Local expansions on graphs

by

J. J. Charatonik and S. Miklos (Wroclaw)

Abstract. A necessary and sufficient condition is proved under which a linear graph admits a local expansion.

§ 1. Introduction. This paper is motivated by a short note of Rosenholtz [11] who studied local expansions on metric continua and proved that every open local expansion on a metric continuum onto itself has a fixed point. Showing that openness of the mapping is essential in the result, he has constructed a fixed point free local expansion on the union of three circles ([11], p. 3 and 4). On the other hand it is easy to point some particular examples of metric continua which do not admit any local expansions onto themselves at all. Such is e.g. the unit segment of reals. Therefore it is very natural to ask about a criterion under which there exists a local expansion of a given metric continuum onto itself:

PROBLEM. Characterize metric continua $X$ which admit a local expansion of $X$ onto itself.

This paper does not answer the problem, however, it is a contribution to the attempt to find such a criterion for some special continua. Namely a partial answer is given by showing a necessary and sufficient condition of the existence of local expansions on linear graphs (i.e. one-dimensional connected polytopes) equipped with a convex metric.

§ 2. Definitions and preliminaries. Let $q$ be a metric on a metric space $X$. The statement that the mapping $f : X \to X$ is a local expansion means that $f$ is continuous and that for each point $x \in X$, there is an open set $U$ containing $x$ and a real number $M > 1$ so that if $y$ and $z$ belong to $U$, then

\[ q(f(y), f(z)) \leq Mq(y, z) \]

(see [11], p. 1). We say that a metric space $X$ admits a local expansion if there exist a metric $q$ that is equivalent to the original one given on $X$, and a surjection $f : X \to X$ satisfying the conditions of the above definition.

Let a metric space $X$ with a metric $q$ be given. Let $x, y, z \in X$. The point $z$ is said to lie between the points $x$ and $y$ provided that $q(x, y) = q(x, z) + q(z, y)$ (cf. [3], p. 317). The point $z$ is said to be a center of the pair $x, y$ provided that $q(x, z)$

References

We say that the set $A \subset X$ is linear if there exists an isometry $\varphi : A \rightarrow R$ of $A$ into the space $R$ of all real numbers, i.e. $\varphi(x) = [\varphi(x) - \varphi(y)]$ for every $x$, $y \in A$ (cf. [6], p. 183). An arc means any subset of $X$ that is homeomorphic to the closed unit segment $[0,1]$ of reals. An arc contained in $X$ is said to be a metric segment if it is linear.

A metric space $X$ is said to be convex in the well-known sense of Menger [7], p. 81) provided that for each two distinct points $x$ and $y$ of $X$ there exists a point $z \in X$ different from $x$ and $y$, which lies between $x$ and $y$. It was proved by Menger ([7], p. 89; see also Aronszajn [2]; cf. [3], p. 41) that in every complete convex metric space $X$ each two points of $X$ can be joined by a metric segment. Moreover, it is known (cf. [9], 2.3, p. 116) that a complete metric space is convex if and only if for every $x, y \in X$ and for every $t$, where $0 \leq t \leq 1$, there exists at least one point $z \in X$ such that

$$g(z, x) = (1-t)g(x, y) \text{ and } g(z, y) = t\cdot g(x, y).$$

If a metric space $X$ equipped with a metric $g$ is convex, then the metric $g$ is called a convex metric.

Let $X$ be a metric space and let $n$ be a positive integer. A point $p \in X$ is said to be of order less than or equal to $n$ provided that if $W$ is any open neighborhood of $p$, there exists an open neighborhood $U$ of $p$ with $U \subset W$ and such that the boundary $\partial U$ of $U$ consists of at most $n$ points. Clearly this amounts to saying that for every positive number $\varepsilon$ there exists a neighborhood $U$ of $p$ of diameter less than $\varepsilon$ whose boundary has at most $n$ points. If $p$ is of order less than or equal to $n$ but not of order less than or equal to $n-1$, then it is said to be of order $n$ (cf. [12], Definition on p. 48 and Note on p. 50), in writing $ord_X X = n$. In particular, if $ord_X X = 1$, then $p$ is called an end point of $X$. We admit that the maximum exists.

A connected set $X$ is said to be semi-locally connected if for each its point $x$ and for each number $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ in $X$ of diameter less than $\varepsilon$ such that $U \cap X$ has only a finite number of components ([12], p. 19).

If $X$ is a connected space and $p$ is a point of $X$ such that $X \setminus \{p\}$ is not connected, then $p$ is called a cut point of $X$ ([12], p. 41). A continuum means a compact connected metric space. A simple closed curve is defined as a continuum homeomorphic to the unit circumference, i.e., to the set of all complex numbers $z$ with $|z| = 1$.

We shall use here some notions from Whyburn's cyclic element theory (see e.g. [12], Chapter IV, p. 64–87). For the reader's convenience we recall here the basic concepts needed in the sequel.

Two points $a$ and $b$ of a connected set $X$ are said to be conjugate provided no point separates $a$ and $b$ in $X$. If $p \in X$ is neither a cut point nor an end point of a connected set $X$, the set consisting of $p$ and of all points of $X$ conjugate to $p$ is called a simple link of $X$ (see [12], p. 64). Simple links are closed subsets of $X$ ([12], 1.4, p. 65). In case when $X$ is a semi-locally connected continuum, simple links coincide with non-degenerate or true cyclic elements of $X$ (see [12], p. 66). A collection $\Gamma$ of sets is said to be coherent if for each proper subcollection $\Gamma'$ of it, an element of $\Gamma'$ intersects an element of $\Gamma \setminus \Gamma'$ (see [8], p. 46). Let a semi-locally connected continuum $X$ be given. A link of $X$ means the union of a maximal coherent family of simple links of $X$ (here "maximal" means that the coherent family is not a proper subfamily of another coherent family of simple links of $X$); compare this definition with one of an $H$-set in ([12], p. 72). A link $L$ of a semi-locally connected continuum $X$ will be called an end link if the set $X' \cap L$ is linear.

