The Set Function $T$ and Contractibility of Continua *)

by

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Presented by K. BORSUK on May 26, 1976

Summary. A sufficient condition of the noncontractibility of Hausdorff continua is proved. It is of particular interest in the case of dendroids.

A continuum means a compact connected Hausdorff space. A dendroid means an arcwise connected and hereditarily unicoherent metric continuum. A mapping means a continuous transformation. A space $X$ is said to be contractible if there exists a homotopy $H: X \times [0, 1] \to X$ which is a deformation (i.e. such that $H(x, 0) = x$ for all $x \in X$) and which contracts $X$ to a point (i.e. $H(X \times \{1\})$ is a one-point set). Given a continuum $X$ and a set $A \subset X$ we define $T(A)$ (or $T_x(A)$ if more than one continuum is under consideration) as the set of all points $x$ of $X$ such that every subcontinuum of $X$ which contains $x$ in its interior must intersect $A$. It is known [5] that $T(a)$ is a subcontinuum of $X$ for each point $a \in X$.

The following theorem is due to Professor Ralph Bennett. It was firstly proved in an unpublished paper [2] by using a false result (Theorem 1 of [4]). We prove here a more general result from which this theorem follows as a corollary.

**Theorem 1 (Bennett).** If a dendroid $X$ contains two points $p$ and $q$ such that

1. $T(p) \cap T(q) \neq \emptyset$

and

2. $p \in X \setminus T(q)$ and $q \in X \setminus T(p)$,

then $X$ is not contractible.

Let $I$ denote the unit interval $[0, 1]$ of reals. We start with the following

**Lemma.** Suppose $S$ in a continuum, $A$ is a closed subset of $S$, and $r \in T(A)$. Suppose $H: S \times I \to S$ is a homotopy such that $H(s, 0) = s$ for each $s \in S$ and such that $H(r, 1) \in S \setminus T(A)$. Then for some $t \in I$, $H(r, t) \in A$.

*) This research was done while the first author was a participant in an exchange grant under the joint sponsorship of the National Academy of Sciences, Washington, D. C., and the Polish Academy of Sciences.
Proof. Define \( f(s) = H(s, 1) \) for each \( s \in S \), and suppose the lemma is false. Then \( H^{-1}(A) \cap ((r) \times I) = \emptyset \), so there exists an open set \( U \subseteq S \) with \( r \in U \) and \( H^{-1}(A) \cap (\overline{U} \times I) = \emptyset \). Let \( W \subseteq S \) be a continuum with \( f(r) \in \text{Int } W \) and \( W \cap A = \emptyset \). Let \( V = U \cap f^{-1}(\text{Int } W) \). Then \( r \in V \) and \( H^{-1}(A) \cap (\overline{V} \times I) = \emptyset \), and \( f(V) \subseteq W \).

Let \( L \subseteq S \times I \) be the continuum \((S \times \{0\}) \cup (\overline{V} \times I)\), and consider the disjoint union \( L \cup W \). Define a continuum \( K \) to be the quotient of the disjoint union, \( L \cup W \), obtained by identifying each \((v, 1) \in V \times \{1\}\) with \( f(v) \in W \), and let \( \eta : L \cup W \to K \) by the identification map. Define a map \( G : K \to S \) by: \( G(\eta(x)) = x \) if \( x \in W \); \( G(\eta(x, t)) = H(x, t) \) if \( (x, t) \in L \). \( G \) is surjective since \( G(s, 0) = s \) for each \( s \in S \), so that, by Lemma 14 of [1], p. 587, \( T \cap (\text{Int } J) = \emptyset \). In particular, \( r \in GT \cap \eta^{-1}(\text{Int } J) = \emptyset \). However, since \( A \cap W = \emptyset \) and \( \eta^{-1}(A) \cap (\overline{V} \times I) = \emptyset \), it follows that \( \eta^{-1}(A) = \emptyset \). Thus, \( \eta^{-1}(A) \cap \eta^{-1}(J) = \emptyset \). Let \( J = \eta((\overline{V} \times I) \cup W) \). \( J \) is a subcontinuum of \( K \), and

\[
M = \eta(W \cup (\overline{V} \times \{0\})) \cup (\overline{V} \times I) = \text{Int}_K J.
\]

Since \( \eta^{-1}(r) \cap \eta(S \times \{0\}) = \emptyset \), it follows that \( \eta^{-1}(r) \cap M = \emptyset \), and since \( \eta(A \times \{0\}) \cap J = \emptyset \), we have \( \eta^{-1}(r) \cap \eta^{-1}(A \times \{0\}) = \emptyset \), a contradiction, and the proof is complete.

