Accessibility and Mappings of Dendroids

by

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Summary. It is proved that every non-degenerate layer of an irreducible continuum $N$ of type $\lambda$ contained in a continuous image $f(X)$ of a dendroid $X$ contains the limit point of a sequence of points that are accessible from the complement $f(X) \setminus N$.

Professor Knaster encouraged me twice to write this paper: first when he was informed of my answering his question posed in a conversation as early as in 1960: why the "Warsaw circle" is not a continuous image of any dendroid, i.e. an arcwise connected and hereditarily unicoherent continuum? Second when Kuperberg published [1] a very interesting (but quite different and --- from a certain point of view --- more general than mine) argument for this problem of Knaster. Owing to Professor Knaster's encouragement I have decided to publish my answer to the problem, even though Kuperberg's answer had already been published, because the methods as well as arguments used in both papers are completely different: those used in [1] belong rather to the algebraic topology, while the ones used in this paper are derived from the set-theoretical topology.

A point $p$ of a compact set $C$ contained in a (metric) continuum $X$ is said to be accessible from $X \setminus C$ if there is an arc $xp$ such that

$$xp \setminus \{p\} \subset X \setminus C.$$

The following lemma is well known. Its proof is given here only for completeness.

**Lemma.** Let $X$ be a continuum and let $C$ be a closed set in $f(X)$, a continuous image of $X$. If a point $p \in f^{-1}(C)$ is accessible from $X \setminus f^{-1}(C)$, then its image $f(p)$ is accessible from $f(X) \setminus C$.

**Proof.** Since $p$ is accessible, there is an arc $xp$ such that $xp \setminus \{p\} \subset X \setminus f^{-1}(C)$. Thus $f(xp \setminus \{p\}) \subset f(X \setminus f^{-1}(C)) = f(X) \setminus C$. Further, $f(xp) = f(xp \setminus \{p\}) \cup \{f(p)\}$, whence $C \cap f(xp) = C \cap f(xp \setminus \{p\}) \cup C \cap \{f(p)\} = \{f(p)\}$. Thus we see that the continuum $f(xp)$ is non-degenerate (since $f(x) \in f(X) \setminus C$ and $f(p) \in C$) and arcwise connected, and that it has only one point; namely $f(p)$, in common with $C$. There exists in $f(xp)$ an arc $f(x)f(p)$ such that $f(x)f(p) \setminus \{f(p)\} \subset f(X) \setminus C$ and this means that $f(p)$ is accessible from $f(X) \setminus C$. 

[239]
Let $X$ denote a dendroid and let a mapping $f: X \to Y$ be continuous and onto. Consider an irreducible continuum $N$ contained in $Y$.

Suppose that $N$ is not an arc. It follows $X \setminus f^{-1}(N) \neq \emptyset$, because otherwise we would have $f(X) = N$, thereby $N$ would be arcwise connected by the arcwise connectedness of $X$, thus it would be an arc by the irreducibility of $N$. Hence there exists a point $c \in X \setminus f^{-1}(N)$.

Put $f^{-1}(N) = \bigcup \{D_i : i \in I\}$, where $D_i$ are components of $f^{-1}(N)$ and $I$ is an arbitrary set of indices. Observe that

1. The set $I$ is infinite.

Indeed, if not, then the irreducible continuum $N$ would be the finite union of arcwise connected continua $f(D_i)$, thus $N$ would be arcwise connected, whence it would be an arc by its irreducibility.

It follows from the hereditary unicoherence of $X$ that for any fixed component $D_i$ of $f^{-1}(N)$ there is only one arc which joins the point $c$ with $D_i$. Thus there exists, for each component $D_i$ of $f^{-1}(N)$, a unique point $p_i \in D_i$ with the property that $cp_i \cap D_i = \{p_i\}$. Call this point $p_i$ the c-point of $D_i$.

