X$ be a space, and let $f : X \to X$ be a mapping of $X$ to itself. A point $x$ of $X$ is said to be:

- a fixed point of $f$ if $f(x) = x$;
- a periodic point of $f$ provided that there is $n \in \mathbb{N}$ such that $f^n(x) = x$; if, moreover, $f^k(x) \neq x$ for all integers $k$ with $1 \leq k < n$, then $x$ is called a periodic point of period $n$;
- a recurrent point of $f$, provided that for each open set $U$ containing $x$ there is $n \in \mathbb{N}$ such that $f^n(x) \in U$;
- an eventually periodic point of period $n \in \mathbb{N}$ for $f$ provided that there exists $m \in \{0\} \cup \mathbb{N}$ such that $f^m(x)$ is a periodic point of $f$ of period $n$;
- an eventually periodic point for $f$ provided that there is $n \in \mathbb{N}$ such that $x$ is an eventually periodic point of period $n \in \mathbb{N}$ for $f$;
- a nonwandering point of $f$ provided that for any open set $U$ containing $x$ there exist $y \in U$ and $n \in \mathbb{N}$ such that $f^n(y) \in U$.

Note that if the orbit of $x$ is defined by $\text{orb}(f; x) = \{ f^n(x) : n \in \{0\} \cup \mathbb{N} \}$, then $x$ is eventually periodic if $\text{orb}(f; x)$ is finite, or equivalently if some element of $\text{orb}(f; x)$ is periodic.

For a mapping $f : X \to X$ the sets of fixed points, periodic points, recurrent points, eventually periodic points and nonwandering points of $f$ will be denoted by $F(f)$, $P(f)$, $R(f)$, $EP(f)$ and $\Omega(f)$, respectively. Notice that

$$F(f) \subset P(f) \subset R(f) \subset \Omega(f) \subset X,$$  

$$P(f) \subset EP(f), \quad f(P(f)) \subset P(f), \quad f(\Omega(f)) \subset \Omega(f).$$  

$$\Omega(f) \text{ is closed.}$$

A space $X$ is said to have:

- the periodic-recurrent property (abbreviated PR-property) provided that for every mapping $f : X \to X$ the equality $\text{cl}(P(f)) = \text{cl}(R(f))$ holds (see [5, Definition 1.4, p. 132]; compare [7]);
- the nonwandering-periodic property (abbreviated $\Omega P$-property) provided that for every mapping $f : X \to X$ the equality $\Omega(f) = P(f)$ holds (equivalently, by (2.1), if and only if $\Omega(f) \subset P(f)$);
- the nonwandering-eventually-periodic property (abbreviated $\Omega EP$-property) provided that for every mapping $f : X \to X$ the inclusion is satisfied

$$\Omega(f) \subset \text{cl}_X(EP(f)).$$
A mapping \( f : X \to Y \) between continua \( X \) and \( Y \) is said to be \textit{monotone} provided that \( f^{-1}(y) \) is connected for each \( y \in Y \). It is called a \textit{retraction} if \( Y \subseteq X \) and the partial mapping \( f|Y : Y \to Y \) is the identity. In this case \( Y \) is called a \textit{retract} of \( X \).

An \textit{arc} means a space homeomorphic to the closed unit interval \([0, 1]\). A \textit{continuum} is a compact connected space. A \textit{graph} is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only at one or both of their end points. A graph which contains no simple closed curve (i.e., which is uniquely arcwise connected) is called a \textit{tree}. A \textit{dendrite} means a locally connected and uniquely arcwise connected continuum. The reader is referred to [6] for more information on dendrites.

A concept of an \textit{order} of a point \( p \) in a continuum \( X \) (in the sense of Menger–Urysohn), written \( \text{ord}(p, X) \), is defined as follows. Let \( n \) stand for a cardinal number. We write:

- \( \text{ord}(p, X) \leq n \) provided that for every \( \varepsilon > 0 \) there is an open neighborhood \( U \) of \( p \) such that \( \text{diam}(U) \leq \varepsilon \) and \( \text{card}(\text{bd}(U)) \leq n \);
- \( \text{ord}(p, X) = n \) provided that \( \text{ord}(p, X) \leq n \) and for each cardinal number \( m < n \) the condition \( \text{ord}(p, X) \leq m \) does not hold;
- \( \text{ord}(p, X) = \omega \) provided that the point \( p \) has arbitrarily small open neighborhoods \( U \) with finite boundaries \( \text{bd}(U) \) and \( \text{card}(\text{bd}(U)) \) is not bounded by any \( n \in \mathbb{N} \).

Thus, for any continuum \( X \) we have

\[ \text{ord}(p, X) \in \{1, 2, \ldots, n, \ldots, \omega, \aleph_0, 2^{\aleph_0}\} \]

(convention: \( \omega < \aleph_0 \)); see [15, §51, I, p. 274].

A point \( p \in X \) is called an \textit{end point} of \( X \) provided that \( \text{ord}(p, X) = 1 \), and it is called a \textit{branch point} of \( X \) provided that \( \text{ord}(p, X) \geq 3 \). For a dendrite \( X \) we denote the sets of end points of \( X \) and of branch points of \( X \) by \( E(X) \) and \( B(X) \), respectively.

In the sequel we will need three special dendrites. The first of them is the dendrite with only one branch point, whose order is \( \omega \). We will denote this dendrite by \( F_\omega \). Note that \( F_\omega \) is just the dendrite \( S \) of [1, Example, p. 33]. The second one, \( W \), is constructed in [2, p. 3] and it is denoted therein by \( W_R \); it is already defined by (1.1) above. Finally, the third needed example is the well-known \textit{Gehman dendrite} \( G \) (see [16, Example 10.39, p. 186]). Note that the infinite binary tree is another name of this dendrite (see, e.g., [10, Example 1.6, p. 45]). Recall that \( G \) can be characterized as the only dendrite whose set of end points is homeomorphic to the Cantor set, and whose branch points are of order 3 only (see [17, p. 100]).

The following three known results will be used.

**Theorem 2.5** [9, Theorem, p. 157]. A continuum \( X \) is a dendrite if and only if each subcontinuum of \( X \) is a monotone retract of \( X \).

The monotone retraction of Theorem 2.5 is the \textit{first point map} of [16, 10.25 and 10.26, p. 176].
Theorem 2.6 [2, Theorem 3.1, p. 3]. A dendrite $X$ is a tree if and only if $X$ contains neither a copy of $F_\infty$ nor of $W$.

The next result is a consequence of Theorem 2.6.

Theorem 2.7. A dendrite $X$ is a tree if and only if $X$ has a finite set $B(X)$ of its branch points, each point of which is of a finite order.

Theorem 2.8 [10, Theorem 1.1, p. 36]. Let $X$ be a tree and $f : X \to X$ be a mapping. If $\Omega(f)$ is finite, then $\Omega(f) = P(f)$.

The assumption that $X$ is a tree is essential in Theorem 2.8 by Example 3.1 below.

Further, we need a lemma on compositions of mappings. Its easy proof is left to the reader (for (2.9.5) see [13, Lemma 2.2, p. 49]).

Lemma 2.9. Let $X$ and $Y$ be spaces with $Y$ being a closed subset of $X$, and let $g : Y \to Y$ be a mapping. If $r : X \to Y$ is a retraction and $f = g \circ r : X \to Y$, then:

\begin{enumerate}
  \item[(2.9.1)] $f^n = g^n \circ r$ for each $n \in \mathbb{N}$;
  \item[(2.9.2)] $P(f) = P(g)$;
  \item[(2.9.3)] $R(f) = R(g)$;
  \item[(2.9.4)] $Y \cap EP(f) = EP(g)$;
  \item[(2.9.5)] $\Omega(f) = \Omega(g)$.
\end{enumerate}

Coming back to the PR-property let us recall that each tree has the property, [20, Theorem 2.6, p. 349], while the circle does not have it, [5, Remarks 1.2, p. 132]. As a consequence of Lemma 2.9 we get the following characterization of trees.

Proposition 2.10. A graph is a tree if and only if it has the PR-property.

Proof. One implication is just the above quoted result [20, Theorem 2.6, p. 349]. Conversely, if a graph $X$ has the PR-property, then it does not contain any simple closed curve, because if a simple closed curve $Y$ is a subset of $X$, then it is a retract of $X$ (since each graph is hereditarily locally connected, so any of its subcontinua is a retract of the graph, see [4, Theorem 6, p. 84]). Taking a retraction $r : X \to Y$, a mapping $g : Y \to Y$ with $\text{cl}(P(g)) \neq \text{cl}(R(g))$ [5, Remarks 1.2, p. 132], and putting $f = g \circ r$ we get $\text{cl}(P(f)) \neq \text{cl}(R(f))$ by Lemma 2.9. \qed

We close this preliminary section with a summary of known results in the area. The above mentioned three properties of spaces, i.e., the PR-, the $\Omega P$- and the $\Omega EP$- property, are related to the dendrites $G$, $F_\infty$ and $W$ by the following propositions.

