Retractions and contractibility in hyperspaces

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Abstract

Given a metric continuum \( X \), let \( 2^X \) denote the hyperspace of all nonempty closed subsets of \( X \). For each positive integer \( k \) let \( C_k(X) \) stand for the hyperspace of members of \( 2^X \) having at most \( k \) components. Consider mappings \( \varphi_B: C_k(X) \to C_{k+m}(X) \) (where \( B \in C_m(X) \)) and \( \psi_B: 2^X \to 2^X \) both defined by \( A \mapsto A \cup B \). We give necessary and sufficient conditions under which these mappings are deformation retractions (under a special convention for \( \varphi_B \)). The conditions are related to the contractibility of the corresponding hyperspaces.

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1. Introduction

Throughout this paper a continuum means a compact connected metric space. Given a continuum \( X \), we let \( 2^X \) denote the hyperspace of all nonempty closed subsets of \( X \) equipped with the Hausdorff metric (see e.g. [4, (0.1), p. 1 and (0.12), p. 10]). We denote by \( C(X) \) the hyperspace of all subcontinua of \( X \), i.e., of all connected elements of \( 2^X \); further, for a given \( k \in \mathbb{N} \), we let \( C_k(X) \) denote the hyperspace of all elements of \( 2^X \) having at most \( k \) components, all equipped with the inherited topology (thus induced by the Hausdorff metric). Therefore, \( C(X) = C_1(X) \) and \( C_k(X) \subset C_{k+1}(X) \) for each \( k \in \mathbb{N} \).

The results of this paper are extensions and generalizations of the ones obtained earlier by the second named author in [5]. Namely, for every \( k, m \in \mathbb{N} \) and \( B \in C_m(X) \) we consider a mapping \( \varphi_B: C_k(X) \to C_{k+m}(X) \) given by \( \varphi_B(A) = A \cup B \) and we give necessary and sufficient conditions under which \( \varphi_B \) is a deformation retraction (under a special convention for \( \varphi_B \)). These conditions are related to the contractibility of the hyperspace \( C_k(X) \). Similarly, for a continuum \( X \) and \( B \in 2^X \) we consider the mapping \( \psi_B: 2^X \to 2^X \) given by \( \psi_B(A) = A \cup B \), and again we give necessary and sufficient conditions under which \( \psi_B \) is a deformation retraction. As previously, these conditions are related to the contractibility of the hyperspace \( 2^X \).

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The paper consists of three sections. After Introduction and Preliminaries, the third section is devoted to characterizations of the contractibility of the hyperspaces $C_k(X)$ and $2^X$. These characterizations are expressed in terms of the mappings $\varphi_B$ and $\psi_B$ of being deformation retractions. As a corollary, some necessary conditions for the contractibility and for having the property of Kelley of the continuum $X$ are obtained.

2. Preliminaries

The symbol $\mathbb{N}$ stands for the set of all positive integers. All considered spaces are assumed to be metric, and all mappings are continuous.

It is known that $2^X$, and for each $n \in \mathbb{N}$, also $C_n(X)$ are arcwise connected continua (for $2^X$ see [4, (1.13), p. 65]; for $C_n(X)$ see [3, Theorem 3.1, p. 240]). Further, for $B \in 2^X$ and $m \in \mathbb{N}$, define the following hyperspaces.

$$2_B^X = \{A \in 2^X: B \subset A\} \quad \text{and} \quad C_m(B, X) = \{A \in C_m(X): B \subset A\}.$$ 

For each nonempty closed subset $A$ of $2^X$ denote by $\bigcup A$ the union of all elements of $A$. Then $\cup: 2^2^X \to 2^X$ is a surjective mapping (see [4, Lemma 1.48, p. 100]; compare [2, Exercise 11.5, p. 91]).

A collection $\mathcal{F}$ of subsets of a space is called a nest if for any $F_1, F_2 \in \mathcal{F}$ we have either $F_1 \subset F_2$ or $F_2 \subset F_1$. Let $X$ be a continuum and let a hyperspace $\mathcal{H} \subset 2^X$ be given. By an order arc in $\mathcal{H}$ we mean an arc, $\alpha$, in $\mathcal{H}$ such that $\alpha$ is a nest. An order arc from $A$ to $B$ is an order arc having $A$ and $B$ as its end points, with $A \subset B$ (see [2, 15.1, p. 119]).

