Local expansions on some curves

by

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Let $X$ be a metric space with the metric $q$. The statement that the mapping $f: X \to X$ is a local expansion means that $f$ is continuous and that for each point $x \in X$ there is an open set $U$ containing $x$ and a real number $M > 1$ so that if $y$ and $z$ belong to $U$, then $q(f(y), f(z)) \geq M \cdot q(y, z)$.

This paper is motivated by a short note of Rosenholtz [2] who studied local expansions on metric continua and proved that every open local expansion of a metric continuum onto itself has a fixed point. Showing that openness of the mapping is essential in the result, he has constructed a fixed point free local expansion on the union of three circles. On the other hand, it is easy to point some particular examples of metric continua which do not admit any local expansions onto themselves at all. Such is e.g. the unit segment of reals. Therefore it is very natural to ask which metric continua $X$ admit a local expansion of $X$ onto $X$, i.e., for which metric continua $X$ there exist a metric $q$ that is equivalent to the original one given on $X$, and a surjection $f: X \to X$ satisfying the conditions of the above definition.

This paper is very far from solving the problem, however, it is a contribution to the attempt to find such a criterion for some special curves (one-dimensional continua). Namely a partial answer is given by showing a necessary and sufficient condition of the existence of local expansions on linear graphs (one-dimensional connected polytopes) equipped with a convex metric. Recall that a metric $q$ on $X$ is said to be convex if for each two distinct points $x$ and $y$ of $X$ there exists a point $z \in X$ different from $x$ and $y$ and such that $q(x, y) = q(x, z) + q(z, y)$. It is known that every linear graph $X$ can be remetrized by a convex metric (it was shown by Borsuk [1], Sections 6 and 7, p. 329–332 that for every polytope $P$ of dimension less than or equal to 2 there exists a metric $q$ such that $(P, q)$ is a convex metric space).

The first and the third authors have obtained the following results, the proofs of which, being rather long and technical, will be published elsewhere. Let a linear graph equipped with a convex metric be given. If the graph
contains a point of the maximal order (in the sense of Menger–Urysohn) which does not disconnect the graph, then a local expansion does exist. If every point of the maximal order in the graph disconnects the graph, but if there is one that closure of every component of its complement contains a simple closed curve, then also the graph admits a local expansion. Invertedly, if there is a local expansion on a linear graph metrized by a convex metric, then either there is a point of the maximal order which does not disconnect the graph, or there is one which disconnects the graph in such a way that the closure of no component of this complementary is a tree. Therefore a characterization of linear graphs that admit local expansions is obtained.

In particular, it follows from the results above that no tree admits a local expansion. But a stronger result has been obtained by the second author. Namely, let us call a mapping \( f : X \to X \) a generalized local expansion if all conditions of the definition of a local expansion are satisfied except for this one change: the real number \( M \) is greater or equal to 1. Then the result is the following. Every generalized local expansion on a dendrite (i.e. a locally connected continuum containing no simple closed curve) metrized by a convex metric is an isometry.

In connection with this the following two questions seem to be natural. 1. Can the result be extended in some way to a larger class of acyclic curves, e.g. to dendroids? 2. Which continua \( X \) have equivalent metrics \( q_x \) with the property that any generalized local expansion of \( X \) onto \( X \) (with respect to \( q_x \)) is an isometry?

References