2. For each c-point $p_{i_0}$ of $D_{i_0}$ which is not accessible from $X \setminus f^{-1}(N)$ there exists a convergent sequence of c-points $p_{i_n}$ of $D_{i_n}$ such that

(a) $\lim_{n \to \infty} p_{i_n} = p_{i_0}$,

(b) $p_{i_n} \in cp_{i_0}$ for each $n = 1, 2, 3, ...$,

(c) $p_{i_n}$ are accessible from $X \setminus f^{-1}(N)$ for each $n = 1, 2, 3, ...$.

In fact, if $p_{i_0}$ is not accessible from $X \setminus f^{-1}(N)$, then for each point $d \in cp_{i_0} \setminus \{p_{i_0}\}$ the arc $dp_{i_0}$ cannot be contained, except for $p_{i_0}$, in $X \setminus f^{-1}(N)$. This means that the arc $dp_{i_0}$ must intersect a component $D_j$ of $f^{-1}(N)$ which is different from $D_{i_0}$. Further, it follows that there is a point $x \in dp_{i_0}$ which lies between $D_j$ and $p_{i_0}$ and is not in $f^{-1}(N)$, i.e., $x \in p_j p_{i_0} \setminus f^{-1}(N)$. Since $f^{-1}(N) \cap p_j p_{i_0}$ is a closed set, hence there is a maximal arc $ab$ contained in $p_j p_{i_0} \setminus f^{-1}(N)$ which has only its end points in $f^{-1}(N)$. Thus the point $b$ is — by definition — one of the c-points $p_i$ defined above, and, moreover, it is accessible from $X \setminus f^{-1}(N)$, the arc $xp$ being contained, except $b$, out of $f^{-1}(N)$. Now let $d_1, d_2, d_3, ...$ be a sequence of points in $cp_{i_0}$ having the point $p_{i_0}$ as its limit. Taking $d_n$ for $d$ we define $p_{i_n}$ as $b$ and see that conditions (a) — (c) are fulfilled. Thus (2) is established.

It follows from (1) and (2) that

3. The set of c-points $p_i$ which are accessible from $X \setminus f^{-1}(N)$ is infinite.

In fact, there are infinitely many c-points $p_i$ by (1). If all of them are accessible from $X \setminus f^{-1}(N)$, then (3) holds. If there exists a c-point $p_{i_0}$ which is not accessible from $X \setminus f^{-1}(N)$, then (3) holds by (2).

Now we shall prove that

4. There are infinitely many points $f(p_i)$ for which the c-point $p_i$ is accessible from $X \setminus f^{-1}(N)$.
Indeed, suppose, on the contrary, that there is only a finite number of different points \( f(p_i) \) with an accessible \( c \)-point \( p_i \). This means that all accessible points \( p_i \) but a finite number of them have a common point \( q \) as their image. Consider an arbitrary component \( D_{i_0} \) of \( f^{-1}(N) \) and the \( c \)-point \( p_{i_0} \) of \( D_{i_0} \) such that \( p_{i_0} \) is not accessible from \( X \setminus f^{-1}(N) \). It was shown in (2) that \( p_{i_0} \) is the limit point of some sequence of points \( p_{i_n} \) which are accessible from \( X \setminus f^{-1}(N) \). It follows from (2) (a) that \( f(p_{i_n}) = \lim_{n \to \infty} f(p_{i_n}) \) and since \( f(p_{i_n}) = q \) for almost all \( n \), we have \( f(p_{i_n}) = q \) for every \( c \)-point \( p_{i_n} \) which is not accessible from \( X \setminus f^{-1}(N) \). Now let \( U \) denote the union of all components \( D_i \) of \( f^{-1}(N) \) for which the \( c \)-point \( p_i \) either \( 1^o \) is not accessible from \( X \setminus f^{-1}(N) \), or \( 2^o \) is accessible from \( X \setminus f^{-1}(N) \) and \( f(p_i) = q \). We shall prove that \( f(U) \) is arcwise connected. Since each component of \( U \) is arcwise connected as a subcontinuum of the dendroid \( X \), it is sufficient to show that for each such component its image under \( f \) contains the point \( q \). In fact, for each component \( D_i \) of \( f^{-1}(N) \) contained in \( U \) we have \( q = f(p_i) \in f(D_i) \) by the definition of \( U \). Consider now another component \( D \) of \( U \). Since \( D \subset \bigcup_{n \to \infty} D_{i_n} \), the \( c \)-points \( p_{i_n} \) of which satisfy \( 1^o \) or \( 2^o \). Choosing a convergent subsequence of points \( p_{i_n} \) and taking its limit point \( p \) we see \( p \in D \) and \( f(p) = q \), whence \( q \in f(D) \). This \( f(U) \) is arcwise connected. Now let us note that the number of components \( D_i \) of \( f^{-1}(N) \) which are not contained in \( U \) is finite by the definition of \( U \) and by the supposition made at the beginning of the proof of (4). Thus the continuum \( N \), being the image of the union of all \( D_i \)'s, is the union of a finite number of arcwise connected continua, namely \( f(U) \) and the finite union of \( f(D_i) \) with \( D_i \) not in \( U \). Therefore \( N \) is arcwise connected, and — being irreducible — it is an arc, contrary to the hypothesis. Thus the proof of (4) is finished.