**Definition 2.1.** For a continuum $X$ and $B \in 2^X$ define $\psi_B: 2^X \to 2_B^X$ by $\psi_B(A) = A \cup B$ for each $A \in 2^X$.

Note that $\psi_B$ is a retraction. It is a generalization of the mapping $\psi_p: 2^X \to 2_p^X = \{A \in 2^X: p \in A\}$ defined for a point $p \in X$ by $\psi_p(A) = A \cup \{p\}$ in [5, p. 277].

**Definition 2.2.** For a continuum $X$, for every $k, m \in \mathbb{N}$ and $B \in C_m(X)$ define $\varphi_B: C_k(X) \to C_{k+m}(X)$ by $\varphi_B(A) = A \cup B$ for each $A \in C_k(X)$.

In this case we cannot ask whether $\varphi_B$ is a retraction, because $C_{k+m}(X)$ is not a subspace of $C_k(X)$. However, we establish the following convention.

**Convention 2.3.** Since $\varphi_B|C_k(B, X)$ is the identity mapping on $C_k(B, X)$, we shall say that $\varphi_B$ is a deformation retraction in $C_{k+m}(X)$ if there exists a homotopy $G: C_k(X) \times [0, 1] \to C_{k+m}(X)$ between $\varphi_B$ and the identity mapping in $C_k(X)$. Finally, we will say that $\varphi_B$ is a strong deformation retraction in $C_{k+m}(X)$ provided that the homotopy $G$ satisfies the condition $G(A, t) = A$ whenever $A \in C_k(B, X)$ and $t \in [0, 1]$.

As previously for the mapping $\psi_B$, the mapping $\varphi_B$ is a generalization of the mapping $\phi_p: C(X) \to C_2(X)$ defined by $\phi_p(A) = A \cup \{p\}$ for some point $p \in X$ and for each $A \in C(X)$ in [5, p. 278]. Thus we will write $\varphi_p$ instead of either $\phi_p$ or $\varphi_{\{p\}}$.

3. Characterizations of contractibility

We start with the following proposition.

**Proposition 3.1.** Let $X$ be a continuum and let $k \in \mathbb{N}$. If $A \in C(2^X)$ and $A \cap C_k(X) \neq \emptyset$, then $\bigcup A \in C_k(X)$.

**Proof.** If $\bigcup A \notin C_k(X)$, then there are $k + 1$ nonempty mutually separated sets $B_1, \ldots, B_{k+1}$ whose union is $\bigcup A$. Let $L \in A \cap C_k(X)$. Then there is some $i \in \{1, \ldots, k + 1\}$ such that $L \cap B_i = \emptyset$. Put

$$A_0 = \{A \in A: A \cap B_i \neq \emptyset\} \quad \text{and} \quad A_1 = \{A \in A: A \cap B_i = \emptyset\}.$$ 

Then $A = A_0 \cup A_1$ is a separation of $A$ in two nonempty mutually separated subsets, contrary to its connectedness. \qed
Note that for \( k = 1 \) the above result is shown in [4, Lemma 1.43, p. 97] (compare also [2, 15.9 (2), p. 124]).

The following two theorems are extensions of [5, Theorem 4.1, p. 280].

**Theorem 3.2.** Let \( X \) be a continuum, let \( k, m \in \mathbb{N} \), and let \( B \in C_m(X) \). If \( \varphi_B : C_k(X) \to C_{k+m}(X) \) is a deformation retraction in \( C_{k+m}(X) \), then \( C_k(X) \) is contractible.

**Proof.** Let \( B \in C_m(X) \) and assume that \( \varphi_B : C_k(X) \to C_{k+m}(X) \) is a deformation retraction in \( C_{k+m}(X) \). Then there exists a homotopy \( \Phi : C_k(X) \times [0, 1] \to C_{k+m}(X) \) such that \( \Phi(A, 0) = A \) and \( \Phi(A, 1) = \varphi_B(A) \) for each \( A \in C_k(X) \).

