GENERALIZED $\varepsilon$-PUSH PROPERTY FOR CERTAIN ATRIODIC CONTINUA

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Abstract. We show that an absolute retract for hereditarily unicoherent continua that contains no simple triod must be an arc-like continuum. More general results are proved for a class of continua having only arcs as their proper subcontinua.

The research presented in this paper is motivated by questions concerning atriodic members of the class $\mathcal{AR}(\mathcal{HU})$ of absolute retracts for hereditarily unicoherent continua. T. Maćkowiak has proved [8] that the simplest indecomposable continuum, called the bucket handle, belongs to the class $\mathcal{AR}(\mathcal{HU})$. More recently the authors together with W. J. Charatonik have shown [3] that any inverse limit of trees with confluent bonding mappings is such a retract. In particular it follows that all continua that can be represented as the inverse limits of arcs with confluent (equivalently: open) bonding mappings, called Knaster type continua, are in the class $\mathcal{AR}(\mathcal{HU})$, and these are all known atriodic members of this class. Atriodic continua in the class $\mathcal{AR}(\mathcal{HU})$ were extensively studied in the previous paper [4]. In particular, it was shown there that if $X$ is an atriodic member of $\mathcal{AR}(\mathcal{HU})$ that additionally is either tree-like or circle-like, then it must be arc-like. It was an open question whether all atriodic members of $\mathcal{AR}(\mathcal{HU})$ must be arc-like (Question 5.5 of [4]). The main result of this paper is to answer this question in the affirmative. To prove this result we essentially use the results of [4] and [11]. The question whether any atriodic member of $\mathcal{AR}(\mathcal{HU})$ is a Knaster type continuum (see Question 4.3) seems to be particularly interesting in view of the results of this paper.

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In the previous papers it was proved that every member of \( \mathbb{A}R(\mathcal{HU}) \) that contains no simple triod has only arcs for its proper subcontinua (see [4]), and that it has the generalized \( \varepsilon \)-push property, Definition 1.1 (see [2]). Though studying of atriodic members of \( \mathbb{A}R(\mathcal{HU}) \) is the main motivation of this paper, our essential research is performed rather for the larger class of continua having the two above mentioned properties.

1. Preliminaries and auxiliary results.

All spaces under consideration are assumed to be metric and all mappings are continuous. The term \textit{compactum} stands for a compact (metric) space, and a \textit{continuum} means a connected compactum. The symbols \( \mathbb{Z} \) and \( \mathbb{N} \) stand for the sets of all integers and of all positive integers, respectively, and \( \mathbb{R} \) denotes the space of real numbers.

An \textit{arc} means a space homeomorphic with the closed unit interval \([0, 1]\), and an \textit{open arc} means an arc without its end points. A point \( p \) of a continuum \( X \) is called an \textit{end point} of \( X \) if for every two subcontinua \( K \) and \( L \) both containing \( p \) either \( K \subset L \) or \( L \subset K \).

Let \( X \) be a metric space with a metric \( d \). We denote by \( B(p, \varepsilon) \) the (open) ball in \( X \) centered at a point \( p \in X \) and having the radius \( \varepsilon \).

For a subset \( A \) of a space \( X \) we denote by \( id | A : A \to A \) the identity mapping defined on \( A \).

Let \( X \) and \( Y \) be spaces, and let \( d \) be a metric in \( X \). For any two mappings \( f, g : Y \to X \) we define
\[
\tilde{d}(f, g) = \sup \{ d(f(y), g(y)) : y \in Y \}.
\]
In particular, for a mapping \( f : A \to B \), where \( A \) and \( B \) are subspaces of \( X \), we define
\[
\tilde{d}(f) = \tilde{d}(f, id | A) = \sup \{ d(x, f(x)) : x \in A \}.
\]

Given a compactum \( X \) with a metric \( d \), let \( C(X) \) denote the hyperspace of all nonempty subcontinua of \( X \) equipped with the Hausdorff metric \( H \) (see e.g. [9, (0.1), p. 1 and (0.12), p. 10]). A mapping \( \omega : C(X) \to [0, \infty) \subset \mathbb{R} \) is called a \textit{Whitney map} for \( C(X) \) (see e.g. [7, Definition 13.1, p. 105]) if
(W.1) \( \omega(A) < \omega(B) \) for every \( A, B \in C(X) \) with \( A \subset B \) and \( A \neq B \);
(W.2) \( \omega(A) = 0 \) if and only if \( A \) is a singleton.