We recall that a connected set $X$ is a linear graph provided it is the union of a finite set $V$ of points, called vertices and a finite number of arcs, called edges, so that the end points of each edge belong to $V$ and that for every two edges their intersection is contained in $V$ (i.e. every two edges have at most end points in common).

If $v \in V$ and $E$ is an edge such that $v \in E$, then we say that $E$ is the edge incident to the vertex $v$. Note that the above definition of a linear graph is equivalent to one given in ([12], p. 182); however the edges in ([12] are defined as open free arcs in $X$ (i.e. arcs without their end points, with the property that each such an arc without its end points is an open subset of the whole space $X$). But it is more convenient to us to understand each edge of a linear graph as an arc, i.e., together with its end points. It is known (see [12], p. 182) that a continuum $X$ is a linear graph if and only if every point of $X$ is of some finite order and almost all points of $X$ are of order less than or equal to 2. Thus it follows that every linear graph is a regular curve, whence we conclude by Whyburn's classification of curves ([12], p. 99 and Corollary (13.21), p. 20) that every linear graph is a semi-locally connected continuum.

A linear graph with a fixed set of vertices is said to be simple if there is at most one edge joining the same pair of vertices (cf. [1], p. 20).

Remark. Let us observe that if a linear graph is given with a convex metric and with a set of its vertices, then we can enlarge the set of vertices taking as new vertices the centers of the pair of end points for every edge, to obtain the same linear graph $X$ (with the same convex metric) which is a simple one. Thus we can assume without loss of generality that the considered linear graphs with convex metric are simple.

An edge of a linear graph is called a bridge if the removal of its interior disconnects the graph (cf. [1], p. 67). In other words, an edge $ab$ of a linear graph $X$ is a bridge provided the set $X \setminus (ab \cup \{a, b\})$ is not connected. In particular, if an edge $E$ is incident to an end point of a linear graph, then $E$ is a bridge. Let us observe that every point of a link $L$ of a linear graph $X$ lies in some simple closed curve contained in $L$. Thus every link of a linear graph is the union of a finite number of simple closed curves, and hence it can be defined as a maximal subgraph of $X$ containing no bridge (thus no end point) of $X$. A linear graph containing no simple closed curve is called a tree. A (finite) sequence $C = (E_1, E_2, \ldots, E_n)$ of edges of a linear graph is said to
be a chain provided \( E_i \cap E_{i+1} \) is a one-point set composed of the common end-
vertex of \( E_i \) and \( E_{i+1} \) for \( i = 1, 2, ..., k-1 \). If \( x \) and \( y \) are vertices such that \( x \in E_i \setminus E_{i-1} \) and \( y \in E_{i+1} \setminus E_i \), then we say that the chain \( C \) joins \( x \) and \( y \), or that \( C \) is a chain from \( x \) to \( y \). Given a finite number \( r \) of chains \( C = \{ E_1', E_1'', ..., E_r' \} \) from \( x \) to \( y \) respectively, with \( y_j = x_{j+1} \) for \( j = 1, 2, ..., r-1 \), we can join them together in one chain \( C \) called the join of chains \( C \) which is defined as a sequence composed of all terms of \( C_i \) ordered linearly with respect to \( j \):

\[
C = \{ E_1', E_2', ..., E_1'', E_1''', ..., E_r' \}
\]

If end vertices of an edge are ordered (i.e. if one of them is distinguished as the beginning and the other as the ending of the edge), then the edge is said to be directed (or oriented). A linear graph is called directed if all its edges are directed (cf. [1], p. 64). By a directed path we mean a chain whose edges are oriented in such a manner that the ending of the edge \( E_i \) coincides with the beginning of \( E_{i+1} \), where \( i = 1, 2, ..., k-1 \). Given a directed path \( p_i \), i.e., a sequence of directed edges \( E_1, E_2, ..., E_k \) with the property mentioned above, we define a sequence \( \sigma(P) \) of consecutive vertices ordered along \( P \), that is

\[
\sigma(P) = \{ x_1, x_2, ..., x_{i+1} \}
\]

where \( x_i \) is the beginning, and \( x_{i+1} \) is the ending of \( E_i \), for every \( i = 1, 2, ..., k \).

If the edges of a directed linear graph are oriented so that each vertex is accessible from each of the other vertices along a directed path, then the graph is called strongly connected ([1], p. 67).

The following lemma is known (see [1], Problem 18, p. 69):

**Lemma 1.** Directions can be assigned to the edges of an arbitrary linear graph without bridges to make it strongly connected.

We shall use this lemma to prove the next one.

**Lemma 2.** If a simple linear graph \( X \) contains no end point, and if \( x \) and \( y \) are vertices of \( X \) (not necessarily distinct), then there is a chain from \( x \) to \( y \) containing all the edges of \( X \).

**Proof.** The proof runs by induction with respect to the number \( n \) of links contained in \( X \). To show the conclusion in case \( n = 1 \), we let \( u \), \( v \), \( a \), \( b \) be distinct vertices in \( X \) and an arbitrary finite sequence of all vertices of \( X \) such that \( u \) is the first and \( v \) is the last term of the sequence:

\[
x = u, v_1, v_2, ..., v_{2j-1}, v_{2j}, ..., v_{2m-1}, y
\]

and, for each \( j = 1, 2, ..., m \) the pair of consecutive vertices \( (v_{2j-1}, v_{2j}) \) is composed of the beginning and the ending of the directed edge \( E_j \) (here \( m \) denotes the number of all edges in \( X \)). Now it is enough to consider a sequence of \( 2m-1 \) directed paths from \( v_i \) to \( v_{i+1} \) (where \( i = 1, 2, ..., 2m-1 \)) such that

(5) the directed path from \( v_{2j-1} \) to \( v_{2j} \) is a one-term sequence, necessarily composed of the directed edge \( E_j \).