Take in \( C_m(X) \) an order arc \( \alpha \) from \( B \) to \( X \), and consider a function \( h : C_k(X) \times [0, 1] \to C_k(X) \) defined by

\[
h(A, t) = \begin{cases} 
  \bigcup \{ \Phi(A, 2s) : s \in [0, t] \}, & \text{if } t \in [0, \frac{1}{2}], \\
  \bigcup \{ \Phi(\{A\} \times [0, 1]) \} \cup \alpha(2t - 1), & \text{if } t \in [\frac{1}{2}, 1].
\end{cases}
\]

Note that for each \( A \in C_k(X) \) and each \( t \in [0, 1] \) we have \( \Phi(\{A\} \times [0, t]) \in C(2^X) \). Applying [4, Lemma 1.48, p. 100] we get

\[
h(A, t) \in C_k(X) \quad \text{for each } (A, t) \in C_k(X) \times [0, \frac{1}{2}].
\]

Furthermore, since \( \Phi(A, 0) = A \in C_k(X) \), Proposition 3.1 implies:

\[
h(A, t) \in C_k(X) \quad \text{for each } (A, t) \in C_k(X) \times [0, 1].
\]

Now, note that \( B \subset \Phi(A, 1) \cap \alpha(0) \subset \bigcup \{ \Phi(\{A\} \times [0, 1]) \} \cap \alpha(2t - 1) \) for each \( (A, t) \in C_k(X) \times [\frac{1}{2}, 1] \). Thus

\[
h(A, t) \in C_k(X) \quad \text{whenever } (A, t) \in C_k(X) \times [\frac{1}{2}, 1].
\]

Hence, according to (3.2.1)–(3.2.4) and by [4, Lemma 1.48, p. 100] we conclude that \( h \) is well defined and continuous. On the other hand, clearly \( h(A, 0) = \Phi(A, 0) = A \) and \( X = \alpha(1) \subset h(A, 1) \) for each \( A \in C_k(X) \). Therefore \( h \) is a contraction in \( C_k(X) \). \( \square \)

**Theorem 3.3.** Let \( X \) be a continuum and let \( k, m \in \mathbb{N} \). If \( C_k(X) \) is contractible, then for each \( B \in C_m(X) \) the mapping \( \varphi_B \) is a deformation retraction in \( C_{k+m}(X) \).

**Proof.** Assume that \( C_k(X) \) is contractible, and let \( B \in C_m(X) \). Then there exists a homotopy \( G : C_k(X) \times [0, 1] \to C_k(X) \) such that \( G(A, 0) = A \) and \( G(A, 1) = D \) for some \( D \in C_k(X) \) and for each \( A \in C_k(X) \). We may assume that \( D = X \). Consider now the function \( g : C_k(X) \times [0, 1] \to C_{k+m}(X) \) given by:

\[
g(A, t) = \begin{cases} 
  G(A, 2t), & \text{if } t \in [0, \frac{1}{2}], \\
  B \cup G(A, 2 - 2t), & \text{if } t \in [\frac{1}{2}, 1].
\end{cases}
\]

Then \( g \) is a mapping. Moreover, \( g(A, 0) = G(A, 0) = A \) and \( g(A, 1) = B \cup G(A, 0) = A \cup B = \varphi_B(A) \) for each \( A \in C_k(X) \). Therefore, \( \varphi_B \) is a deformation retraction in \( C_{k+m}(X) \). \( \square \)

Theorems 3.2 and 3.3 imply the following corollary.

**Corollary 3.4.** The following assertions are equivalent for a continuum \( X \) and for \( k \in \mathbb{N} \):

1. The hyperspace \( C_k(X) \) is contractible;
2. For each \( m \in \mathbb{N} \) and for each \( B \in C_m(X) \) the mapping \( \varphi_B : C_k(X) \to C_{k+m}(X) \) (defined in Definition 2.2) is a deformation retraction in \( C_{k+m}(X) \);
3. For each \( m \in \mathbb{N} \) there exists \( B \in C_m(X) \) such that the mapping \( \varphi_B : C_k(X) \to C_{k+m}(X) \) is a deformation retraction in \( C_{k+m}(X) \);
(3.4.4) there exists $m \in \mathbb{N}$ such that for each $B \in C_m(X)$ the mapping $\varphi_B : C_k(X) \to C_{k+m}(X)$ is a deformation retraction in $C_{k+m}(X)$;

(3.4.5) there exist $m \in \mathbb{N}$ and $B \in C_m(X)$ such that the mapping $\varphi_B : C_k(X) \to C_{k+m}(X)$ is a deformation retraction in $C_{k+m}(X)$.

Taking in Corollary 3.4 above $k = m = 1$ and $B = \{p\}$ for some $p \in X$ we get an extension of [5, Theorem 4.1, p. 280] as a corollary.