Proposition 2.11. A dendrite $X$ has the PR-property if and only if $X$ does not contain any copy of the Gehman dendrite $G$ (see [11, Theorem 2, p. 222]).
Proposition 2.12. A dendrite $X$ has the $\Omega$EP-property if and only if $X$ does not contain any copy of the dendrite $W$ (see Corollary 5.14 in this paper).

Proposition 2.13. A dendrite $X$ with finitely many branch points contains a topological copy of the dendrite $F_\omega$ if and only if for every two positive integers $j, k$ with $j < k$ there exists a mapping $f_{j,k} : X \to X$ such that $j = \text{card}(P(f_{j,k}))$ and $k = \text{card}(\Omega(f_{j,k}))$ (see Corollary 3.4 in this paper).

In connection with Proposition 2.13 the following problem can be posed.

Problem 2.14. Give an internal (i.e., structural) characterization of dendrites with $\Omega$P-property.

3. The $\Omega$P-property—dendrites with finitely many branch points

Define

$$S = \left\{ r \exp\left(\frac{2\pi i}{n}\right) \in \mathbb{C} : r \in \left[0, \frac{1}{n}\right] \text{ for } n \in \mathbb{N}\right\}$$

and note that $S$ is homeomorphic to $F_\omega$. The following result is known.

Example 3.1 [1, Example, p. 33]. For every two integers $j, k \in \mathbb{N}$ with $j < k$ there is a mapping $g_{j,k} : S \to S$ such that

(3.1.1) $j = \text{card}(P(g_{j,k})) < \text{card}(\Omega(g_{j,k})) = k$.

The above result can be strengthened as follows.

Theorem 3.2. If a dendrite $X$ contains a point of order $\omega$, then

(3.2.1) for every two integers $j, k \in \mathbb{N}$ with $j < k$ there is a mapping $f_{j,k} : X \to X$ such that

(3.2.2) $j = \text{card}(P(f_{j,k})) < \text{card}(\Omega(f_{j,k})) = k$.

Proof. Let $p \in X$ be a point of order $\omega$. Then there exists a subdendrite $F_\omega \subset X$ such that $p$ is the (only) branch point of $F_\omega$. Put $S = F_\omega$, choose two integers $j, k \in \mathbb{N}$ with $j < k$, and let $g_{j,k} : S \to S$ be the mapping of Example 3.1. Thus we have (3.1.1). By Theorem 2.5 there exists a monotone retraction $r : X \to S$. Define $f_{j,k} = g_{j,k} \circ r : X \to S$. Then $P(f_{j,k}) = P(g_{j,k})$ and $\Omega(f_{j,k}) = \Omega(g_{j,k})$ according to equalities (2.9.2) and (2.9.5) of Lemma 2.9, respectively. Therefore (3.2.2) follows from (3.1.1). \qed

The converse implication to that of Theorem 3.2 is also true under the assumption that the dendrite $X$ has finitely many branch points.
Theorem 3.3. Let a dendrite $X$ have a finite set $B(X)$ of its branch points. If there are $j, k \in \mathbb{N}$ with $j < k$ for which there exists a mapping $f_{j,k} : X \to X$ such that (3.2.2) holds, then $X$ contains a point of order $\omega$.

Proof. Suppose on the contrary that there is no point of order $\omega$ in $X$. Thus each point of $X$ is of a finite order, and since $B(X)$ is finite, it follows by Theorem 2.6 that $X$ is a tree. Since $\text{card}(\Omega(f_{j,k})) = k$ (see (3.2.2)) it follows that $\Omega(f_{j,k})$ is finite. Applying Theorem 2.8 we get $\text{P}(f_{j,k}) = \Omega(f_{j,k})$, whence $\text{card}(\text{P}(f_{j,k})) = \text{card}(\Omega(f_{j,k}))$, contrary to (3.2.2). \[\square\]

As a consequence of Theorems 3.2 and 3.3 we get the following corollary.

Corollary 3.4. Let a dendrite $X$ have a finite set $B(X)$ of its branch points. Then the following conditions are equivalent:

(3.4.1) there are $j, k \in \mathbb{N}$ with $j < k$ for which there exists a mapping $f_{j,k} : X \to X$ such that (3.2.2) holds;
(3.4.2) for every two integers $j, k \in \mathbb{N}$ with $j < k$ there is a mapping $f_{j,k} : X \to X$ such that (3.2.2) holds;
(3.4.3) $X$ contains a point of order $\omega$ (equivalently, a copy of $F_\omega$).

Remark 3.5. Let us note that the assumption of finiteness of $\text{card}(B(X))$ is essential in Theorem 3.3, and thus in Corollary 3.4. Namely a dendrite $X_1$ and a mapping $g_1 : X_1 \to X_1$ are constructed in [10, Example 1.5, p. 44] such that $B(X_1)$ is infinite, $X_1$ does not contain any point of order $\omega$, and $2 = \text{card}(\text{P}(g_1)) < \text{card}(\Omega(g_1)) = 3$ (see below, Example 4.1).

Corollary 3.4 implies the next one, that is also a consequence of Theorem 2.8.

Corollary 3.6. Let $X$ be a tree and $f : X \to X$ be a mapping. If $\Omega(f)$ is finite, then $\text{card}(\text{P}(f)) = \text{card}(\Omega(f))$.

It is not known to the authors if the assumption of the finiteness of the set $\Omega(f)$ is or is not essential in Theorem 2.8 and Corollary 3.4. Thus we have the following question.

Question 3.7. Do there exist a tree $X$ and a mapping $f : X \to X$ such that $\Omega(f)$ is infinite while $\text{P}(f)$ is finite?

4. The $\Omega$P-property—dendrites with infinitely many branch points

Let $m \in \mathbb{N}$. Recall that the wedge of $m$ pointed spaces $(S_1, p_1), \ldots, (S_m, p_m)$, denoted by $S_1 \wedge \cdots \wedge S_m$, is the one-point union $S_1 \cup \cdots \cup S_m$, where all the distinguished points $p_1, \ldots, p_m$ are identified to a point. If $Y$ is a space and for each $\alpha \in \{1, \ldots, m\}$ a mapping $f_{\alpha} : S_\alpha \to Y$ is given, then a mapping $f : S_1 \wedge \cdots \wedge S_m \to Y$ defined by $f(x) = f_{\alpha}(x)$ for $x \in S_\alpha$ is called the combination of the mappings $f_1, \ldots, f_m$ and is denoted by $f_1 \triangledown \cdots \triangledown f_m$ (see [8, p. 71]).
To unify notation for further purposes, we need to redefine Example 1.5 of [10, p. 44] as follows.

**Example 4.1** [10, Example 1.5, p. 44]. There are a dendrite $X$ and a mapping $g : X \to X$ such that

(4.1.1) the set $B(X)$ of branch points of $X$ is infinite;

(4.1.2) $X$ does not contain any topological copy of the dendrite $F_\omega$;

(4.1.3) $2 = \text{card}(P(g)) < \text{card}(\Omega(g)) = 3$.

**Proof.** Let $W$ be dendrite defined in the plane $\mathbb{R}^2$ by (1.1). Put $a = (0, 0)$, and for each $n \in \mathbb{N}$ denote $a_n = (\frac{1}{n}, \frac{1}{n})$ and $b_n = (\frac{1}{n}, 0)$. Then

$$W = ab_1 \cup \bigcup \{a_nb_n : n \in \mathbb{N}\}.$$ 

Take two copies $W^{(1)}$ and $W^{(2)}$ of the dendrite $W$. Let $x^{(1)} \in W^{(1)}$ and $x^{(2)} \in W^{(2)}$ be the copies of a point $x \in W$. Then the needed dendrite $X$ (which is denoted by $X_1$ in [10, Example 1.5, p. 44]) is defined by

(4.1.4) $X = W^{(1)} \cap W^{(2)}$, with $W^{(1)} \cap W^{(2)} = \{p\}$, where $p = b_1^{(1)} = b_1^{(2)}$.

The mapping $g : X \to X$ of Example 1.5 of [10, p. 44] (which is denoted by $g_1$ in the quoted example in [10]) can be seen as the combination $g^{(1)} \cup g^{(2)}$ with $g^{(1)} : W^{(1)} \to W^{(2)}$ and $g^{(2)} : W^{(2)} \to W^{(2)} \cup \{p\}$ determined by the following conditions:

(4.1.5) $g^{(1)}(p) = p$;

(4.1.6) $g^{(1)}(a_0^{(1)}) = a_0^{(2)}$;

(4.1.7) $g^{(1)}(a_n^{(1)}) = a_n^{(2)}$ for each $n \in \mathbb{N}$;

(4.1.8) $g^{(1)}(b_n^{(1)}) = b_n^{(2)}$ for each $n \in \mathbb{N}$;

(4.1.9) $g^{(2)}(p) = p$;

(4.1.10) $g^{(2)}(a_0^{(2)}) = a_0^{(2)}$;

(4.1.11) $g^{(2)}(a_n^{(2)}) = a_{n-1}^{(2)}$ is a linear homeomorphism with $g^{(2)}(a_n^{(2)}) = a_{n-1}^{(2)}$ for each $n \in \mathbb{N}$ and $n > 1$;

(4.1.12) $g^{(2)}(a_1^{(2)}) = a_1^{(1)}$;

(4.1.13) $g^{(2)}(b_n^{(2)}) = b_{n-1}^{(2)}$ with $g^{(2)}(b_n^{(2)}) = b_{n-1}^{(2)}$ for each $n \in \mathbb{N}$ and $n > 1$;

(4.1.14) $g^{(2)}(b_1^{(2)}) = \{p\}$.