This condition (5) guarantees that each edge of \( X \) does occur indeed in the chain to be defined. We arrange all edges of \( X \) in the sequence

\[
E_1, E_2, ..., E_{2m-1}, E_{2m}
\]

and, for each \( i = 1, 2, ..., 2m-1 \) the directed path from \( v_i \) to \( v_{i+1} \) is defined as the last edge of the directed path from \( v_{2j-1} \) to \( v_{2j} \). By a directed path we mean a chain whose edges are oriented in such a manner that the ending of the edge \( E_i \) coincides with the beginning of \( E_{i+1} \), where \( i = 1, 2, ..., k-1 \). Given a directed path \( p_i \), i.e., a sequence of directed edges \( E_1, E_2, ..., E_k \) with the property mentioned above, we define a sequence \( \sigma(P) \) of consecutive vertices ordered along \( P \), that is

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\[
x = u, v_1, v_2, ..., v_{2j-1}, v_{2j}, ..., v_{2m-1}, y
\]

and, for each \( j = 1, 2, ..., m \) the pair of consecutive vertices \( (v_{2j-1}, v_{2j}) \) is composed of the beginning and the ending of the directed edge \( E_j \) (here \( m \) denotes the number of all edges in \( X \)). Now it is enough to consider a sequence of \( 2m-1 \) directed paths from \( v_i \) to \( v_{i+1} \) (where \( i = 1, 2, ..., 2m-1 \)) such that

(5) the directed path from \( v_{2j-1} \) to \( v_{2j} \) is a one-term sequence, necessarily composed of the directed edge \( E_j \).
denote by \( P(S; p) \) one of the two possible directed paths from \( p \) to \( p' \) such that the union of all its edges is equal to \( S \).

We shall need also the following

**Lemma 3.** Let a simple linear graph \( X \) be given which is not a simple closed curve. Let \( c \in X \) be a point of the maximal order in \( X \), i.e., \( \text{ord}_X(c) = \text{ord}_X, \) such that for every component of \( X - \{c\} \) its closure contains a simple closed curve. Then for any two edges \( ca \) and \( cb \) (not necessarily different) there exists a directed path from \( c \) to \( c \) in such that \( ca \) (directed from \( c \) to \( a \)) is the first, and \( bc \) (directed from \( b \) to \( c \)) is the last edge of \( P \). Moreover, the number of edges of \( P \) can be arbitrarily large.

**Proof.** Let \( v \) be an arbitrary natural number. First, assume that \( a \) and \( b \) both lie in the same component \( K \) of \( X - \{c\} \). We consider the following two cases:

(i) \( a \neq b \), and (ii) \( a = b \).

In case (i) there exists a nondegenerate arc \( ab \in K \). Denote by \( P_0 \) the join of \( P(c; a, c), P(ab; a, b) \) and \( P(bc; b, c) \), and define \( P \) as the join of the directed paths \( P_1, P_2, \ldots, P_v \), where \( P_i = P_0 \) for \( i = 1, 2, \ldots, v \).

In case (ii) we discuss the three possibilities:

1° \( \text{ord}_K(c) = 1; \) 2° \( \text{ord}_K(c) = 2; \) 3° \( \text{ord}_K(c) \geq 3. \)

If 1° holds, take a simple closed curve \( S \subseteq K \) and an arc \( cd \) such that \( ca = cd \) and \( cd \cap S = \{d\} \). Put \( P_0 = P(cd; c, d) \) and \( P_i = P(S; d) \) for \( i = 1, 2, \ldots, v \), and define \( P \) as the join of \( P_0, P_1, P_2, \ldots, P_v, P_v^{-1} \).

If 2° holds, let \( ca \) and \( cb \) be the two different edges in \( K \) incident to the vertex \( c \). Then \( a \neq c \) and, as in case (i) above, there is a directed path \( P_0 \) from \( c \) to \( c \) having \( ca \) as its first and \( x \) as its last edge. Since \( X \) is not a simple closed curve, we have \( \text{ord}_X(c) > 2 \), and by \( \text{ord}_K(c) = 2 \) we see that there is another component, say \( K' \), of \( X - \{c\} \), and by hypothesis there is a simple closed curve \( S' \) in \( K' \). Let \( cd' \) be an arc in \( K' \), joining \( S' \) with \( S' \), i.e., such that \( cd' \cap S' = \{d'\} \) (this arc can be degenerate if \( c \in S' \)). Let \( P' \) be the join of \( P(cd'; c, d'), P(S'; d') \) and \( P(cd'; d', c) \). Define \( P \) as the join of the directed paths \( P_1, P_2, \ldots, P_v, P', P', P_0^{-1} \), where \( P_i = P_0 \) for \( i = 1, 2, \ldots, v \).

If 3° holds, then there are three different edges \( ca, cb \), and \( cy \) in \( K \). Thus the vertices \( a, x \), and \( y \) are distinct and, as in case (i) above, there is a directed path \( P_0 \) from \( c \) to \( c \) having \( ca \) (directed from \( c \) to \( a \)) as its first and \( x \) (directed from \( x \) to \( c \)) as its last edge, and a directed path \( P' \) from \( c \) to \( c \) having \( cy \) (directed from \( c \) to \( y \)) and \( ca \) (directed from \( c \) to \( a \)) as its first and last edges correspondingly. Putting \( P_i = P_0 \) for \( i = 1, 2, \ldots, v \), we define \( P \) as the join of \( P_1, P_2, \ldots, P_v, P' \).