**Corollary 3.5.** Let $X$ be a continuum. Then the following assertions are equivalent:

(3.5.1) the hyperspace $C(X)$ is contractible;

(3.5.2) for each point $p \in X$ the mapping $\varphi_p : C(X) \to C_2(X)$ defined by $\varphi_p(A) = A \cup \{p\}$ is a deformation retraction in $C_2(X)$;

(3.5.3) there is a point $p \in X$ such that the mapping $\varphi_p : C(X) \to C_2(X)$ is a deformation retraction in $C_2(X)$.

The next two results generalize [5, Theorem 4.2, p. 281].

**Theorem 3.6.** Let $X$ be a continuum. If there is $B \in 2^X$ such that $\psi_B$ is a deformation retraction, then $2^X$ is contractible.

**Proof.** Let $B \in 2^X$ and assume that $\psi_B : 2^X \to 2^X_B$ is a deformation retraction. Then there exists a homotopy $\Psi : 2^X \times [0, 1] \to 2^X_B$ such that for each $A \in 2^X$ we have $\Psi(A, 0) = A$ and $\Psi(A, 1) = \psi_B(A) = A \cup B$. Take in $2^X$ an order arc $\alpha$ from $B$ to $X$, and consider a function $h : 2^X \times [0, 1] \to 2^X$ defined by

$$h(A, t) = \begin{cases} \bigcup \{\Psi(A, 2s) : s \in [0, t]\}, & \text{if } t \in [0, \frac{1}{2}], \\ \bigcup \{\Psi([A] \times [0, 1])\} \cup \alpha(2t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

As previously (in the corresponding part of the proof of Theorem 3.2) it can be verified that $h$ is well defined and continuous. Furthermore, $h(A, 0) = \Psi(A, 0) = A$ and $X = \alpha(1) \subset h(A, 1)$ for each $A \in 2^X$, whence it follows that $h$ is a contraction in $2^X$. □

**Theorem 3.7.** Let $X$ be a continuum. If the hyperspace $2^X$ is contractible, then for each $B \in 2^X$ the mapping $\psi_B$ is a deformation retraction.

**Proof.** Assume that $2^X$ is contractible and let $B \in 2^X$. Then there exists a homotopy $G : 2^X \times [0, 1] \to 2^X$ such that $G(A, 0) = A$ and $G(A, 1) = D$ for some $D \in 2^X$ and for each $A \in 2^X$. Without loss of generality we may assume that $D = B$. Consider the function $g : 2^X \times [0, 1] \to 2^X$ given by the same formula as previously, that is, by (3.3.1). Then $g$ is a mapping such that $g(A, 0) = G(A, 0) = A$ and $g(A, 1) = B \cup G(A, 0) = A \cup B = \psi_B(A) = \psi_B(A)$ for each $A \in 2^X$. Therefore, $\psi_B$ is a deformation retraction in $2^X$. □

As a consequence of [5, Theorem 4.2, p. 281] and of Theorems 3.6 and 3.7 above we get a corollary.

**Corollary 3.8.** The following assertions are equivalent for a continuum $X$:

(3.8.1) the hyperspace $2^X$ is contractible;

(3.8.2) for each point $p \in X$ the mapping $\psi_p : 2^X \to 2^X_p$ defined by $\psi_p(A) = A \cup \{p\}$ is a deformation retraction in $2^X$;

(3.8.3) there exists a point $p \in X$ such that the mapping $\psi_p$ is a deformation retraction in $2^X$;

(3.8.4) for each $B \in 2^X$ the mapping $\psi_B$ defined in Definition 2.1 is a deformation retraction in $2^X$;

(3.8.5) there exists $B \in 2^X$ such that the mapping $\psi_B$ is a deformation retraction in $2^X$.

Since for each continuum $X$ contractibility of the hyperspaces $2^X$ and $C_k(X)$ for each $k \in \mathbb{N}$ are equivalent (see [3, Theorem 3.7, p. 241]) Corollaries 3.4, 3.5 and 3.8 can be summarized as follows.
Theorem 3.9. For each continuum $X$ and for each $k \in \mathbb{N}$ all 13 conditions (3.4.1)–(3.4.5), (3.5.1)–(3.5.3) and (3.8.1)–(3.8.5) are equivalent.

Since each contractible continuum $X$ has contractible hyperspaces $2^X$ and $C(X)$, see [2, Corollary 20.2, p. 166], we get the following result as a consequence of Theorem 3.9.