A \textit{tree} means a graph containing no simple closed curve. A continuum \( X \) is said to be \textit{unicoherent} if the intersection of every two of its subcontinua whose union is \( X \) is connected. \( X \) is said to be \textit{hereditarily unicoherent} if all of its
subcontinua are unicoherent. A solenoid is defined as a continuum which is an inverse limit of a sequence of circles with open bonding mappings. Note that a circle is a solenoid.

A continuum is said to be tree-like (arc-like) provided that it is the inverse limit of an inverse sequence of trees (arcs, respectively).

A continuum $X$ is called a triod provided that there is a subcontinuum $Z$ of $X$ such that $X \setminus Z$ is the union of three mutually disjoint nonempty open sets in $X$. A continuum is said to be atriodic provided that it does not contain any triod. It is known that each arc-like continuum is atriodic (see e.g. [10, Corollary 12.7, p. 233]). Recall that a simple triod is the union of three arcs emanating from a single point and otherwise disjoint from one another.

In the rest of this section we collect concepts and results used in the body of the paper, mostly introduced and studied in our very recent papers, and therefore perhaps not known to the reader. In this way we help the reader in understanding our arguments applied in proofs of results in the next sections.

We start with recalling the following two concepts.

**Definition 1.1.** A continuum $X$ is said to have the generalized $\varepsilon$-push property provided that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every two points $x, y \in X$ with $d(x, y) < \delta$ there exists a mapping $f : X \to X$ satisfying $f(x) = y$ and $\bar{d}(f) < \varepsilon$ (see [2, Definition 2.20] and [1, (2.2)], where continua having this property are studied).

A continuum $X$ is said to have the property of Kelley provided that for each point $x \in X$, for each subcontinuum $K$ of $X$ containing $x$ and for each sequence of points $x_n$ converging to $x$ there exists a sequence of subcontinua $K_n$ of $X$ containing $x_n$ and converging to the continuum $K$ (see e.g. [9, Definition 16.10, p. 538]).

It is known that each continuum having the generalized $\varepsilon$-push property has the the property of Kelley (see [2, Property 3.1] and compare [1, Property 2.4, p. 130]).

A continuum $X$ is said to have the arc approximation property provided that for each point $x \in X$, for each subcontinuum $K$ of $X$ containing $x$ there exists a sequence of arcwise connected subcontinua $K_n$ of $X$ containing $x$ and converging to the continuum $K$ (see [5, Section 3, p. 113]).
2. The no end point case

Let a Whitney map \( \omega \) for a compactum \( X \) be fixed. For any points \( x, y \in X \) lying in the same component of \( X \), define
\[
\sigma(x, y) = \inf\{ \omega(K) : K \text{ is a continuum and } x, y \in K \}.
\]

Let \( Z \) be a class of spaces that are homeomorphic with the product \( Z \times [0, 1] \) for some compact, 0-dimensional set \( Z \). We say that a compactum \( X \) has a local product \( Z \)-structure provided that for each point \( p \in X \) there exists a closed neighborhood \( U(p) \) belonging to \( Z \).

Let us observe the following assertion.

(2.1) If \( X \in Z \), then for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever a point \( y \) belongs to an arc \( xz \) in \( X \) and \( \sigma(x, y) = \sigma(y, z) = \varepsilon \), then for each mapping \( \varphi : xz \to X \) with \( \tilde{d}(\varphi, \text{id}|xz) < \delta \) the point \( \varphi(y) \) lies in the open arc from \( \varphi(x) \) to \( \varphi(z) \).

**Lemma 2.1.** Let \( X \) be a compactum containing no simple closed curve and having a local product \( Z \)-structure. Then for each sufficiently small \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever a point \( y \) belongs to an arc \( xz \) in \( X \) and \( \sigma(x, y) = \sigma(y, z) = \varepsilon \), then for each mapping \( \varphi : xz \to X \) with \( \tilde{d}(\varphi, \text{id}|xz) < \delta \) the point \( \varphi(y) \) lies in the open arc from \( \varphi(x) \) to \( \varphi(z) \).