**Theorem 1.** Let a continuum \( Y \) be a continuous image of a dendroid and let an irreducible continuum \( N \) contained in \( Y \) not be an arc. Then the set of all points of \( N \) which are accessible from \( Y \setminus N \) is infinite.

**Proof.** Let a continuous mapping \( f \) map a dendroid \( X \) onto \( Y \). Since there exist infinitely many points of \( f^{-1}(N) \) which are accessible from \( X \setminus f^{-1}(N) \) by (3) and since their accessibility implies the accessibility of their images under \( f \) from \( Y \setminus N \) by the lemma, the conclusion of the theorem follows from (4).

The further part of the paper deals with a particular case when the irreducible continuum \( N \) is of type \( \lambda \) (see [2], §48, III, p. 197, footnote). It is known (see e.g. [2], §48, IV, p. 199–204) that if the irreducible continuum \( N \) is of type \( \lambda \), then it can be upper semi-continuously decomposed into nowhere dense continua (which are maximal with respect to this property) such that the decomposition space is an arc. We call this decomposition of \( N \) Kuratowski’s decomposition. Elements of this decomposition are called layers of \( N \). In particular, if \( N \) is irreducible between two points, then the layers containing these points are called the end layers of \( N \).

We shall prove that
(5) If the irreducible continuum $N$ is of type $A$ and is not an arc, then the set of all points of $N$ which are accessible from $Y \setminus N$ is not contained in the union of the two end layers of $N$.

Indeed, suppose — on the contrary — that the set of all points of $N$ which are accessible from $Y \setminus N$ is contained in the union of the two end layers of $N$. Consider an arbitrary component $D_{i_0}$ of $f^{-1}(N)$. If the c-point $p_{i_0}$ of $D_{i_0}$ is accessible from $X \setminus f^{-1}(N)$, then it follows from the lemma that $f(p_{i_0})$ is accessible from $Y \setminus N$, thus $f(p_{i_0})$ is in the union of the two end layers $T_0$ and $T_1$ of $N$, whence

\[ f(D_{i_0}) \cap (T_0 \cup T_1) \neq \emptyset. \]

If the c-point $p_{i_0}$ of $D_{i_0}$ is not accessible from $X \setminus f^{-1}(N)$, then it follows from (2) that there exists a sequence of c-points $p_{i_n}$ satisfying (a)—(c). In particular, it follows from (c) and the lemma that the points $f(p_{i_n})$ are accessible from $Y \setminus N$, whence $f(p_{i_n}) \in T_0 \cup T_1$. The union $T_0 \cup T_1$ being a closed set, it follows from (a) by the continuity of $f$ that $f(p_{i_n}) \in T_0 \cup T_1$; thus (*) is proved for every index $i_0 \in I$. In other words, we see that for every component $D_i$ of $f^{-1}(N)$ its image $f(D_i)$ intersects either $T_0$ or $T_1$. This implies that $N$ is an arc as an irreducible continuum which is the union of two arcwise connected continua $f(D_{i_0})$ and $f(D_{i_1})$ with $A \in f$, $I(D_{ik}) \cap T_k : f : 0, k = 0$ or $1$. Thus (5) is established.