Corollary 3.10. If a continuum $X$ is contractible and if $k \in \mathbb{N}$, then each of the 13 conditions (3.4.1)–(3.4.5), (3.5.1)–(3.5.3) and (3.8.1)–(3.8.5) is satisfied.

A continuum $X$ (with a metric $d$) is said to have the property of Kelley provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every two points $a, b \in X$ with $d(a, b) < \delta$ and for each subcontinuum $A$ of $X$ containing $a$ there exists a subcontinuum $B$ of $X$ containing $b$ and satisfying $\text{dist}(A, B) < \varepsilon$ (where dist means the Hausdorff metric; see for example [2, p. 167]). Since each continuum $X$ having the property of Kelley has contractible hyperspaces $2^X$ and $C(X)$, see [4, Theorem 16.15, p. 544] (compare also [3, Corollary 3.8, p. 241]), Theorem 3.9 implies a corollary.

Corollary 3.11. If a continuum $X$ has the property of Kelley and if $k \in \mathbb{N}$, then each of the 13 conditions (3.4.1)–(3.4.5), (3.5.1)–(3.5.3) and (3.8.1)–(3.8.5) is satisfied.

A weaker form of the property of Kelley, called the semi-Kelley property has been introduced in [1, Definition 3.16, p. 79], and several results known for continua having the property of Kelley have been generalized to continua satisfying the weaker condition. In connection with this, the following question can be asked.

Question 3.12. Are the conditions (3.4.1)–(3.4.5), (3.5.1)–(3.5.3) and (3.8.1)–(3.8.5) satisfied for continua having the semi-Kelley property?

Remark 3.13. The conclusion of Theorem 3.3 cannot be sharpened to “strong deformation retraction” in place of “deformation retraction” (see Convention 2.3). An example showing this is presented below in Example 3.14. The same example shows that a similar strengthened form is not true for Theorem 3.7 (since contractibility of $C_m(X)$ is equivalent to the one of $2^X$, see [3, Theorem 3.7, p. 241]).

Example 3.14. There is a continuum $X$ such that:

(3.14.1) for each integer $m \in \mathbb{N}$ the hyperspace $C_m(X)$ is contractible;
(3.14.2) for each integer $m \in \mathbb{N}$ and for each $B \in C_m(X)$ the mapping $\varphi_B : C_m(X) \to C_{2m}(X)$ is a deformation retraction in $C_{2m}(X)$;
(3.14.3) there exists a subcontinuum $Y$ of $X$ such that $\text{int}(Y) = \emptyset$ and for each point $p \in Y$ the mapping $\varphi_p : C(X) \to C_2(X)$ is not a strong deformation retraction in $C_2(X)$.

Proof. Let $X$ be the circle with a spiral, that is, $X = S^1 \cup S$, where $S^1$ is the unit circle in the plane, and $S$ is the spiral given in polar coordinates $(\rho, \theta)$ by

$$S = \left\{ (\rho, \theta) : \rho = 1 + \frac{1}{1 + \theta} \text{ and } \theta \geq 0 \right\},$$

see [2, Fig. 14, p. 51]. Thus $S$ approximates $S^1$. It can easily be observed that $X$ has the property of Kelley (defined here before Corollary 3.11) and thus $C_m(X)$ is contractible for each $m \in \mathbb{N}$ according to condition (3.4.1) of this corollary (another argument can be used for contractibility of $C_m(X)$, namely that $C(X)$ is homeomorphic to the cone over $X$, [2, Example 7.1, p. 53], so it is contractible, whence $C_m(X)$ is contractible for each $m \in \mathbb{N}$, see [3, Theorem 3.7, p. 241]). So (3.14.1) is shown.

It follows from Theorem 3.3 that for each $m \in \mathbb{N}$ and for each $B \in C_m(X)$ the mapping $\varphi_B : C_m(X) \to C_{2m}(X)$ is a deformation retraction in $C_{2m}(X)$. Thus (3.14.2) holds.

To show (3.14.3) define $Y$ as the limit circle $S^1$, and take $p \in Y$. To see that $\varphi_p$ is not a strong deformation retraction in $C_2(X)$ observe that there is no local basis of $p$ being a nested countable family of subcontinua of $X$ containing $p$, and therefore the conclusion follows from [5, Theorem 4.4, p. 281]. The proof is complete. \[\square\]
References