Indeed, by the compactness of \( X \) there are compacta \( K_1, \ldots, K_n \) in \( X \) with \( K_i \in Z \) for each \( i \in \{1, \ldots, n\} \) and such that \( X = \text{int}(K_1) \cup \cdots \cup \text{int}(K_n) \). Since by (2.1) the conclusion of the lemma is satisfied for each compactum \( K_i \), hence using the Lebesgue number of the covering \( \{\text{int}(K_1), \ldots, \text{int}(K_n)\} \) of \( X \) so that \( \varepsilon \) and \( \delta \) guarantee that the points \( x, y, z, \varphi(x), \varphi(y), \varphi(z) \) and the corresponding arcs between \( x \) and \( z \) as well as between \( \varphi(x) \) and \( \varphi(z) \) lie in one set \( K_i \) for some \( i \), we get the conclusion.

Let a nondegenerate continuum \( X \) different from a simple closed curve be such that

(2.2) each proper nondegenerate subcontinuum of \( X \) is an arc,
(2.3) \( X \) does not have end points,

and let \( \mathcal{F} \) be the collection of all one-to-one mappings from \( \mathbb{R} \) onto arc components of \( X \). For every two points \( x, y \in X \) define
\[
\rho(x, y) = \inf\{ \tilde{d}(f, g) : f, g \in \mathcal{F} \text{ with } f(0) = x \text{ and } g(0) = y \}.
\]

It is known that \( \rho \) is a metric on \( X \) (see [11, p. 137]) and that
(2.4) $\rho(x, y) \geq d(x, y)$ for all $x, y \in X$,
(see [11, (6), p. 137]), whence the topology induced by $\rho$ on $X$ is finer than the one induced by $d$.

Lemma 2.2. Let a nondegenerate continuum $X$ different from a simple closed curve satisfy conditions (2.2) and (2.3) and has the generalized $\varepsilon$-push property. Then the metrics $\rho$ and $d$ on $X$ are equivalent.

Proof. Observe that condition (2.2) obviously implies that the continuum $X$ has the arc approximation property. Since $X$ has the generalized $\varepsilon$-push property, it has the property of Kelley [2, Property 3.1]. Therefore $X$ has the property of Kelley, the property that is just the conjunction of the above two (see [2, Definition 3.3 and Proposition 3.4]). Now we are ready to apply Theorem 4.4 of [4] which says that if a continuum has the arc property of Kelley and contains no simple triod, then each non-end point of $X$ has a neighborhood homeomorphic to the Cartesian product of a compact 0-dimensional set and an arc. Consequently, $X$ has a local product $\mathcal{Z}$-structure, and we can use Lemma 2.1.

To prove the lemma observe that, in view (2.4), it suffices to show that for each sequence $\{x_n\}$ of points of $X$

$$\lim d(x_n, x_0) = 0 \implies \lim \rho(x_n, x_0) = 0.$$  

So, let a sequence $\{x_n\}$ satisfies $\lim d(x_n, x) = 0$, and let $\eta > 0$ be given. Let $\varepsilon > 0$ be so small that the conclusion of Lemma 2.1 holds, and that each continuum $K \subset X$ with $\omega(K) < 2\varepsilon$ has its diameter less than $\frac{\eta}{2}$. We inductively choose points $p_0, p_1, p_{-1}, p_2, p_{-2}, \ldots$ such that $p_0 = x_0, p_{k+1} \neq p_{k-1}$, and $\sigma(p_k, p_{k+1}) = \varepsilon$ for all $k \in \mathbb{Z}$. Observe that, up to the reflection of indices about 0, this choice is unique by the condition (W.1) of the definition of the Whitney map $\omega$. Note also that the points $p_k$ belong to single arc component $A$ of $X$. Let a positive $\delta < \frac{\eta}{4}$ be a number guaranteed by Lemma 2.1 and $q_0 = x_n$ be so close to $x_0 = p_0$ that there is a mapping $\varphi : X \to X$ that satisfies $\varphi(x_0) = x_n$ for all $n \in \mathbb{N}$ and $\bar{d}(\varphi, \text{id}_{X}) < \delta$. Let $B = \varphi(A)$. For each $k \in \mathbb{Z}$ let $q_k = \varphi(p_k)$, and $h_k : p_k p_{k+1} \to q_k q_{k+1}$ be a homeomorphism. Define a function $h : A \to B$ such that $h|p_k p_{k+1} = h_k$ for each $k \in \mathbb{Z}$. By Lemma 2.1 the point $q_k$ is in $B$ and it lies in the open arc from $q_i$ to $q_j$ whenever $i < k < j$. Note that $B$ is the arc component of $X$ containing $x_n$. Therefore the function $h : A \to B$ is one-to-one. Let a mapping $f : \mathbb{R} \to A$ be in $\mathcal{F}$ with $f(0) = x_0$. Then the composition $g = h \circ f : \mathbb{R} \to B$ is a continuous surjection such that $g(0) = x_n = q_0$. In particular, $g \in \mathcal{F}$. We prove that