**Theorem 2.** Let a continuum $Y$ be a continuous image of a dendroid and let an irreducible continuum $N \subset Y$ be of type $A$. Then for every non-degenerate layer $T$ of $N$ there exists a convergent sequence of points $y_n$ in $N$ which are accessible from $Y \setminus N$, belong to different layers of $N$ and whose limit point is in $T$.

**Proof.** Let $N = \bigcup \{T_t : 0 \leq t \leq 1\}$ be Kuratowski's decomposition of $N$ into layers. Consider first a particular case $T = T_0$, i.e. assume that the end layer $T_0$ is non-degenerate. It is known that for every natural $n$ the continuum $N_n = \bigcup \{T_t : 0 \leq t < 1/n\}$ is irreducible, thus it is not an arc, $T_0$ being non-degenerate. Therefore by Theorem 1 it has a non-empty set of points which are accessible from $Y \setminus N_n$. This set cannot be contained in $T_0$ according to (5). Thus, for every $n$, there is a point $y_n$ of $N_n \subset N$ which is accessible from $Y \setminus N_n$, i.e., there is an arc $q_n y_n$ such that $q_n y_n \subset Y \setminus N_n$. The arc $q_n y_n$ cannot be contained in $N$ for every natural $n$, because if so, then the point $y_n$ is in the end layer of $N_n$ that is different from $T_0$. Thus, taking a suitable convergent subsequence if necessary, we get a convergent sequence of points $y_n$ accessible from $Y \setminus N$, belonging to different layers of $N$ and with the limit in $T_0$.

Consider next the general case of an arbitrary non-degenerate layer $T = T_{t_0}$, where $0 \leq t_0 \leq 1$. It is known that the continua

\[ N_L = \bigcup \{T_t : 0 \leq t < t_0\} \quad \text{and} \quad N_R = \bigcup \{T_t : t_0 < t \leq 1\} \]

(the subscripts $L$ and $R$ stand for left and right, respectively) are irreducible and at least one of them is not an arc and has a non-degenerate end layer. Thus applying the particular case to $N_L$ or $N_R$ for $N$ we get the conclusion of the Theorem.
Remark that the point $\lim_{n \to \infty} y_n$ need not be accessible from $Y \setminus N$, as it can be seen from the following example. Put in Cartesian coordinates in the plane $X = \{(x, 0) : 0 \leq x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\} \cup \bigcup_{n=1}^{\infty} \{(1/n, y) : -1 \leq y \leq 1\}$ and let $f$ identify points $(1/n, 1)$ with $(1/(n+1), 1)$ for $n = 1, 3, 5, ...$ and $(1/n, -1)$ with $(1/(n+1), -1)$ for $n = 2, 4, 6, ...$. Thus $X$ is a dendroid, $f$ is continuous and $Y = f(X)$ is homeomorphic to the union $J \cup N$ of the unit segment $J = \{(x, 0) : 0 \leq x \leq 1\}$ with the closure $N$ of the graph of sin $(1/x)$-curve for $x > 0$. The points $y_n$ (which are elements of $J \cap N$ except for the origin) are accessible from $Y \setminus N$, but their limit — the origin — is not.

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REFERENCES

Я. Е. Харатоник, Достигимость и отображения дендроидов

Содержание. Доказывается, что каждый не вырожденный слой неприводимого континуума $N$ типа $\lambda$ в непрерывном образе $f(X)$ дендроида $X$ содержит предельную точку сходящейся последовательности точек достижимых из дополнения $f(X) \setminus N$. 