Then for $g = g^{(1)} \cup g^{(2)}$ we have

(4.1.15) $P(g) = \{a_0^{(2)}, p\} \subset \Omega(g) = \{a_0^{(1)}, a_0^{(2)}, p\}$. 


Thus the required conditions (4.1.1)–(4.1.3) follow from the construction. The proof is complete.

The idea of the above construction leads to the following extension of the result in [10, Example 1.5, p. 44]. In this extension a dendrite $X$ and a self-mapping $g : X \to X$ are constructed so that the numbers $\text{card}(F(g))$ and $\text{card}(\Omega(g) \setminus P(g))$ can be made arbitrarily large.

**Theorem 4.2.** For every two numbers $j, k \in \mathbb{N}$ there exists a dendrite $X$ with a point $p \in X$ and a mapping $g : X \to X$ such that

1. $X$ contains a copy of the dendrite $W$ defined by (1.1) (thus $B(X)$ is infinite);
2. $X$ does not contain any point of order $\omega$;
3. each point $x \in B(X) \setminus \{p\}$ is of order 3 in $X$;
4. $1 + k = \text{card}(F(g)) = \text{card}(P(g)) < \text{card}(\Omega(g)) = 1 + j + k$.

**Proof.** For each $m \in \mathbb{N}$ let $W^{(m)}$ be a copy of the dendrite $W$ defined by (1.1), and for any point $x \in W$ denote by $x^{(m)}$ its copy in $W^{(m)}$. Fix numbers $j, k \in \mathbb{N}$ and define

$$X = W^{(1)} \land \cdots \land W^{(j)} \land W^{(j+1)} \land \cdots \land W^{(j+k)},$$

where $p = b^{(1)}_1 = b^{(1)}_j$ for $\alpha, \beta \in \{1, \ldots, j\}$ with $\alpha \neq \beta$ is the only common point of $W^{(\alpha)}$ and $W^{(\beta)}$. Therefore $X$ is a dendrite satisfying conditions (4.2.1)–(4.2.4).

Let mappings

$$g^{(1)}_\alpha : W^{(\alpha)} \to W^{(\alpha+1)} \quad \text{for} \quad \alpha \in \{1, \ldots, j\}$$

be determined by the same conditions (4.1.5)–(4.1.8) applied to the dendrites $W^{(\alpha)}$ and $W^{(\alpha+1)}$ in place of $W^{(1)}$ and $W^{(2)}$. Similarly, let mappings

$$g^{(2)}_\beta : W^{(\beta)} \to W^{(\beta)} \cup \overline{pa^{(1)}} \quad \text{for} \quad \beta \in \{j + 1, \ldots, j + k\}$$

satisfy the conditions (4.1.9)–(4.1.14) applied to $W^{(\beta)}$ and $W^{(\beta)} \cup \overline{pa^{(1)}}$ in place of $W^{(2)}$ and $W^{(2)} \cup \overline{pa^{(1)}}$.

Observe that

$$F(g^{(1)}_\alpha) = P(g^{(1)}_\alpha) = \{p\} \quad \text{for} \quad \alpha \in \{1, \ldots, j\} \quad \text{and} \quad F(g^{(2)}_\beta) = P(g^{(2)}_\beta) = \{p, a^{(\beta)}\} \quad \text{for} \quad \beta \in \{j + 1, \ldots, j + k\}.$$ 

Define

$$g = g^{(1)}_1 \lor \cdots \lor g^{(1)}_j \lor g^{(2)}_{j+1} \lor \cdots \lor g^{(2)}_{j+k}.$$ 

Thus $g : X \to X$ is a well defined surjection satisfying, by its definition and according to (4.2.6), the following conditions.

1. $F(g) = P(g) = \{p, b^{(j+1)}, \ldots, b^{(j+k)}\}$;
These conditions imply (4.2.5). The proof is complete. □

Similarly to Theorem 3.3, the converse implication to that of Theorem 4.2 is also true.

Theorem 4.3. If a dendrite $X$ does not contain any copy of $F_\omega$ and if there are numbers $m, m' \in \mathbb{N}$ with $m < m'$ for which there exists a mapping $f : X \to X$ such that

\[(4.3.1) \quad m = \text{card}(P(f)) < \text{card}(\Omega(f)) = m',\]

then the set $B(X)$ of branch points of $X$ is infinite.

Proof. Suppose on the contrary that $X$ has a finite set $B(X)$. Since there is no copy of $F_\omega$ in $X$, the dendrite $X$ is a tree according to Theorem 2.7. Since for the mapping $f : X \to X$ the set $\Omega(f)$ is finite by the assumption, the equality $P(f) = \Omega(f)$ holds by Theorem 2.8, which contradicts to (4.3.1). □

Remark 4.4. Note that in the previous section the discussed inequality $P(f) \neq \Omega(f)$ was shown to be true for some mapping $f$ of each dendrite $X$ having a finite set $B(X)$ and containing the dendrite $S = F_\omega$ (see Theorem 3.2). In the present section we have a weaker result: the constructed dendrite $X$ depends on the choice of numbers $j$ and $k$ mentioned in Theorem 4.2. So, one can ask whether the result can be improved in the sense that each dendrite $X$ which satisfies (4.2.1) and (4.2.2) would admit a mapping $f : X \to X$ with $\text{card}(P(f)) < \text{card}(\Omega(f))$. The next result shows that this is not the case.

Theorem 4.5. For each dendrite $X$ there is a mapping $f : X \to X$ such that $\Omega(f)$ is finite and $P(f) = \Omega(f)$.

Proof. Choose an arc $A \subset X$, and let $p$ and $q$ be the end points of $A$. By Theorem 2.5 there is a monotone retraction $r : X \to A$. Fix a homeomorphism $g : A \to [0, 1]$ and let $h : [0, 1] \to [0, 1]$ be a homeomorphism such that $h(0) = 0, h(1) = 1$ and $h(t) > t$ for $0 < t < 1$ (for example, $h(t) = \sqrt{t}$). Define a mapping $f : X \to A \subset X$ putting $f = g^{-1} \circ h \circ g \circ r$. It is easy to verify that $\Omega(f) = \{p, q\} = P(f)$. □

The following result is related to the previous one.

Theorem 4.6. Let $X$ be a dendrite such that

\[(4.6.1) \quad \text{for each mapping } f : X \to X \text{ the equality } \text{card}(P(f)) = \text{card}(\Omega(f)) \text{ holds.}\]

Then $X$ is a tree.

Proof. Consider two cases.

Case 1. $X$ has a point of order $\omega$. 
Then, according to Theorem 3.2 there is a mapping \( f : X \to X \) such that \( \text{card}(P(f)) < \text{card}(\Omega(f)) \), contrary to (4.6.1).

**Case 2.** \( X \) does not have a point of order \( \omega \).

In this case we discuss two subcases.

**Subcase 2(a).** \( X \) contains a homeomorphic copy of the dendrite \( W \).

In Example 5.7 below a mapping \( f : W \to W \) is defined such that \( EP(f) = \{(0,0),(2,0)\} \) (see Property 8 of the proof of Example 5.7). Observe that, by conditions (13) and (14) of the definition of \( f \), the two elements of \( EP(f) \) are fixed points of \( f \). Thus by (2.1) and (2.2) we have

\[
\left\{(0,0),(2,0)\right\} \subset F(f) \subset P(f) \subset EP(f) = \left\{(0,0),(2,0)\right\}
\]

whence it follows that \( P(f) = EP(f) \), and therefore \( \text{card}(P(f)) = 2 \). On the other hand, since \( P(f) \subset \Omega(f) \) by (2.1) and since \( (1,1) \in \Omega(f) \setminus P(f) \) according to Property 3 of the proof of Example 5.7, it follows that \( \Omega(f) \) consists of three points at least: the two points of \( P(f) \) and of (1.1). Thus \( \text{card}(\Omega(f)) \geq 3 \), and consequently \( \text{card}(P(f)) < \text{card}(\Omega(f)) \).

By Theorem 2.5 there is a monotone retraction \( r : X \to W \). Define \( f_s = f \circ r : X \to W \subset X \). Since \( P(f) = P(f_s) \) and \( \Omega(f) = \Omega(f_s) \) according to (2.9.2) and (2.9.5) of Lemma 2.9, respectively, it follows that \( \text{card}(P(f_s)) < \text{card}(\Omega(f_s)) \), again contrary to (4.6.1).