$$\bar{d}(f, g) < \eta,$$  

which will complete the proof.
Indeed, it suffices to show that \( d(h, \text{id}|X) < \eta \). Let \( x \in A \). Then \( x \in p_k p_{k+1} \) for some \( k \in \mathbb{Z} \), whence \( h(x) \in q_k q_{k+1} \subset \varphi(p_k p_{k+1}) \). Thus there is a point \( y \in p_k p_{k+1} \) such that \( \varphi(y) = h(x) \). Observe that \( \omega(p_k p_{k+1}) = \varepsilon < 2\varepsilon \), and hence \( \text{diam}(p_k p_{k+1}) < \frac{\eta}{2} \). Since \( d(\varphi, \text{id}|X) < \delta < \frac{\eta}{4} \), we also have \( d(y, \varphi(y)) < \frac{\eta}{4} \). Therefore
\[
d(x, h(x)) = d(x, \varphi(y)) \leq d(x, y) + d(y, \varphi(y)) < \frac{\eta}{2} + \frac{\eta}{4} < \eta,
\]
which completes the proof. \( \square \)

**Theorem 2.3.** Let a continuum \( X \) satisfy the three following conditions:

(2.3.1) \( X \) has the generalized \( \epsilon \)-push property;
(2.3.2) each proper nondegenerate subcontinuum of \( X \) is an arc;
(2.3.3) \( X \) does not have any end point.

Then \( X \) is a solenoid.

**Proof.** Since all the assumptions of Lemma 2.2 are satisfied, the metrics \( \rho \) and \( d \) are equivalent by that lemma, so \( X \) is a solenoid according to [11, Theorem 15, the equivalence of (a) and (d), p. 146]. \( \square \)

It is well known that each solenoid satisfies all the conditions of Theorem 2.3. In particular, the generalized \( \epsilon \)-push property is a consequence of the \( \epsilon \)-push property for all homogeneous continua (see [12, (1), p. 397] and [6, Lemma 4, p. 37]). Thus we have the following characterization of solenoids.

**Corollary 2.4.** A continuum \( X \) is a solenoid if and only if it satisfies conditions (2.3.1) – (2.3.3).

3. The End Point Case

Recall that, for a given continuum \( X \), the set of all end points of \( X \) is denoted by \( E(X) \). We start with the following lemma.

**Lemma 3.1.** Let a continuum \( X \) satisfy the two following conditions:

(3.1.1) \( X \) has the generalized \( \epsilon \)-push property;
(3.1.2) \( X \) has only arcs as its proper subcontinua.

Then \( X \) has a basis \( B \) of open sets such that for each \( B \in B \) each component of \( X \setminus B \) is an arc \( ab \) such that \( ab \cap \text{bd}(B) \subset \{a, b\} \subset \text{bd}(B) \cup E(X) \).

**Proof.** It follows from (3.1.2) by [5, Fact 3.8, p. 115] that the continuum \( X \) has the arc approximation property, and from (3.1.1) by [2, Property 3.1] that \( X \) has the property of Kelley. Thus \( X \) has the arc property of Kelley, see [2, Proposition
3.4]. Therefore [4, Theorem 4.4] implies that each point of the set \( X \setminus E(X) \) has a neighborhood homeomorphic to the Cartesian product of a compact 0-dimensional set and an arc. Consequently, for each \( x \in X \setminus E(X) \) and each \( \varepsilon > 0 \) there is a neighborhood \( V(x, \varepsilon) \) of \( x \) in \( X \) such that, for some compact 0-dimensional set \( Z \),

(a) \( \text{cl}(V(x, \varepsilon)) \subset B(x, \varepsilon) \setminus E(X) \),
(b) \( \text{cl}(V(x, \varepsilon)) \) is homeomorphic to the product \( Z \times [0, 1] \),
(c) \( V(x, \varepsilon) \) is homeomorphic to the product \( Z \times (0, 1) \) under the same homeomorphism.