-A second, assume that \( a \) and \( b \) lie in two different components \( K \) and \( K' \) of \( X - \{c\} \). Using the same argumentation as previously one can find two directed paths \( P_0 \) and \( P' \), both from \( c \) to \( c \), such that \( ca \) (oriented from \( c \) to \( a \)) is the first edge of \( P_0 \) and all edges of \( P_0 \) are contained in \( K \), and that \( bc \) (oriented from \( b \) to \( c \)) is the last edge of \( P' \), and all edges of \( P' \) are in \( K' \). Let \( P_i \) for \( i = 1, 2, \ldots, v \), be the join of \( P_0 \) and \( P' \). Then we define \( P \) as the join of \( P_1, P_2, \ldots, P_v \). The proof of the lemma is finished.

§ 3. **Standard mappings.** It is known that every linear graph \( X \) can be remetrized by a convex metric: it was shown by Borsuk [4], Section 6 and 7, p. 329–332, that for every polytope \( P \) of dimension less than or equal to 2 there exists a metric \( q \) such that \((P, q)\) is a convex (and locally strongly convex) metric space.

Let \( B_1 \) and \( B_2 \) denote two closed intervals of reals. A surjection \( g : B_1 \rightarrow B_2 \) is called affine if there are reals \( \alpha \neq 0 \) and \( \beta \) such that \( g(x) = \alpha x + \beta \) for every \( x \in B_1 \). For \( k = 1 \) and 2 let \( X_k \) be a convex metric space and let \( A_k \) be a metric segment contained in \( X_k \). Thus there is an isometry \( \varphi_k : A_k \rightarrow B_k \), where \( B_k \) is a closed interval of reals. We shall say that a surjection \( f : A_1 \rightarrow A_2 \) is linear if there exists an affine surjection \( g : B_1 \rightarrow B_2 \) such that \( f = g \circ \varphi_1 \). We say that a mapping \( f : X_1 \rightarrow X_2 \) of \( X_1 \) into \( X_2 \) is piecewise linear if there is a partition of the domain \( X_1 \) into a finite number \( n \) of metric segments \( A'_j \), i.e., \( X_1 = \{A'_j | j = 1, 2, \ldots, n\} \) such that \( f(A'_j) \) is a metric segment in \( X_2 \) and that the restrictions \( f|A'_j \) are linear for every \( f = 1, 2, \ldots, n \). Obviously each piecewise linear mapping is continuous.

Let \( Y \) be a complete convex metric space with a metric \( q_2 \). Consider a finite sequence of \( n+1 \) not necessarily distinct points

\[ b_0, b_1, \ldots, b_n \]

in \( Y \), and for every \( j = 0, 1, \ldots, n-1 \) take an arc \( b_j b_{j+1} \), which is a metric segment. Further, let a metric segment \( pg \) be given in a complete convex metric space \( X \) with a metric \( q_1 \), and put

\[ \mu = \left[ \sum_{j=0}^{n-1} q_2(b_j, b_{j+1}) \right] q_1(p, q). \]

We define \( n+1 \) different points \( p = a_0, a_1, \ldots, a_n = \varphi \) in the metric segment \( pg \) putting for every \( k = 1, 2, \ldots, n \)

\[ \nu q_2(p, a_j) = \sum_{j=0}^{k-1} q_2(b_j, b_{j+1}). \]

Thus we have \( \mu q_1(a_j, a_{j+1}) = q_2(b_j, b_{j+1}) \) for every \( j = 0, 1, \ldots, n-1 \). The mapping

\[ f : pg \rightarrow \{b_j b_{j+1} | j = 0, 1, \ldots, n-1\} \]

defined in such a way that the partial mapping

\[ f|_{a_j a_{j+1}} : a_j a_{j+1} \rightarrow b_j b_{j+1} \]

is linear, with

\[ f(a_j) = b_j \]

for every \( j = 0, 1, \ldots, n-1 \), will be called the **standard mapping** associated with sequence (7). In particular we see that for the standard mapping \( f \) just defined we have

\[ f(p) = b_0 \quad \text{and} \quad f(\varphi) = b_n. \]

\[ 1 \]
The number $\mu$ defined by (8) will be called the coefficient of the standard mapping $f$, and will be denoted by $\tau(f)$.

It is manifestly evident that the standard mapping $f$ is piecewise linear, and thereby continuous. Further, it follows from the linearity of the partial mapping (11) that, for every $j = 0, 1, \ldots, n-1$,

$$\text{if } y, z \in a_ja_{j+1}, \text{ then } e_\delta(f(y), f(z)) = \mu_1(y, z).$$

Hence we have

**STATEMENT 1.** Let $X$ and $Y$ be simple linear graphs with convex metrics $q_1$ and $q_2$ respectively and let an edge $pq$ be given in $X$. Let $f: pq \to Y$ be a standard mapping associated with sequence (7) of distinct points $b_j$ such that for every $j = 1, 2, \ldots, n-1$ the points $b_j$ lies between $b_{j-1}$ and $b_{j+1}$, and let $\mu = \tau(f)$. Then for every point $x \in pq \setminus \{p, q\}$ there exists a neighborhood $U$ of $x$ such that if $y, z \in U$, then