**Subcase 2(b).** \( X \) does not contain any homeomorphic copy of the dendrite \( W \).

Then \( X \) is a tree according to Theorem 2.6.

The proof is complete. \( \square \)

It would be interesting to know if the converse to Theorem 4.6 is true. So, we have a question.

**Question 4.7.** Is it true that for each tree \( X \) assertion (4.6.1) holds?

Note that Theorem 2.8 and Problem 2.14 are related to the above question.

### 5. The \( \Omega EP \)-property

The concept of the \( \Omega EP \)-property of a space \( X \) has been introduced in Section 2 by requiring that inclusion (2.4) holds for each mapping \( f : X \to X \). We start this section with the following proposition.

**Proposition 5.1.** The \( \Omega EP \)-property is preserved under retractions, i.e., if a space \( X \) having the \( \Omega EP \)-property contains a closed subspace \( Y \) which is a retract of \( X \), then \( Y \) has the \( \Omega EP \)-property, too.

**Proof.** Let \( g : Y \to Y \) be a mapping satisfying \( \Omega(g) \not\subseteq \text{cl}(EP(g)) \), and let \( r : X \to Y \) be a retraction. Define a mapping \( f : X \to X \) by \( f = g \circ r \). Then for each \( n \in \mathbb{N} \) the equality \( f^n = g^n \circ r \) holds by (2.9.1). Let a point \( y \in Y \) be given with \( y \in \Omega(g) \setminus \text{cl}(EP(g)) \).
By (2.9.4) and (2.9.5) it follows that $y \in \Omega(f) \setminus \text{cl}(EP(f))$. Thus $\Omega(f) \not\subset \text{cl}(EP(f))$, and the proof is finished. 

In [3, Theorem C, p. 228] it is shown that the closed unit interval has the $\Omega$EP-property. In [10, Theorem 1.2, p. 36 and Example 1.4, p. 44] it is proved that every tree has the property, while the circle does not have it. Thus, analogously to Theorem 2.7, we have the following characterization.

**Theorem 5.2.** A graph is a tree if and only if it has the $\Omega$EP-property.

**Proof.** One implication is just the above quoted result [10, Theorem 1.2, p. 36]. Conversely, if a graph $X$ has the $\Omega$EP-property, then it does not contain any simple closed curve, because if a simple closed curve $Y$ is a subset of $X$, then it is a retract of $X$ (since each graph is hereditarily locally connected, so any of its subcontinua is a retract of the graph, see [4, Theorem 6, p. 84]). Taking a retraction $r : X \to Y$, a mapping $g : Y \to Y$ with $\Omega(g) \not\subset \text{cl}(EP(g))$, [10, Example 1.4, p. 44], and putting $f = g \circ r$ we get $\Omega(f) \not\subset \text{cl}(EP(f))$ by Lemma 2.9.

The result for trees has been extended to dendrites $X$ with a finite set $B(X)$ of its branch points in [1, Theorem 2, p. 30]. The assumption on finiteness of $B(X)$ is essential because, as it is shown in [10, Example 1.6, p. 45], the Gehman dendrite $G$ admits a mapping $g : G \to G$ for which the inclusion in matter does not hold.

**Proposition 5.3.** If a continuum $X$ contains a copy $G$ of the Gehman dendrite, then it does not have the $\Omega$EP-property, i.e., there is a mapping $f : X \to X$ such that

$$\Omega(f) \not\subset \text{cl}(EP(f)).$$

**Proof.** Since each dendrite is an absolute retract (see [15, §53, III, Theorem 16, p. 344]), there is a retraction $r : X \to G$. Let $g : G \to G$ be the above quoted mapping of [10, Example 1.6, p. 45]. Putting $f = g \circ r : X \to G$ we get (5.3.1) according to Lemma 2.9.

**Example 5.4.** The $\Omega$EP-property is not preserved under the inverse limit operation of trees.

**Proof.** Indeed, the Gehman dendrite $G$ is the inverse limit of an increasing sequence of trees $T_n \subset G$ with monotone retractions $r_n : T_{n+1} \to T_n$ as bonding mappings. Then each $T_n$ has the $\Omega$EP-property by [10, Theorem 1.2, p. 36], while $G$ does not have the property by [10, Example 1.6, p. 45].

The above example directs our attention to inverse limits of arcs, and thus it motivates the following questions.

**Question 5.5.** Is the $\Omega$EP-property preserved under the inverse limit of arcs? In other words: can the above quoted result of Block in [3, Theorem C, p. 228] be generalized to arc-like continua? The following two related questions are of a particular interest. Does (a) the
\[ \sin(1/x) \text{-curve (b) the simplest indecomposable continuum (see [15, §48, V, Example 1, p. 204]) have the } \Omega \text{EP-property?} \]

Containing a copy of the Gehman dendrite does not characterize dendrites which do not have the \( \Omega \)EP-property, i.e., the inverse implication to that of Proposition 5.3 is not true, because the dendrite \( W \) defined by (1.1) (which does not contain any copy of the Gehman dendrite) does not have the property. To show this, we redefine \( W \) as follows.

For each \( n \in \mathbb{N} \) let
\[
L_n = \left\{ 2 - \frac{2}{2n} \right\} \times \left[ 0, \frac{1}{2n-1} \right] \quad \text{and} \quad M_n = \left\{ 2 - \frac{2}{2n+1} \right\} \times \left[ 0, \frac{1}{2n-1} \right].
\]

Define
\[
W = [0,2] \times \{0\} \cup \bigcup_{n \in \mathbb{N}} \{L_n \cup M_n:\ n \in \mathbb{N}\}, \quad (5.6)
\]
and note that the dendrites defined by (1.1) and (5.6) are homeomorphic.

**Example 5.7.** There exists a mapping \( f : W \to W \) such that \( \Omega(f) \not\subset \text{cl}_W(\text{EP}(f)) \).

**Proof.** For each \( n \in \mathbb{N} \) and \( j \in \{0, 4, 6, 7, 8\} \) put (in the plane equipped with the Cartesian coordinates)
\[
a(n, j) = \left( 2 - \frac{2}{2n}, \frac{j}{8}, \frac{1}{2n-1} \right), \quad b(n, j) = \left( 2 - \frac{2}{2n+1}, \frac{j}{8}, \frac{1}{2n-1} \right), \quad c_n = \left( \frac{1}{n+2}, 0 \right).
\]

Further, let
\[
a(0, 0) = c_1, \quad c_0 = a(1, 0), \quad \text{and} \quad Y = [0,2] \times \{0\}.
\]

We will use the following convention. Given points \( x, y, p, q \in W \) and a mapping \( f : W \to W \), the notation \( xy \xrightarrow{f} pq \) means that \( f(x) = p, f(y) = q \) and \( f \) sends the arc \( xy \) in \( W \) linearly onto the arc \( pq \) in \( W \), both arcs being parametrized by the length of arc. In particular, if the arcs \( xy \) and \( pq \) have the same length, then \( f|_{xy} : xy \to pq \) is an isometry (with the metric in \( W \) given by the length of arc). In this case, i.e., if \( f|_{xy} \) is an isometry such that \( f(x) = p \) and \( f(y) = q \), we write \( xy \xrightarrow{f} pq \) (observe the difference in notation between “\( \xrightarrow{f} \)” and “\( \xrightarrow{\to} \)”). The mapping \( f : W \to W \) is defined by the following 14 conditions, where \( n \in \mathbb{N} \).

(1) \( a(n, 7)a(n, 8) \xrightarrow{f} a(n+1, 6)a(n+1, 8); \)
(2) \( a(n, 6)a(n, 7) \xrightarrow{f} b(n, 4)a(n+1, 6); \)
(3) \( a(n, 4)a(n, 6) \xrightarrow{f} b(n, 0)b(n, 4); \)
(4) \( a(n, 0)a(n, 4) \xrightarrow{f} a(n-1, 0)b(n, 0); \)
(5) \( b(n, 4)b(n, 8) \xrightarrow{f} b(n-1, 6)b(n-1, 8) \) (for \( n \geq 2 \)).
It is easy to verify that, under these conditions, \( f \) is well-defined and continuous. We prove a series of properties of \( f \).

**Property 1.** For each \( n \geq 3 \) we have

\[
a(1, 8) \left( 1, 1 - \frac{1}{2^n} \right) \xrightarrow{f^{n-2}} a(n - 1, 8) \left( 2 - \frac{2}{2(n-1)}, \frac{1}{2^{n-2}} - \frac{1}{2^n} \right).
\]

We prove Property 1 by induction. For \( n = 3 \) we have to show that

\[
a(1, 8) \left( 1, 1 - \frac{1}{8} \right) \xrightarrow{f^{3-2}} a(2, 8) \left( 2 - \frac{2}{2 \cdot 2}, \frac{1}{2} - \frac{1}{8} \right).
\]

Note that \( 1, 1 - \frac{1}{8} = a(1, 7) \) and \( 2 - \frac{2}{2 \cdot 2}, \frac{1}{2} - \frac{1}{8} = a(2, 6) \). Since length of \( a(1, 8) a(1, 7) \) and \( a(2, 8) a(2, 6) \) is equal to \( \frac{1}{8} \) and \( f \) sends \( a(1, 8) a(1, 7) \) linearly onto \( a(2, 8) a(2, 6) \), we are done.

