A Characterization of the Pseudo-Arc

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Summary. It is proved that a nondegenerate chainable continuum is the pseudo-arc if and only if it is homogeneous with respect to open mappings.

A topological space $X$ is said to be homogeneous with respect to a class $\mathcal{H}$ of mappings if for every two points $p$ and $q$ of $X$ there is a mapping $f$ of $X$ onto itself such that $f$ is in $\mathcal{H}$ and $f(p)=q$. All spaces considered are assumed to be metric, and all mappings are continuous. The aim of this note is to prove the following

**Theorem.** Each nondegenerate chainable continuum which is homogeneous with respect to open mappings is the pseudo-arc.

**Proof.** The basic ideas of this proof, which is quite elementary, come from the proofs of the Bing theorem of [4], p. 345, and of the Rosenholtz Theorem 1.0 of [6], p. 259.

Let $X$ be a continuum satisfying the assumptions of the theorem. First we show that $X$ has an end point $p$ (in the sense of [3], 5. (C), p. 660). For each positive integer $n$ let $x_n$ be a point of $X$ such that a $1/n$-chain covers $X$ and an end link of this chain contains $x_n$. Some subsequence of the sequence $\{x_n\}$ converges to a point $x$. Then the point $x$ has the following property:

$\ast$ for each neighborhood $U$ of $x$ and for each positive number $\delta$ there is a $\delta$-chain covering $X$, one of whose end links intersects $X$ and lies in $U$.

Now let $q$ be an arbitrary point of $X$. We intend to show that $q$ has property $(\ast)$. The continuum $X$ being homogeneous with respect to open mappings, there exists an open mapping $f$ of $X$ onto itself with $f(x)=q$. Let $V$ be a neighborhood of $q$ and let $\varepsilon$ be a positive number. Since $f$ is continuous at $x$, there exists a neighborhood $U$ of $x$ such that $f(U)\subseteq V$. Since $f$ is uniformly continuous, there is a positive number $\delta$ such that if the distance between two points of $X$ is less than $\delta$, then the distance between their images under $f$ is less than $\varepsilon$. The point $x$ having property $(\ast)$, there exists a $\delta$-chain $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ covering $X$ such that $C_1 \subseteq U$.

Define $D_1 = f(C_1)$, $D_2 = f(C_2)$, and $D_{j+2} = f(C_{j+2}) \setminus f(\bigcup_{k=1}^{j} C_k)$ for $j = 1, 2, \ldots, m \leq n$ continuing as long as the $D_j$'s remain nonempty. We will show that $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$
forms an $\varepsilon$-chain in $X$. First, since $f$ is an open mapping and all $C_k$'s are open sets, each $D_k$ is clearly open, and, by the definition of $\delta$, has diameter less than $\varepsilon$. Next, since $/is an open mapping and all $C_k$'s are open sets, each $D_k$ is clearly open, and, by the definition of $J$, has diameter less than $\varepsilon$.

Next, since the $C_k$'s are links of the $\delta$-chain $\mathcal{C}$, for each $k$ we have $\mathcal{C}_k \subseteq C_{k-1} \cup C_k \cup C_{k+1}$, whence $\bigcup_{k=1}^{j+1} \mathcal{C}_k \subseteq \bigcup_{k=1}^{j} C_k$. Thus, for each $j$, the link $D_j$ contains $f(C_j) \setminus \bigcup_{k=1}^{j-1} f(C_k)$.

Therefore we see that $X = f\left(\bigcup_{j=1}^{n} C_j\right) = f(C_1) \cup (f(C_2) \setminus f(C_1)) \cup (f(C_3) \setminus (f(C_1) \cup \cup f(C_2))) \cup \ldots \cup f((C_n) \setminus \bigcup_{k=1}^{n-1} f(C_k)) \subseteq D_1 \cup D_2 \cup \ldots \cup D_m$. Thus we have proved that $\mathcal{D}$ covers $X$. Finally, it is clear that $D_j$ does not intersect $D_k$ if $j$ and $k$ differ by more than one. Thus by connectivity of $X$ it follows that, for each $j$, $D_j$ intersects $D_{j+1}$.

Therefore $\mathcal{D}$ is an $\varepsilon$-chain covering $X$ such that $f(C_1) = D_1 \subseteq \mathcal{C}$. Thus we have proved that $\mathcal{D}$ has property $(\star)$.

Let $E_1$ be an end link of a 1-chain covering $X$ such that $E_1$ contains a point $p_1$ of $X$. Since $p_1$ has property $(\star)$, there is an end link $E_2$ of a 1/2-chain covering $X$ such that $E_2 \subseteq E_1$ and a point $p_2 \in X$ is in $E_2$. Also, there is an end link $E_3$ of a 1/3-chain covering $X$ such that $E_3 \subseteq E_2$ and $E_3$ contains a point $p_3$ of $X$. Similarly we obtain $E_4, E_5, \ldots$ Then the point $p$ which is the intersection of all links $E_i$ for $i = 1, 2, \ldots$ is an end point of $X$.

Since under an open mapping the image of each end point of $X$ is again an end point (see [6], Corollary 1.2, p. 260) and since $X$ is homogeneous with respect to open mappings, we see that all points of $X$ are end points. This fact characterizes the pseudo-arc up to homeomorphisms (see [3], Theorem 16, p. 662 and Theorem 1, p. 653; cf. [2], Theorem 1, p. 44). Thus the proof is finished.

**COROLLARY.** $X$ nondegenerate chainable continuum is the pseudo-arc if and only if it is homogeneous with respect to the class of open mappings.

In fact, one way is shown by the above theorem. The opposite way follows from the homogeneity of the pseudo-arc with respect to homeomorphisms (see [1], Theorem 13, p. 740; cf. [5]).

It is shown in [4] that if a nondegenerate chainable continuum $X$ is homogeneous (with respect to the class $\mathcal{H}$ of homeomorphisms), then $X$ is the pseudo-arc. The above theorem states a similar fact for the class $\mathcal{M}$ of open mappings. Thus the following problem seems to be natural:

**Problem.** Find the largest class $\mathcal{M}$ of mappings with the property that if a nondegenerate chainable continuum $X$ is homogeneous with respect to $\mathcal{M}$, then $X$ is the pseudo-arc.

**REFERENCES**


Я. Е. Харатоник, Характеристика псевдодуги

Содержание. Доказывается, что невырожденный змеевидный континуум является превдо-
dугой тогда и только тогда, когда он однороден относительно открытых отображений-