$$e_\delta(f(y), f(z)) = \mu_1(y, z).$$

Indeed, if $x$ is an interior point of a segment $a_ja_{j+1}$ for some $j = 0, 1, \ldots, n-1$, where the points $a_i$ are defined by equality (9), then taking $U = a_ja_{j+1} \setminus \{a_j, a_{j+1}\}$ we see that $U$ is a neighborhood of $x$ and that the conclusion follows from (14). If $x = a_j$ for some $j = 1, 2, \ldots, n-1$, then take $U = a_ja_{j+1} \setminus \{a_j, a_{j+1}\}$. If both $y$ and $z$ are in $a_ja_{j+1}$ or in $a_ja_{j+1}$, the argumentation for (15) is the same as previously. If $y \in a_ja_{j+1}$ and $z \in a_{j+1}a_{j+2}$, then $f(y) \in b_{j+1}b_j$ and $f(z) \in b_jb_{j+1}$. Since $b_j$ lies between $b_{j+1}$ and $b_{j+2}$ and since $b_{j+1} \neq b_{j+2}$ by assumption, we conclude from (19), 22, p. 116 that $b_{j+1}b_j \cup b_jb_{j+1}$ is a metric segment. Hence we have by (12) and (14) that

$$e_\delta(f(y), f(z)) = \mu_1(f(y), b_j) + e_\delta(b_j, f(z)) = e_\delta(f(y), f(a_j)) + e_\delta(f(a_j), f(z)) = \mu_1(y, a_j) + \mu_2(a_j, z) = \mu_1(y, z).$$

and (15) follows. Let

**STATEMENT 2.** Let $X$ and $Y$ be simple linear graphs with convex metrics $q_1$ and $q_2$ respectively. Given a vertex $p \in X$ of a finite order $n$, let $pq_1, pq_2, \ldots, pq_n$ be edges in $X$ incident to the point $p$. For every $i = 1, 2, \ldots, n$ let $f_i : pq_i \to Y$ be a st: n lard mapping with the coefficient $\tau(f_i) = \mu_1 > 1$ and such that there exists a point $q_i \in Y$ with $f_i(p) = q_i$. Assume further that for every $i = 1, 2, \ldots, n$ there is an edge $rr_i$ in $Y$ such that $rr_i < f_i(pq_i)$ and that $rr_i$ are distinct for distinct indices $i$. Let a mapping $f: \bigcup \{pq_i\}_{i = 1, 2, \ldots, n} \to Y$ be defined by

$$f|pq_i = f_i.$$

Then there is an open neighborhood $U$ of $p$ and a real number $M > 1$ such that

$$e_\delta(f(y), f(z)) \geq M \cdot \mu_1(y, z).$$

**Proof.** For $i = 1, 2, \ldots, n$ let $a_i$ be the nearest to $p$ point of the edge $pq_i$ such that $f_i(a_i) = r_i$. Let $U$ be the component of the open set $X \setminus \{a_i\}_{i = 1, 2, \ldots, n}$ which contains the point $p$. In other words, we put $U = \bigcup \{pq_i\}_{i = 1, 2, \ldots, n}$. Hence $U$ is a neighborhood of $p$ in $X$. Further put $M = \min \{\mu_i : i = 1, 2, \ldots, n\}$. Since $\mu_i > 1$, for every $i$, we have $M > 1$. Take two arbitrary points $y$ and $z$ of $U$ and consider the case of $y$ and $z$ in the same are $pq_i$ for some $i = 1, 2, \ldots, n$. Then by (14) we have that

$$e_\delta(f(y), f(z)) = e_\delta(f(y), f(a_i)) = \mu_1(y, z) \geq M \cdot \mu_1(y, z),$$

and (17) follows. Secondly, let $y \in pq_i$ and $z \in pq_j$, where $i \neq j$. Since $rr_i$ and $rr_j$ are distinct metric segments by hypothesis, we see that 2.2 of [9], p. 116 can be applied, and thus $rr_i \cup rr_j$ is a metric segment. It follows from assumptions that $f(y) = f_i(a_i) \in f(pq_i)$ and $f(z) = f_j(a_j) \in f(pq_j)$, and since $f(p) = r$, we have by (14) and (16) that

$$e_\delta(f(y), f(z)) = e_\delta(f_i(r_i), r_i) + e_\delta(r_i, f(z)) = e_\delta(f(y), f(p)) + e_\delta(f(p), f(z)) = e_\delta(f(y), f(p)) + e_\delta(f_j(f(p)), f(z)) = \mu_1(y, p) + \mu_1(p, z) \geq M \cdot \mu_1(y, z),$$

and thereby (17) follows in this case, too. Thus the proof is complete.

**§ 4. Existence of local expansions-sufficiency.** Now let a linear graph $X$ endowed with a convex metric $q$ be given. We can assume without loss of generality that (i) every edge of $X$ is a metric segment of $X$, and that (ii) every such edge is uniquely determined by two its end points (i.e. that there is at most one edge between the two vertices). Indeed, both (i) and (ii) can be realized simply by enlarging of the number of vertices in a proper way, e.g. by taking as new vertices the centers of the pair of the end points of every edge. In other words condition (ii) says that the linear graph $X$ is simple (compare the remark in § 2). Since now, in the present paragraph, $X$ always will mean a convex metric space being a simple linear graph that satisfies condition (i).

Now we are going to prove some sufficient conditions for the existence of a local expansion on such an $X$. The existence depends heavily on the structure of the linear graph. Namely if the graph contains a point of the maximal order which does not disconnect the graph, then a local expansion does exist. If every point of the maximal order in the graph disconnects the graph, but if there is one such that the closure of every component of its complement contains a simple closed curve, then also the graph admits a local expansion.

Invectedly, it will be shown in sixth paragraph that if there exists a local expansion on a linear graph metrized by a convex metric, then either there is a point of the maximal order which does not disconnect the graph, or there is one which
disconnects the graph in such a way that the closure of no component of its com­
plementary is a tree. Therefore a characterization of linear graphs that admit local
expansions will be obtained.