Assume now that Property 1 is valid for some integer \( n \geq 3 \). Since \( a(1, 8) (1, 1 - \frac{1}{2^n}) \subset a(1, 8) (1, 1 - \frac{1}{2^n}) \), it follows from the assumption that

\[
a(1, 8) \left( 1, 1 - \frac{1}{2^n+1} \right) \xrightarrow{f^{n-2}} a(n - 1, 8) \left( 2 - \frac{2}{2(n-1)}, \frac{1}{2^{n-2}} - \frac{1}{2^n+1} \right)
= a(n - 1, 8) a(n - 1, 7),
\]

and, by (1),

\[
a(n - 1, 8) a(n - 1, 7) \xrightarrow{f} a(n, 8) a(n, 6) = a(n, 8) \left( 2 - \frac{2}{2^n}, \frac{3}{4}, \frac{1}{2^n-1} \right).
\]

Therefore

\[
a(1, 8) \left( 1, 1 - \frac{1}{2^n+1} \right) \xrightarrow{f^{n-1}} a(n, 8) \left( 2 - \frac{2}{2^n}, \frac{3}{4}, \frac{1}{2^n-1} \right)
= a(n, 8) \left( 2 - \frac{2}{2^n}, \frac{1}{2^n-1} - \frac{1}{2^n+1} \right).
\]

as needed. The proof of Property 1 is complete.
Property 2. For each \( n \geq 2 \) we have
\[
 b(n, 4)b(n, 8) \mapsto \left( \frac{4}{3} - \frac{1}{2^n} \right) b(1, 8).
\]

Indeed, according to (5) we have \( b(n, 4)b(n, 8) \mapsto b(n - 1, 6)b(n - 1, 8) \). Since \( b(n - 1, 6)b(n - 1, 8) \subseteq b(n - 1, 4)b(n - 1, 8) \) and \( b(n - 1, 4)b(n - 1, 8) \mapsto b(n - 2, 6)b(n - 2, 8) \) again by (5), we conclude that
\[
 b(n, 4)b(n, 8) \mapsto \left( 2 - \frac{2}{2(n - 2) + 1} \cdot \frac{1}{2^{n-2}} - \frac{1}{2^n} \right) b(n - 2, 8).
\]

Proceeding in this way \( n - 1 \) times we get the needed assertion. So, Property 2 is shown.

Property 3. \( (1, 1) \in \Omega(f) \).

According to the definition of \( \Omega(f) \) it is enough to prove that for \( n \geq 3 \) we have \( f^{2n-2}((1, 1 - \frac{1}{2^n})) = (1, 1 - \frac{1}{2^n}) \). In fact, by Property 1 we get
\[
 f^{n-2}\left(1, 1 - \frac{1}{2^n}\right) = \left( 2 - \frac{2}{2(n - 1)} \cdot \frac{1}{2^{n-2}} - \frac{1}{2^n} \right) = a(n - 1, 6).
\]

It follows from (3) that \( f(a(n - 1, 6)) = b(n - 1, 4) \). By Property 2 we have \( f^{n-2}(b(n - 1, 4)) = (\frac{4}{3} - \frac{1}{2^n}) \). It follows from (7) that \( b(1, 0)b(1, 8) \mapsto a(1, 0)a(1, 8) \), whence \( f((\frac{4}{3}, 1 - \frac{1}{2^n})) = (1, 1 - \frac{1}{2^n}) \). Therefore \( f^{2n-2}((1, 1 - \frac{1}{2^n})) = (1, 1 - \frac{1}{2^n}) \), as needed. Thus the argument for Property 3 is complete.

Let us order the straight line segment \( Y = (0, 0)(2, 0) \) in a natural way by \( < \) with \( (0, 0) < (2, 0) \).

Property 4. The following four conditions are satisfied:

(P4.1) \( f(Y \setminus \{(0, 0), (2, 0)\}) \subset Y \setminus \{(0, 0), (2, 0)\} \);
(P4.2) \( f(p) < p \) for each \( p \in Y \setminus \{(0, 0), (2, 0)\} \);
(P4.3) if \( q \in W \) is such that \( f^m(q) \in Y \setminus \{(0, 0), (2, 0)\} \) for some \( m \in \mathbb{N} \), then \( q \notin EP(f) \);
(P4.4) \( Y \setminus \{(0, 0), (2, 0)\} \cap EP(f) = \emptyset \).

Really, (P4.1) and (P4.2) follow from the equality
\[
 Y \setminus \{(0, 0), (2, 0)\} = \bigcup \{c_n c_{n-1} \cup a(n, 0)b(n, 0) \cup b(n, 0)a(n + 1, 0) : n \in \mathbb{N}\}
\]
and parts (8)–(12) of the definition of \( f \). (P4.3) and (P4.4) are consequences of (P4.2).

Property 5. The following three conditions are true:

(P5.1) \( f(b(1, 8)) = a(1, 8) \);
We have to show that (P5.2) \( f(a(n, 8)) = a(n + 1, 8) \) and \( f(b(n + 1, 8)) = b(n, 8) \) for each \( n \in \mathbb{N} \);
(P5.3) \([a(n, 8), b(n, 8)] \subset W \setminus EP(f)\) for each \( n \in \mathbb{N} \).

Indeed, (P5.1) follows from (7); (P5.2) is a consequence of (1) and (5); and the two imply (P5.3).

As a consequence of (4) and Property 4 we get the next one.

Property 6. For each \( n \in \mathbb{N} \) the following two assertions hold:

(P6.1) \( f(a(n, 0)a(n, 4)) \subset Y \setminus \{(0, 0), (2, 0)\})\);
(P6.2) \( a(n, 0)a(n, 4) \cap EP(f) = \emptyset \).

Property 7. \((a(n, 0)a(n, 7) \cup b(n, 0)b(n, 6)) \cap EP(f) = \emptyset \) for each \( n \in \mathbb{N} \).

We prove Property 7 by induction. According to Property 6 it is sufficient to show that \((a(n, 4)a(n, 7) \cup b(n, 0)b(n, 6)) \cap EP(f) = \emptyset \) for each \( n \in \mathbb{N} \).

Since the arcs \( a(1, 0)a(1, 8) \) and \( b(1, 0)b(1, 8) \) have the same length, (7) implies \( b(1, 0)b(1, 8) \sim a(1, 0)a(1, 8) \). Thus \( f(b(1, 0)b(1, 6)) = a(1, 0)a(1, 6) \). Hence, in order to finish the first step of induction, we only need to show that \( a(1, 4)a(1, 7) \cap EP(f) = \emptyset \). By (3), (7) and (4) we see that \( f^3(a(1, 4)a(1, 6)) = a(0, 0)b(1, 0) \subset Y \setminus \{(0, 0), (2, 0)\})\).

Thus (P4.3) implies

(P7.a) \( a(1, 4)a(1, 6) \cap EP(f) = \emptyset \).

Hence we only need to consider the arc \( a(1, 6)a(1, 7) \).

It follows from (2) that
\[
\begin{align*}
f(a(1, 6)a(1, 7)) &= b(1, 4)a(2, 6) \\
&= a(2, 6)a(2, 4) \cup a(2, 4)a(2, 0) \cup a(2, 0)b(1, 0) \cup b(1, 0)b(1, 4) .
\end{align*}
\]

Since \( b(1, 0)b(1, 4) \sim a(1, 0)a(1, 4) \) according to (7), it follows from Properties 4 and 6, applying (4), that it suffices to consider the arc \( a(2, 4)a(2, 6) \).

By (3), (6) and (7) we get \( f^3(a(2, 4)a(2, 6)) = a(1, 0)a(1, 6) \). Recall that Property 6 and assertion (P7.a) above imply \( a(1, 0)a(1, 6) \cap EP(f) = \emptyset \). Thus \( a(2, 4)a(2, 6) \cap EP(f) = \emptyset \). This completes the first step of induction.

For further purposes note that the first step of induction implies \( a(1, 0)a(1, 7) \cap EP(f) = \emptyset \), i.e.,

(P7.b) \( (1, 0)(1, \frac{7}{2}) \cap EP(f) = \emptyset \).

Now take some \( n \in \mathbb{N} \) and assume that Property 7 is true for each positive integer \( m \leq n \). We have to show that

(P7.c) \( (a(n + 1, 0)a(n + 1, 7) \cup b(n + 1, 0)b(n + 1, 6)) \cap EP(f) = \emptyset \).
This will be divided in two steps. First we will show that

(P7.d) \( b(n + 1, 0)b(n + 1, 6) \cap EP(f) = \emptyset \),

and next that

(P7.e) \( a(n + 1, 0)a(n + 1, 7) \cap EP(f) = \emptyset \).