Thus it follows that the set \( E(X) \) is closed, so it is compact, and consequently it is totally disconnected, so 0-dimensional.

Let \( x \in X \) and \( \varepsilon > 0 \) be fixed. We will construct a neighborhood \( B \) of \( x \) in \( X \) such that \( x \in B \subset B(x, \varepsilon) \) that satisfies the conclusion, which will complete the proof.

Since \( E(X) \) is closed and 0-dimensional, then there is a neighborhood \( W \) of \( x \) in \( X \) such that \( x \in W \subset \text{cl}(W) \subset B(x, \varepsilon) \) and \( \text{bd}(W) \cap E(X) = \emptyset \). Let

\[
\delta = \frac{1}{2} \inf \{d(z, e) : z \in \text{bd}(W) \text{ and } e \in E(X) \cup \text{bd}(B(x, \varepsilon))\}.
\]

For each point \( z \in \text{bd}(W) \) we choose a neighborhood \( V(z, \delta) \) as described above. Since the set \( \text{bd}(W) \) is compact, there are points \( z_1, \ldots, z_k \in \text{bd}(W) \) such that \( \text{bd}(W) \subset V(z_1, \delta) \cup \cdots \cup V(z_k, \delta) \). Define

\[
B = W \cup V(z_1, \delta) \cup \cdots \cup V(z_k, \delta).
\]

Then \( B \) is an open neighborhood of \( x \) in \( X \) and \( B \subset B(x, \varepsilon) \). To see that \( B \) satisfies the conclusion note that

\[
\text{bd}(B) \subset \text{bd}(V(z_1, \delta) \cup \cdots \cup V(z_k, \delta)) \subset \text{bd}(V(z_1, \delta)) \cup \cdots \cup \text{bd}(V(z_k, \delta)).
\]

Thus any component \( C = ab \) can intersect \( \text{bd}(B) \) only at some boundary points of the sets \( V(z_1, \delta) \). The reader can observe that such intersection can occur only at the end points \( a, b \) of \( ab \) because \( X \) contains no simple triod. This completes the proof. \( \square \)

**Theorem 3.2.** Let a continuum \( X \) satisfy the three following conditions:

(3.1.1) \( X \) has the generalized \( \varepsilon \)-push property;
(3.1.2) \( X \) has only arcs as its proper subcontinua;
(3.2.1) \( X \) contains an end point \( p \).

Then for each \( \varepsilon > 0 \) there is a mapping \( f_\varepsilon : X \to A \), where \( A \) is an arc in \( X \) with \( p \) as its end point, such that \( d(f_\varepsilon) < \varepsilon \).
Proof. Fix an $\varepsilon > 0$. Let $\delta > 0$ be a number guaranteed by the generalized $\varepsilon$-push property for the number $\varepsilon$ (see Definition 1.1). Let $B$ be a basis of open sets in $X$ guaranteed by Lemma 3.1. We choose a neighborhood $U$ of $p$ such that $U \in B$ and $\text{cl}(U) \subset B(p, \delta)$. Let $C$ be a component of $X \setminus U$. By (3.1.2) $C$ is an arc $ab$ for some $a, b \in X$. Denote by $A_p$ the arc component of $X$ containing the point $p$. We will define a mapping $g_C : C \to A_p$ such that $\tilde{d}(g_C) < \varepsilon$. According to the definition of $B$ we consider two cases.

Case 1. Assume that only one end point of $C$, say $a$, belongs to $\text{bd}(U)$, and the other is an end point of $X$, i.e., $b \in E(X) \setminus \text{bd}(U)$.

In this case we let $g_C$ be a mapping guaranteed by the generalized $\varepsilon$-push property such that $g_C(a) = p$ and $\tilde{d}(g_C) < \varepsilon$ (see Definition 1.1).

Case 2. Assume that $a, b \in \text{bd}(U)$.

Since $d(a, p), d(b, p) < \delta$, then, by (3.1.1), there are mappings $g_a, g_b : X \to X$ such that $g_a(a) = p = g_b(b)$ and $\tilde{d}(g_a), \tilde{d}(g_b) < \varepsilon$. Let $S = g_a(ab) \cup g_b(ab)$. Observe that $S$ is an arc with $p$ as an end point, and either $S = g_a(ab)$ or $S = g_b(ab)$. By [10, Corollary 12.26, p. 252] there is a point $e \in ab$ such that $g_a(e) = g_b(e)$. Let $ae$ and $eb$ be subarcs of the arc $ab$. Define $g_C : ab \to S$ by

$$g(x) = \begin{cases} g_a(x) & \text{for } x \in ae, \\ g_b(x) & \text{for } x \in eb. \end{cases}$$

Hence $\tilde{d}(g_C) < \varepsilon$.