Corollary 1. If \( S \) is a continuum and \( A \) and \( B \) are closed subsets of \( S \) such that \( A \cap T(B) = \emptyset \), \( B \cap T(A) = \emptyset \) and \( T(A) \cap T(B) \neq \emptyset \), then \( S \) is not contractible.

Proof. Suppose \( H : S \times I \to S \) is a contraction. Without loss of generality suppose \( H(S \times \{1\}) \in A \). Let \( r \in T(A) \cap T(B) \). Since \( H(r, 1) \in A \), \( H(r, 1) \in S \setminus T(B) \) and by the lemma there is a \( t \in I \) such that \( H(r, t) \in B \). Let \( t_A, t_B \) be the smallest elements of \( I \) such that \( H(r, t_A) \in A \) and \( H(r, t_B) \in B \). Either \( t_A < t_B \) or \( t_B < t_A \), since \( t_A = t_B \) is impossible. If \( t_A < t_B \), applying the lemma to \( H|S_\times [0, t_A] \) yields an element \( t' < t_A \) such that \( H(r, t') \in B \), a contradiction since \( t' < t_B \). Similarly, if \( t_B < t_A \), applying the lemma to \( H|S_\times [0, t_B] \) yields a \( t'' < t_B < t_A \) such that \( H(t'', r) \in A \), a contradiction, and the corollary is proved.

Remark. It is easy to see that both conditions (1) and (2) are essential in Theorem 1. In fact, let \( X \) be the disjoint union of two harmonic fans with the unique accumulation point on their bases joined by an arc otherwise missing the fans. Then \( X \) satisfies (2) but not (1), and is contractible. Alternatively, if \( X \) is taken as the union of two harmonic fans \( P \) and \( Q \) with the limit continuum of \( Q \) contained in that of \( P \), but missing the vertex of \( P \), then \( X \) satisfies (1) but not (2), and is contractible (regardless of the orientation of the two fans).

The following theorem is known (see [6]):

**Theorem 2.** There exists a plane dendroid \( X \) which contains a subdendroid \( M \subseteq X \) and two points \( p \) and \( q \) such that

\[
T_M(p) \cap T_M(q) \neq \emptyset,
\]

\[
p \in M \setminus T_M(q) \quad \text{and} \quad q \in M \setminus T_M(p),
\]

and \( X \) is contractible.
Proof. Take as $X$ the union of a harmonic fan $P$ with the vertex $p$ and the unique accumulation point $q$ on its base and of a harmonic fan $Q$ with the vertex $q$ and the limit continuum of $Q$ contained in that of $P$ but missing $p$. If $r$ is a point of the limit continuum $pq$ of $P$ with the limit continuum of $Q$ equal to $qr$, we define $M$ as the union of $Q$ and of a subfan of $P$ having $pr$ as its limit continuum. The homotopy $H: X \times I \to X$ which is a contraction can be patterned after the homotopy $h$ which contracts the harmonic hooked fan $F'_h$ in [3], p. 31.

Since conditions (3) and (4) correspond to (1) and (2) of Theorem 1, we have the following

**Corollary 2.** Let $X$, $M$, $p$ and $q$ be as in Theorem 2. Then $M$ is not contractible, but $M$ is contractible in $X$.

**Problem.** Does there exist a fan $X$ containing a subfan $M \subseteq X$ and satisfying all the conditions of Theorem 2?

It appears likely to the authors that the techniques herein may be applied to prove more general results about homotopies and other mappings of product continua.

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