By (5) and (6) we see that \( f|b(n + 1, 0)b(n + 1, 8) : b(n + 1, 0)b(n + 1, 8) \to b(n, 0)b(n, 8) \) is a homeomorphism that sends \( b(n + 1, 0) \) to \( b(n, 0) \) and \( b(n + 1, 8) \) to \( b(n, 8) \). By Property 2 it follows that \( f^n(b(n + 1, 6)) = (1, 1 - \frac{1}{2^{n+2}}) \). Therefore \( f^n(b(n + 1, 0)b(n + 1, 6)) = (1, 0)(1, 1 - \frac{1}{2^{n+2}}) \). Thus, in order to prove (P7.d) we only need to show that

(P7.f) \( (0,1, 1 - \frac{1}{2^{n+2}}) \cap EP(f) = \emptyset \).

To do this, we again apply induction. The equality (P7.b) is the first step in this induction. Now, take \( m \in \{2, \ldots, n\} \) and consider the arc of the form \((1, 1 - \frac{1}{2^{m+2}})(1, 1 - \frac{1}{2^{m+1}})\). By Property 1 we get \((1, 1 - \frac{1}{2^{m+2}})a(1,8) \xrightarrow{f^{m-1}} (2 - \frac{2}{2^{m+1}} \cdot \frac{1}{2^{m+2}} - \frac{1}{2^{m+1}})a(m,8)\). So,

\[
(1, 1 - \frac{1}{2^{m+2}})(1, 1 - \frac{1}{2^{m+1}}) \xrightarrow{f^{m-1}} (2 - \frac{2}{2^{m+1}} \cdot \frac{1}{2^{m+2}} - \frac{1}{2^{m+1}})(2 - \frac{2}{2^{m+1}} \cdot \frac{1}{2^{m+2}} - \frac{1}{2^{m+1}} + (\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}})) \xrightarrow{f} (1, 1 - \frac{1}{2^{m+1}})(1, 1 - \frac{1}{2^{m+2}}) \xrightarrow{f^{m-1}} (2 - \frac{2}{2^{m+1}} \cdot \frac{1}{2^{m+2}} - \frac{1}{2^{m+1}} + (\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}})) \xrightarrow{f} a(m,6)a(m,7) \cap EP(f) = \emptyset \).

Thus, in order to prove (P7.e) we only need to consider the arc \( a(n + 1, 4)a(n + 1, 7) \). According to (3) we have \( f(a(n+1,4)a(n+1,6)) = b(n + 1, 0)b(n + 1, 4) \). Since (P7.d) implies

(P7.g) \( b(n + 1, 0)b(n + 1, 4) \cap EP(f) = \emptyset \),

it follows that \( a(n + 1, 4)a(n + 1, 6) \cap EP(f) = \emptyset \). It remains to consider the arc \( a(n + 1, 6)a(n + 1, 7) \).

By (2) we have

\[
f(a(n + 1,6)a(n + 1,7)) = b(n + 1,4)a(n + 2,6) = a(n + 2,6)a(n + 2,4) \cup a(n + 2,4)a(n + 2,0) \cup a(n + 2,0)b(n + 1,0) \cup b(n + 1,0)b(n + 1,4)
\]

Applying Properties 6 and 4 and (P7.g) to the second, the third and the fourth member of this union we see that it remains to prove that

(P7.h) \( a(n + 2,4)a(n + 2,6) \cap EP(f) = \emptyset \).
By (3) it follows that \( f(a(n + 2, 4)a(n + 2, 6)) = b(n + 2, 0)b(n + 2, 4); \) by (6) we get \( f(b(n + 2, 0)b(n + 2, 4)) = b(n + 1, 0)b(n + 1, 6), \) whence (P7.h) follows by (P7.d). Thus (P7.e) is shown. So, we have finished the induction, and consequently (P7.c) is true.

This finishes the proof of Property 7.

**Property 8.** \( EP(f) = \{(0, 0), (2, 0)\}. \)

By (13) and (14) we get \( \{(0, 0), (2, 0)\} \subset EP(f). \) To show that no other point of \( W \) is in \( EP(f) \) we have to consider the arcs \( a(n, 0)a(n, 8) \) and \( b(n, 0)b(n, 8) \).

By (5), (6) and (7) we get \( f^n(b(n, 0)b(n, 8)) = a(1, 0)a(1, 8). \) Further, by Properties 5 and 7, we only need to prove that \( (a(n, 7)a(n, 8)) \cap EP(f) = \emptyset. \)

Take a point \( p \in a(n, 7)a(n, 8) \). Thus \( f(p) \in a(n + 1, 6)a(n + 1, 8). \) Therefore,

- if \( m = 3, \) then \( f(p) \in a(n + 1, 6)a(n + 1, 7); \)
- if \( m = 4, \) then \( f^2(p) \in a(n + 2, 6)a(n + 2, 7); \)
- if \( m = 5, \) then \( f^3(p) \in a(n + 3, 6)a(n + 3, 7). \)

In all these cases \( p \notin EP(f) \) by Property 7. Proceeding in this way we conclude that \( p \notin EP(f). \) This completes the proof of Property 8.

Properties 3 and 8 imply \( \Omega(f) \not\subset cl_W(EP(f)). \) The proof is finished. \( \square \)

Example 5.7 can be reformulated as follows.

**Proposition 5.8.** The dendrite \( W \) does not have the \( \Omega EP \)-property.

Analogously to Proposition 5.3 we have a corollary, whose proof is exactly the same as the one of Proposition 5.3.

**Corollary 5.9.** If a continuum \( X \) contains a homeomorphic copy of the dendrite \( W, \) then it does not have the \( \Omega EP \)-property.

**Remark 5.10.** Since the Gehman dendrite contains topologically a copy of the dendrite \( W, \) Proposition 5.3 and Corollary 5.4 can be viewed as a consequence of Corollary 5.9.

The opposite implication to that of Corollary 5.9 is true for dendrites. To prove it we need a lemma (Lemma 5.12 below). The lemma is stated (in fact, without proof) in [1, Lemma 1, p. 32]. As the only argument the reader is refereed to the proof of [20, Lemma 2.8, p. 349], whose proof is based on a proof of another result. In these circumstances the authors decide, for the reader convenience, to present a complete argument. It should be underlined, however, that the argument follows along the lines of the proofs of [20, Theorem 2.6 and Lemma 2.8, p. 349].
Recall that for a dendrite $X$ the symbol $B(X)$ means the set of all branch points of $X$. For each arc $J$ in a dendrite $X$, let $r : X \to J$ be the natural retraction, that is, $r(x)$ is the only point in $J$ such that $x r(x) \cap J = \{r(x)\}$. Given an arc $J = ab$ in a dendrite, we put $J_0 = J \setminus \{a, b\}$. Further, for any set $M \subset X$ let $M'$ denote the set of all accumulation points of $M$.

We start with an auxiliary result.

**Lemma 5.11.** Let a mapping $f : X \to X$ of a dendrite $X$ be given, and let an arc $J$ in $X$ ordered by the natural fixed order $<$. If there exists a point $z$ in the arc $J$ such that $x r(x) \cap J = \{r(x)\}$, then there is a point $z$ in the arc $J$ such that $z = r(f(z))$ and $z \neq p$. Since $z \in J_0$ and $J \cap B(X) = \emptyset$, the point $z$ is not the image of any other point under $r$ (except $z$ itself). Thus $z = f(z)$. This contradicts the fact that $J \cap B(f) = \emptyset$, and proves that $q < r(f^n(q))$ for each $q \in J_0$.

**Proof.** We proceed by induction.

**Step 1.** $n = 1$. Notice that the mapping $r \circ f$ sends the arc $J$ to itself. Since $p < f(p) = r(f(p))$, if there exists a point $q \in J_0$ such that $r(f(q)) \leq q$, then there is a point $z$ in the arc $pq \subset J$ such that $z = r(f(z))$ and $z \neq p$. Since $z \in J_0$ and $J \cap B(X) = \emptyset$, the point $z$ is not the image of any other point under $r$ (except $z$ itself). Thus $z = f(z)$. This contradicts the fact that $J \cap B(f) = \emptyset$, and proves that $q < r(f^n(q))$ for each $q \in J_0$.

**Step 2.** Assume $q < r(f^n(q))$ for some $n \in \mathbb{N}$ and each $q \in J_0$. In particular, $p < f(p) < r(f^{n+1}(p))$. Now, we can repeat the argument in Step 1 using the point $p$ and the mapping $f^{n+1}$. In this way we conclude that $q < r(f^{n+1}(q))$ for each $q \in J_0$. Thus the induction is complete and so the lemma is proved. □

**Lemma 5.12.** Let a mapping $f : X \to X$ of a dendrite $X$ be given. If an arc $J$ in $X$ satisfies the equalities (5.11.1), then $J_0 \cap f^n(J_0 \cap \Omega(f)) = \emptyset$ for each $n \in \mathbb{N}$.

**Proof.** Suppose on the contrary that there exist a point $x \in J_0 \cap \Omega(f)$ and a number $n \in \mathbb{N}$ such that $f^n(x) \in J_0$. Fix an order $<$ in the arc $J$. We may assume that $x < f^n(x)$ and that $n$ is the minimal number with the property that $f^n(x) \in J_0$.