For each component $C$ of $X \setminus U$ the set $g_C(C)$ is an arc $A_C$ contained in $A_p$. The mapping $g_C : C \to g_C(C) = A_C \subset A_p$ can be extended to a mapping $\tilde{g}_C : \tilde{C} \to A_C$, where $\tilde{C}$ is a closed and open subset of $X \setminus U$, and $\tilde{d}(\tilde{g}_C) < \varepsilon$.

By the compactness of $X \setminus U$ there are its components $C_1, \ldots, C_k$ such that $X \setminus U = \tilde{C}_1 \cup \cdots \cup \tilde{C}_k$. We may modify the sets $\tilde{C}_1, \ldots, \tilde{C}_k$ to their corresponding mutually disjoint subsets $D_1, \ldots, D_k$ such that each $D_i$ is open and closed in $X \setminus U$ and $D_1 \cup \cdots \cup D_k = X \setminus U$. Therefore, to keep our notation simple, we assume that the sets $\tilde{C}_1, \ldots, \tilde{C}_k$ are mutually exclusive.

Define

$$f_\varepsilon(x) = \begin{cases} \tilde{g}_{C_i}(x) & \text{for } x \in \tilde{C}_i, \\ p & \text{for } x \in U. \end{cases}$$

Note that $f_\varepsilon$ is a well defined mapping from $X$ to itself such that $\tilde{d}(f_\varepsilon) < \varepsilon$ and $f_\varepsilon(X) = A_{C_1} \cup \cdots \cup A_{C_k}$, and this union is an arc in $X$ having $p$ as its end point. This completes the proof. \qed
As an immediate consequence of Theorem 3.2 we get the following result.

**Corollary 3.3.** Let a continuum $X$ satisfy the three following conditions:

(3.1.1) $X$ has the generalized $\varepsilon$-push property;
(3.1.2) $X$ has only arcs as its proper subcontinua;
(3.2.1) $X$ contains an end point.

Then $X$ is arc-like.

4. **Summary of main results; questions**

Theorem 2.3 and Corollary 3.3 can be summarized as follows.

**Corollary 4.1.** Let a continuum $X$ have the generalized $\varepsilon$-push property and all proper nondegenerate subcontinua of $X$ be arcs.

(4.1.1) If $X$ has no end point, then it is a solenoid.
(4.1.2) If $X$ has an end point, then it is arc-like.

The question whether all atriodic members of $\mathbb{A}R(\mathbb{H}U)$ are arc-like has been central in the former study of these spaces, see [4, Question 5.5]. This question motivated and inspired the results presented in this paper. In the next corollary we answer this question in the affirmative.

**Corollary 4.2.** Each absolute retract for the class of hereditarily unicoherent continua that contains no simple triod is an arc-like continuum. Moreover, such a continuum must contain an end point.

**Proof.** Let $X$ be in $\mathbb{A}R(\mathbb{H}U)$. By [2, Corollary 2.22] $X$ has the generalized $\varepsilon$-push property. Applying [4, Theorem 2.4 and Proposition 2.3] we infer that $X$ has the arc approximation property. Therefore, by the implication from (3.4.1) to (3.4.3) of Theorem 3.4 in [4], each proper nondegenerate subcontinuum of $X$ is an arc. Moreover, by [4, Corollary 5.8], $X$ is not a solenoid. Therefore, by Corollary 4.1, $X$ is arc-like, and it contains an end point. \qed

We end this paper with the two following questions that naturally appear in view of Corollary 4.2. Note that they are interrelated in the sense that a positive answer to any of them would lead to a negative answer to the other.

**Question 4.3.** Is every nondegenerate atriodic member of $\mathbb{A}R(\mathbb{H}U)$ a Knaster type continuum?

**Question 4.4.** Let $X$ be the (simplest) arc-like continuum having three end points, see e.g. [10, 1.10, p. 7]. Is $X$ in $\mathbb{A}R(\mathbb{H}U)$?
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