THEOREM 1. Let a simple linear graph \( X \) metrized by a convex metric be given.
If there is a point \( c \in X \) of the maximal order in \( X \), i.e.,
\[
\text{ord}_c X = \text{ord} X
\]
such that for every component of \( X \setminus \{c\} \) its closure contains a simple closed
curve, then there exists a local expansion \( \phi: X \to X \) of \( X \) onto itself.

Proof (\( \dagger \)). Since the unit circle \( \{z: |z| = 1\} \) admits a local expansion, e.g. \( z \to z^2 \),
we may assume that \( X \) is not a simple closed curve. To describe the local expansion \( \phi \)
mentioned in the conclusion of the theorem we distinguish some subsets of the
graph \( X \). Every of them will be the union of some edges of \( X \).

The set \( X_1 \) is defined as the union of all end edges of \( X \): an edge \( E \) of \( X \) is contained
in \( X_1 \) if and only if there is an end point of \( X \) which belongs to \( E \). Note that
we can assume without loss of generality that
\[
(19) \quad \text{all edges of } X \text{ contained in } X_1 \text{ are disjoint.}
\]
In fact, take two edges \( p_1q_1 \) and \( p_2q_2 \) contained in \( X_1 \), where \( p_1 \) and \( p_2 \) are
end points of \( X \). If \( q_1 = q_2 \), i.e., if these edges have a common vertex \( q \), then we take
centers \( r_1 \) and \( r_2 \) of the pairs \((p_1, q)\) and \((p_2, q)\) respectively as some new vertices of \( X \).
Such an improvement does not violate conditions (i) and (ii) assumed on \( X \), and it
let us to consider only the new edges \( p_1r_1 \) and \( p_2r_2 \) as contained in \( X_1 \). Obviously
they are disjoint, and therefore we may assume that the set \( X_1 \) satisfies (19) indeed.

Observe that if the graph \( X \) has no end point, then the set \( X_1 \) is empty by
definition.

Further, define \( X_2 \) as the union of all end edges \( E \) of \( X \) such that \( E \) is contained
either in some link of \( X \) or in some arc whose end points are in two different links
of \( X \). Thus \( X_2 \) is a subgraph of \( X \) which contains no end point of itself. Note that \( X_2 \)
can never be empty, because \( X \) contains a simple closed curve by assumption, and
every edge of \( X \) which lies in a simple closed curve is contained in \( X_2 \). Apart from this
we have \( c \in X_2 \). Finally put \( X_3 = X \setminus (X_1 \cup X_2) \). Thus
\[
(20) \quad X = X_1 \cup X_2 \cup X_3.
\]

Now we are going to define \( \phi: X \to X \). To describe \( \phi|X_1: X_1 \to X \) we define
it separately on each end edge of \( X \) the set \( X_1 \) is composed of. Let \( pq \) be such an
edge, where \( p \) denotes an end point of \( X \). We denote by \( T \) the closure of the
component containing the point \( p \) of the set \( X \setminus X_2 \), and by \( b \) the only point of the
intersection \( T \cap X_2 \). Let \( pc \) be an arbitrary but fixed arc from \( p \) to \( c \). Obviously we have

\( \dagger \) The authors thank Krzysztof Oniślanowski for valuable suggestions concerning a part
of this proof.

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\( pq < pb < pc \) and \( pb \cap X_2 = \{b\} \). Observe that it can happen \( q = b \) or \( b = c \) but
never \( q = c \) because otherwise we would have a component \( pc \setminus \{c\} \) of \( X \setminus \{c\} \) such
that its closure \( pc = pq \) would contain no simple closed curve, contrary to the
hypothesis. Thus
\[
(21) \quad pq \text{ is a proper subset of } pc.
\]

Take the sequence of all vertices of \( X \) lying in the arc \( pc \) and ordered from \( p \)
to \( c \):
\[
p, q, b, c.
\]
and define
\[
(22) \quad f|pq: pq \to pc
\]
as the standard mapping associated with sequence (22). It follows from (13) by the
definition of a standard mapping that
\[
(23) \quad f(p) = p \quad \text{and} \quad f(q) = c.
\]

Further, since \( f|pq \) maps \( pq \) onto \( pc \) by its definition, we conclude from (21) that
\[
(24) \quad \tau(f|pq) > 1 \quad \text{for every edge } pq \subseteq X_1.
\]
Since \( T \) is a tree by definition, with \( p \) and \( b \) as its end points, hence each point of
\( T \) lies in some arc \( p' \) for some end point \( p' \neq b \) of \( T \), whence we conclude
\[
(25) \quad T = f(X_1 \cap T).
\]
Taking the union over all end points \( p \) of \( X \) we infer from (26) that
\[
(26) \quad X_1 \cup X_3 = f(X_1).
\]
To describe the mapping \( f|X_2 \cup X_3 \) we need some preliminary constructions.
First, we arrange all vertices of \( X \) which are not end points of \( X \) in a (finite) sequence
\[
(27) \quad c = v_1, v_2, ..., v_n
\]
such that \( i_1 \neq i_2 \) implies \( v_{i_1} \neq v_{i_2} \). Second, we set up all edges of \( X \) contained in \( X_2 \)
or in \( X_3 \) in a sequence
\[
(28) \quad E_1, E_2, ..., E_j, ..., E_n
\]
such that \( i_1 \neq i_2 \) implies \( E_{i_1} \neq E_{i_2} \) and that \( E_1 \) is an edge incident to the vertex
\( c = v_1 \). Now we are going to assign to each edge \( E_j \) of (29) a directed path \( P_j \) from \( c \)
to \( c \) in \( X_2 \). This is done by the finite induction. Then we define \( f|E_j: E_j \to X \) as
the standard mapping associated with the sequence \( \sigma(P_j) \). This is realized in such a way
that not only the auxiliary partial mappings \( f|E_j \) are local expansions, but also the
resulting mapping \( f: X \to X \) is. For this purpose we use Statements 1 and 2. To
guarantee that all assumptions of Statement 2 are fulfilled, we define, again by the finite induction with respect to \( j \) and simultaneously with \( P_j \), a sequence of families \( M_j \) of some edges incident to the vertex \( c \) (here \( i = 1, \ldots, m \) and \( j = 0, 1, 2, \ldots, n \)), and we choose the directed paths in such a way that either the first or the last edge of the path just being defined is taken from out of the corresponding family \( M_j \) in a proper manner.