We prove that

$$f^n(x) = f(x), f^2(x), \ldots, f^n(x) \text{ are pairwise different.}$$

Indeed, suppose on the contrary that there exist nonnegative integers $i, j$ such that $0 \leq i < j \leq n$ and $f^i(x) = f^j(x)$. Since $x$ is not a periodic point of $f$, we have $1 < i$. Further, $j < n$ by the minimality of $n$. Then the complete orbit of $x$ under $f$ is the set $F = \{f(x), f^2(x), \ldots, f^n(x)\}$. Thus $f^n(x)$ belongs to this orbit, but $f^n(x) \in J_0$ and, again by the minimality of $n$, we see that $f^n(x)$ is not in $F$. This contradiction proves (5.12.1).

Therefore, by (5.12.1), we can choose disjoint open connected subsets $V_0, V_1, \ldots, V_n$ of $X$ such that $f^i(x) \in V_i$ for each $i \in \{0, 1, \ldots, n\}$. We may also assume that $V_0$ and $V_n$ are subintervals of $J_0$. Thus $p < q$ for each $p \in V_0$ and each $q \in V_n$. 
Put $V = V_0 \cap f^{-1}(V_1) \cap \cdots \cap f^{-n}(V_n)$. Then $V$ is open in $X$ and $x \in V$. Since $x \in \Omega(f)$, there exist a point $y \in V$ and a number $m \in \mathbb{N}$ such that $f^m(y) \in V$. Since $f^i(y) \in V_i$ for each $i \in \{1, \ldots, n\}$ and $V_i \cap V = \emptyset$, it follows that $m > n$. Notice that $f^n(y) \in V_n$. Let $w = f^n(y)$. Then $w \in V_n$ and $f^{m-n}(w) = f^m(y) \in V \subset V_0$. So, $f^{m-n}(w) < w$.

Let $r : X \to J$ be the natural retraction. By Lemma 5.11 applied to the mapping $f^{m-n}$ we have $r((f^{m-n})^k(q)) < q$ for each $k \in \mathbb{N}$ and each $q \in J_0$. But Lemma 5.11 applied to the mapping $f^n$ gives $r((f^n)^s(q)) > q$ for each $s \in \mathbb{N}$ and each $q \in J_0$ (recall that $x < f^n(x)$). Thus, if we put $k = n$ and $s = m - n$, we obtain a contradiction. The proof is complete.

Note that Lemma 5.12 can be reformulated as follows.

**Lemma 5.12a.** Let a mapping $f : X \to X$ of a dendrite $X$ and an arc $J \subset X$ be given such that $J \cap P(f) = \emptyset$. If there are a point $c \in J_0 \cap \Omega(f)$ and a positive integer $n$ such that $f^n(c) \in J_0$, then $J \cap B(X) \neq \emptyset$.

**Theorem 5.13.** If a dendrite $X$ does not contain any topological copy of the dendrite $W$, then $X$ has the $\Omega EP$-property.

**Proof.** Since $X$ does not contain any copy of $W$, it follows that

(5.13.1) for each arc $J \subset X$ the set $J \cap B(X)$ is finite.

In order to prove the theorem, suppose on the contrary that there are a mapping $f : X \to X$ and a point $p \in X$ such that $p \in \Omega(f) \setminus \text{cl}(\text{EP}(f))$. Let $V$ be the component of $X \setminus \text{cl}(\text{EP}(f))$ containing $p$. Then $V$ is an open and connected subset of $X$.

Since $p \in \Omega(f)$, there exist $q \in V$ and $m \in \mathbb{N}$ such that $f^m(q) \in V$. Define $g = f^m$. Then $q \in V \cap g^{-1}(V)$, whence $V \cap g(V) \neq \emptyset$. This implies that $g(V) \cap g^2(V) \neq \emptyset$, $g^2(V) \cap g^3(V) \neq \emptyset$, and so on. Therefore the set

$$T = \bigcup \{g^n(V) : n \in \{0\} \cup \mathbb{N}\}$$

is connected. Since $V \cap \text{EP}(f) = \emptyset$, we get $g^n(V) \cap \text{EP}(f) = \emptyset$ for each $n \in \mathbb{N}$. Thus

(5.13.2) $T \cap \text{EP}(f) = \emptyset$.

Let $M = \{p, g(p), g^2(p), g^3(p), \ldots\}$. Since $p \notin \text{EP}(f)$, the elements of $M$ are mutually distinct, and thus

(5.13.3) $M' = \{x \in X : \text{there exists a sequence } n_1 < n_2 < \cdots \text{ in } \mathbb{N} \text{ such that } \lim_k g^{n_k}(p) = x\}$.

The remaining part of the proof is divided in ten claims.

**Claim 1.** For each arc $J \subset X$ the set $J \cap M$ is finite.
Suppose on the contrary that there is an arc $J$ in $X$ such that $J \cap M$ is infinite. Since each sequence in $[0, 1]$ contains either an increasing or a decreasing subsequence, we may assume that there exist $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $g^{n_1}(p) < g^{n_2}(p) < g^{n_3}(p) < \cdots$, where $< \in$ is a natural order for the arc $J$. We may also assume that $g^{n_1}(p)$ is not an end point of $J$. Since $p \in \Omega(f)$, it follows that $g^{n_k}(p) \in \Omega(f)$ for each $k \in \mathbb{N}$. Choose points $a, b \in J$ so that $g^{n_1}(p) < a < g^{n_2}(p) < g^{n_3}(p) < b < g^{n_4}(p)$. Since $T$ is a connected subset of the dendrite $X$, it is arcwise connected (see [16, Proposition 10.9, p. 169]). Thus the arc $g^{n_1}(p)g^{n_4}(p)$ is contained in $T$ for each $k \in \mathbb{N}$. Since $ab \subset g^{n_1}(p)g^{n_4}(p) \subset T$, and since $P(f) \subset EP(f)$ according to (2.2), it follows from (5.13.2) that $ab \cap P(f) = \emptyset$. Hence the arc $ab$ satisfies all the assumptions of Lemma 5.12(a), and thereby it contains a branch point of $X$. Thus there exists $r_1 \in J \cap B(X)$ such that $r_1 < g^{n_5}(p)$. Repeating the argument, we can find a point $r_2 \in J \cap B(X)$ such that $r_1 < r_2 < g^{n_6}(p)$. Following this procedure, we can find infinitely many elements in $J \cap B(X)$. This contradicts (5.13.1) and completes the proof of Claim 1.

Given $x \in M'$, let a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ be such that $\lim_k g^{n_k}(p) = x$.

**Claim 2.** For each arc $J \subset X$ with $x \in J$ the set $\{k \in \mathbb{N} : xg^{n_k}(p) \cap J \neq \{x\}\}$ is finite.

Suppose, contrary to Claim 2, that the above mentioned set is infinite. Thus we may assume that for each $k \in \mathbb{N}$ we have $xg^{n_k}(p) \cap J \neq \{x\}$ and $g^{n_k}(p) \notin J$ (by Claim 1). For each $k \in \mathbb{N}$ let $y_k \in J$ be such that $g^{n_k}(p)y_k \cap J = \{y_k\}$. Since $\lim_k g^{n_k}(p) = x$, we get $\lim_k y_k = x$. Thus we may assume that $y_k$ is not an end point of $J$ for each $k \in \mathbb{N}$. So, each $y_k$ is a branch point of $X$, and since $y_k \neq x$, the arc $J$ contains infinitely many branch points of $X$. This contradicts (5.13.1) and ends the proof of Claim 2.

**Claim 3.** $M' \subset B(X) \setminus EP(f)$.

Take a point $x \in M'$. Fix a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\lim_k g^{n_k}(p) = x$. Since the points $p, g(p), g^2(p), \ldots$ are pairwise different, we may assume that $x \neq g^{n_k}(p)$ for each $k \in \mathbb{N}$. Applying Claim 2 to the arc $xg^{n_1}(p)$ we may assume that $xg^{n_k}(p) \cap \chi g^{n_k}(p) = \{x\}$ for each $k \geq 2$. Thus $x$ belongs to the arc $g^{n_1}(p)g^{n_2}(p)$ and $x \notin \{g^{n_1}(p), g^{n_2}(p)\}$. Since $g^{n_1}(p), g^{n_2}(p) \in T$ and $T$ is arcwise connected, it follows that $x \in T$. Thus $x \notin EP(f)$ by (5.13.2). Now applying Claim 2 to the arc $g^{n_1}(p)g^{n_2}(p)$ we infer that there is $k \in \mathbb{N}$ such that $xg^{n_k}(p) \cap \chi g^{n_k}(p) = \{x\}$. Since $g^{n_k}(p) \neq x$, it follows that $x \in B(X)$, as required.

**Claim 4.** The set $M'$ is at most countable.

Indeed, by Claim 3 we have $M' \subset B(X)$. Since $B(X)$ is countable (see, e.g., [16, Theorem 10.23, p. 174]), the conclusion follows.