To begin with, for every \( i = 1, 2, \ldots, m \) we define \( M_0 \) as follows. If a vertex \( v_i \) from sequence (28) belongs to an end edge \( pv_i \) (i.e. if \( v_i \in X_1 \cap (X_2 \cup X_3) \)) we put \( M_0 = \{ cx \} \), where \( x \) is the last point in (22) that is different from \( c \). Otherwise we put \( M_0 = \varnothing \). Now recall that \( X_2 \) is a subgraph of \( X \) which contains no end point of itself, whence by Lemma 2 there is a chain \( C \) from \( c \) to \( c \) containing all edges of \( X_2 \). Choose an orientation on \( C \) to get a fixed directed path \( P \) from \( c \) to \( c \). Consider the first edge \( E_1 \) in (29) and let \( E_1 = cv_i \). If the last edge of \( P \) is not a member of \( M_0 \) then we define \( P_0 = P \). In the opposite case we take an arbitrary edge \( cv_i \) incident to \( c \) and distinct from \( E_1 \), we construct a directed path \( P' \) having \( cv_i \) (directed from \( c \) to \( y \)) as the first edge and \( cv_i \) (directed from \( v_i \) to \( c \)) as the last one (such a path does exist in \( X_2 \) by Lemma 3), and we define \( P_0 \) as the join of \( P \) and \( P' \). Next we define \( f(E_1) = f(v_i v_i : v_i v_i \rightarrow X) \) as the standard mapping associated with the sequence \( \sigma(P_0) \). Since the directed path \( P \) runs over all edges of \( X_2 \) by construction, the same holds for \( P_0 \), and it follows from the definition of \( f \) that

\[
\tau(f(E_1)) > 1.
\]

And since \( E_1 \) is a proper subset of \( X_2 \), we conclude from (30) that

\[
\tau(f(E_1)) > 1.
\]

Further, if \( A \) and \( B \) denote the first and the last edge of the directed path \( P_1 \) respectively, we put

\[
M_1^1 = M_0^1 \cup \{ A \}, \quad M_1^2 = M_0^2 \cup \{ B \} \quad \text{and} \quad M_1^3 = M_0^3
\]

for every \( i \in \{ 1, 2, \ldots, m \} \setminus \{ 1, r \} \).

Now let us fix some \( k \in \{ 1, 2, \ldots, n-1 \} \) and assume that for all \( j \in \{ 1, 2, \ldots, k \} \) there are defined directed paths \( P_j \) from \( c \) to \( c \) and standard mappings \( f(E_j) : E_j \rightarrow X \) associated with the sequences \( \sigma(P_j) \) such that

\[
\tau(f(E_j)) > 1.
\]

Furthermore, we assume that, for every pair of standard mappings defined on two edges incident to the same vertex, the assumptions of Statement 2 are satisfied, i.e., the following assertion is assumed:

\[
\text{For every vertex } v_i, \text{ where } i = 1, 2, \ldots, m, \text{ if two distinct edges } E_{j_1} \text{ and } E_{j_2} \text{ with } 1 \leq j_1 < j_2 \leq k \text{ are both incident to } v_i \text{ and considered as directed from } v_i \text{ (it means that the vertex } v_i \text{ is the common beginning of } E_{j_1} \text{ and } E_{j_2} \text{), then the first directed edges of directed paths } P_{j_1} \text{ and } P_{j_2} \text{ are distinct, too. Further, if } p_{v_i} \text{ is an edge contained in } X_1, \text{ then the first directed edge of } P_{j_1} \text{ and the element of } M_1^i \text{ also are distinct.}
\]

Moreover, we assume that

\[
\text{for every } i = 1, 2, \ldots, m \text{ the family } M_i^0 \text{ consists of all edges incident to } c \text{ which are either the first edges of a directed path } P_j \text{ for some } j < k \text{ if } E_j \text{ is considered as directed from } v_i \text{ or the last edges of } P_j, \text{ if the corresponding edge } E_j \text{ is considered as directed to } v_i, \text{ or } — \text{ finally } — \text{ belong to } M_0^i.
\]

Now we are ready to go to the next step, i.e. to define \( P_{k+1} \). Suppose \( E_{k+1} = v_i v_i \) where \( v_i \) and \( v_i \) are some vertices taken from sequence (28). Note that, by the induction hypotheses (34) and (35), we have card \( M_0^j \leq \operatorname{ord}_X - 1 < \operatorname{ord} X \) and similarly card \( M_0^j \leq \operatorname{ord} X \). Therefore we can find an edge \( c \) which is not in \( M_0^j \) and an edge \( c \) not in \( M_0^j \). By Lemma 3 there is a directed path \( P_{k+1} \) from \( c \) to \( c \) such that \( c \) (directed from \( c \) to \( a \)) is the first, and \( bc \) (directed from \( b \) to \( c \)) is the last edge of \( P_{k+1} \), and that the number of edges of this path can be arbitrarily large. This condition guarantees that for the mapping \( f(E_{k+1}) = f(v_i v_i : v_i v_i \rightarrow X) \) defined as the standard mapping associated with the sequence \( \sigma(f(E_{k+1})) \) we have

\[
\tau(f(E_{k+1})) > 1.
\]

Finally, put

\[
M_{k+1}^1 = M_0^1 \cup \{ ca \}, \quad M_{k+1}^2 = M_0^2 \cup \{ cb \} \quad \text{and} \quad M_{k+1}^3 = M_0^3
\]

for \( i = 1, 2, \ldots, m \setminus \{ \tau, r \} \).