**Claim 5.** $g(M') \subset M'$.

Take $x \in M'$. By (5.13.3) there is a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ be such that $\lim_k g^{n_k}(p) = x$. Thus $g(x) = \lim_k g^{n_k+1}(p) \in M'$ again by (5.13.3). Therefore the required inclusion follows.

Now we construct, by transfinite induction, a family $\{C_\alpha : \alpha < \omega_1\}$ of closed subsets $C_\alpha$ of $X$. 

Let $C_0 = M'$. To define $C_{α+1}$ consider two cases. If $C_α = \emptyset$, put $C_{α+1} = \emptyset$. If $C_α \neq \emptyset$, fix an element $x_α \in C_α$ and define $C_{α+1} = \{(x_α, g(x_α), g^2(x_α), \ldots)\}$. Finally, if $γ < ω_1$ is a limit ordinal, define $C_γ = \bigcap\{C_β: β < γ\}$. It is easy to verify that $C_α$ is closed for each $α$.

**Claim 6.** $C_α \subset M' \subset B(X) \setminus EP(f)$ for each ordinal $α < ω_1$.

The latter inclusion is shown in Claim 3. We prove the former by transfinite induction. For $α = 0$ it is nothing to prove. Take an ordinal $β > 0$ and assume that the inclusion holds for each $α < β$. Consider first the case when $β = α + 1$ for some ordinal $α$. If $C_α = \emptyset$, then $C_{α+1} = \emptyset \subset M'$. If $C_α \neq \emptyset$, then $C_{α+1} = \{(x_α, g(x_α), g^2(x_α), \ldots)\}$, where $x_α \in C_α \subset M'$. By Claim 5 we see that $g^n(x_α) \in M'$ for each $n \in \mathbb{N}$, whence $C_{α+1} \subset M'$. Finally, let $β$ be a limit ordinal. Then $C_β \subset C_0 = M'$ by the definition. So, Claim 6 is shown.

**Claim 7.** $g(C_α) \subset C_α$ for each $α < ω_1$.

We prove Claim 7 again by transfinite induction. For $α = 0$ the inclusion is proved in Claim 5.

Take $β > 0$ and assume that the inclusion holds for each $α < β$. If $β$ is not a limit ordinal, let $β = α + 1$ for some $α$. If $C_α = \emptyset$, then $g(C_β) = \emptyset = C_β$ by the definition. If $C_α \neq \emptyset$, then $C_β = \{(x_α, g(x_α), g^2(x_α), \ldots)\}$ for some $x_α \in C_α$. Given $x \in C_β$, there exists a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $x = \lim_k g^n(x_α)$. Thus $g(x) = \lim_k g^{n+1}(p)$. By Claim 6 it follows that $x_α \notin EP(f)$. Thus the elements $x_α, g(x_α), g^2(x_α), \ldots$ are pairwise different. Hence $g(x) \in C_β$. Therefore $g(C_β) \subset C_β$.

Let $β$ be a limit ordinal. Then $C_β = \bigcap\{C_α: α < β\}$. Thus $g(C_β) = g(\bigcap\{C_α: α < β\}) \subset \bigcap\{g(C_α): α < β\} \subset \bigcap\{C_α: α < β\} = C_β$. This completes the induction argument. So Claim 7 is shown.

**Claim 8.** $C_{α+1} \subset (C_α)'$ for each ordinal $α < ω_1$.

Really, if $C_α = \emptyset$, then $C_{α+1} = \emptyset$ by the definition, and the inclusion obviously holds. If $C_α \neq \emptyset$, then $C_{α+1} = \{(x_α, g(x_α), g^2(x_α), \ldots)\}$, where $x_α \in C_α$. Since $\{x_α, g(x_α), g^2(x_α), \ldots\} \subset C_α$ by Claim 7, the inclusion follows.

**Claim 9.** If $α < β$, then $C_β \subset C_α$.

To show the implication we apply transfinite induction with respect to an ordinal number $γ$ such that $α < β ≤ γ$. If $γ = 0$, then the implication is true in an empty way. Let $γ > 0$ and assume that the implication is true for each $λ < γ$. Take $α$ and $β$ such that $α < β ≤ γ$. If $β < γ$, then $C_β \subset C_α$ by the inductive hypothesis. So, let $β = γ$. If $γ = η + 1$ for some $η$, we only need to show that $C_{η+1} \subset C_γ$. If $C_η = \emptyset$, then $C_{η+1} = \emptyset$ by definition, so the inclusion holds. If $C_η \neq \emptyset$, then $C_{η+1} \subset (C_η)'$ by Claim 8, and $(C_η)' \subset C_γ$ since $C_η$ is closed. Thus the needed inclusion follows. Finally, if $γ$ is a limit ordinal, then $C_γ = C_β \subset C_α$ just by the definition. This completes the proof of Claim 9.

Claim 9 shows that the transfinite sequence $\{C_α: α < ω_1\}$ is decreasing. Since all its members are compact, there exists a countable ordinal $α_0$ such that $C_{α_0} = C_{α_0+1}$ (see
We may assume that $\alpha_0$ is the first ordinal having this property.

Claim 10. $C_{\alpha_0} = \emptyset$, and $\alpha_0$ is not a limit ordinal.

Suppose on the contrary that $C_{\alpha_0} \neq \emptyset$. We show that $C_{\alpha_0}$ is a perfect set, i.e., that $C_{\alpha_0} = (C_{\alpha_0})'$ by Claim 8, and $(C_{\alpha_0})' \subset C_{\alpha_0}$ since $C_{\alpha_0}$ is closed.

Further, $C_{\alpha_0} \subset C_0 = M'$ by Claim 9, whence it follows that $C_{\alpha_0}$ is at most countable by Claim 4, and thus it is totally disconnected. Consequently, $C_{\alpha_0}$ is homeomorphic to the Cantor set (see, e.g., [19, Corollary 30.4, p. 217]; compare also [8, Exercise 6.2.A(c), p. 370]). But this contradicts the fact that $C_{\alpha_0}$ is at most countable. Therefore $\alpha_0$ is not a limit ordinal. The proof of Claim 10 is complete.

Now we are ready to obtain the final contradiction. By Claim 10 we have $\alpha_0 = \beta_0 + 1$ for some ordinal $\beta_0$. Minimality of $C_{\alpha_0}$ implies that $C_{\beta_0} \neq \emptyset$, whence $C_{\alpha_0} = \{x_{\beta_0}, g(x_{\beta_0}), g^2(x_{\beta_0}), \ldots\}$ for some $x_{\beta_0} \in C_{\beta_0}$. Since $x_{\beta_0} \notin \text{EP}(f)$ by Claim 6, the set $\{x_{\beta_0}, g(x_{\beta_0}), g^2(x_{\beta_0}), \ldots\}$ is infinite, and therefore $C_{\alpha_0} \neq \emptyset$. This contradicts Claim 10 and completes the proof of the theorem.

Corollary 5.9 and Theorem 5.13 imply the following characterization of the $\Omega\text{-EP}$-property for dendrites.

Corollary 5.14. A dendrite has the $\Omega\text{-EP}$-property if and only if it does not contain any topological copy of the dendrite $W$.

A continuum which is arcwise connected and hereditarily unicoherent is called a dendroid. It is well known that each dendroid is hereditarily decomposable, thus one-dimensional, and that each locally connected dendroid is a dendrite (compare, for example, [16, Exercises 10.58 and 11.54, p. 192 and 226, respectively]). Therefore dendroids form the nearest (in a sense) class of curves containing the class of dendrites. An important example of a dendroid which is not a dendrite is the Cantor fan, i.e., the cone over the Cantor set. Let $C \subset [0, 1]$ be the standard Cantor ternary set, and let $p = (1/2, 1) \in \mathbb{R}^2$. For each $c \in C$ let $L_c$ denote the straight line segment joining $p$ with $(c, 0)$. Then the Cantor fan $F_C$ is defined as the union

$$F_C = \bigcup \{L_c : c \in C\}.$$
as it is constructed in [10, proof of Example 1.6, p. 45], we extend it to the needed mapping $f : FC \to FC$ defined so that $f(p) = p$, and if $q \in LC \setminus \{p\}$ for some $c \in C$ (note that such a $c$ is uniquely determined), then $f(q) \in L_d$, where $(d, 0) = h((c, 0))$ and $\pi_y(f(q)) = \pi_y(q)$ (here $\pi_y$ denotes the projection of a point in the plane to its second coordinate). It can be observed that then we have $\Omega(f) = FC$ and $EP(f) = \{p\}$, whence the conclusion follows.

Theorem 5.15 and Proposition 5.1 imply a corollary.

**Corollary 5.16.** If a continuum $X$ contains the Cantor fan $FC$ as its retract, then $X$ does not have the $\Omega$EP-property.

**Remark 5.17.** Observe that the Cantor fan does not contain the dendrite $W$, thus the assumption in Theorem 5.13 that the continuum $X$ is a dendrite is indispensable, and it cannot be weakened to being a dendroid.

**References**