Therefore the inductive procedure is finished, both for the families \( M_i \) (see (32), (35) and (37)) and for the directed paths \( P_j \), and so the mapping \( f \) is defined on each edge \( E_{k+1} \) of \( X_2 \cup X_3 \) with the property \( \tau(f(E)) > 1 \) (see (31), (33) and (36)), from which we see by (20) and (25) that

\[
\tau(f(E)) > 1 \quad \text{for each edge } E \text{ of } X.
\]

Let us recall that \( f(X_1) \) has already been defined, separately for each edge of \( X_1 \) (which are disjoint — see (19)), in such a way that each end point of \( X_1 \) is a fixed point and each vertex of \( X \) which is not an end point is mapped onto the point \( e \) (see (24)). Similarly every vertex in \( X_2 \cup X_3 \) is sent to \( c \) under \( f(X_2) \cup X_3 \). Thus mappings \( f(X_1) \) and \( f(X_2 \cup X_3) \) do agree on the common part \( X_1 \cap (X_2 \cup X_3) \) which is a subset of the set of vertices of \( X \), and it follows from (20) that \( f \) is well-defined on the whole \( X \). The continuity of \( f \) follows from continuity of every partial mapping \( f(E) \), where \( E \) is an edge of \( X \) (see [5], Theorem 9.4, p. 83). It follows from (20), (27) and (30) that \( f \) is surjective.

To see that \( f \) is a local expansion it is enough to apply Statements 1 and 2. In fact, let us observe that the inequality \( \tau(f(E)) > 1 \) holds for all edges \( E \) of \( X \). If a point \( x \in X \) is not a vertex of \( X \), then a suitable neighborhood \( U \) mentioned in the definition of a local expansion (see the very beginning of § 2) exists by Statement 1.
If \( x \) is a vertex of \( X \), then \( U \) is constructed in Statement 2, and the existence of the constant \( M > 1 \) follows from \( (38) \) in both the cases. Thus the proof is complete.

**§ 5. Some properties of local expansions on linear graphs.** To show that the condition mentioned in Theorem 1 is not only sufficient but also necessary for a linear graph to admit a local expansion, we shall use some properties of these mappings. We will prove them now.

**Proposition 1.** Let \( f: X \to X \) be a local expansion on a metric space \( X \). Then \( f \) is locally one-to-one: for every point \( x \in X \) and for the open neighborhood \( U \) of \( x \) that exists by the definition of the local expansion, the restricted mapping \( f|U: U \to f(U) \) is one-to-one.

Indeed, let \( U \) be as in the definition of the local expansion (see § 2). Then, if \( y \) and \( z \) are distinct points of \( U \), we have \( g(y, z) > 0 \), whence \( g(f(y), f(z)) > 0 \) by \((1)\), and the conclusion follows.

**Proposition 2.** Let \( f: X \to X \) be a local expansion on a metric space \( X \). Then for each simple closed curve \( S \) contained in \( X \) its image \( f(S) \) does not contain end points of itself.

To see this, let \( p \in f(S) \) and let a point \( x \in S \) be such that \( p = f(x) \). Take the open neighborhood \( U \) of \( x \) that exists by the definition of the local expansion, and choose two points \( x_1, x_2 \) in \( S \cap U \) both different from \( x \) and such that \( x \) lies in the arc \( x_1 x_2 \) of \( S \cap U \). Then the arc \( x_1 x_2 \) is mapped homeomorphically under \( f \) (see Proposition 2) onto the arc \( f(x_1)f(x_2) \) which contains \( p = f(x) \) with \( f(x_1) \neq p \neq f(x_2) \), whence \( p \) cannot be an end point of \( f(S) \).

**Proposition 4.** Let \( f: X \to X \) be a local expansion on a metric space \( X \). Then for each arc \( ab \) contained in \( X \) no point of \( ab\setminus\{a, b\} \) can be mapped on an end point of \( f(ab) \).

Indeed, as in the proof of the previous proposition, we take \( p \in f(ab)\setminus\{f(a), f(b)\} \) and \( x \in ab\setminus\{a, b\} \) such that \( p = f(x) \); in the open neighborhood \( U \) of \( x \) as in the definition of the local expansion we choose two points \( x_1, x_2 \) of \( ab\setminus\{a, b\} \) such that \( x \) lies between them. The rest of the argumentation is exactly the same as for the proof of Proposition 3.

**Proposition 5.** Let \( f: X \to X \) be a local expansion on a linear graph \( X \). Then for every point \( x \in X \) we have

\[
\mbox{ord}_X X \leq \mbox{ord}_f(x) X.
\]

To see this, let us consider the points \( x \) and \( f(x) \) as vertices of the graph \( X \). Let \( n = \mbox{ord}_X X \) and \( m = \mbox{ord}_f(x) X \) be the numbers of edges of \( X \) incident to \( x \) and to \( f(x) \) respectively. Further, let \( U \) be the open neighborhood of \( x \) as in the definition of the local expansion \( f \). For every \( i = 1, 2, \ldots, n \) choose exactly one point \( a_i \in U \cap E_i \), where \( E_i \) is an edge of \( X \) incident to \( x \). Thus, according to Propositions 1 and 2, the union \( \bigcup \{ f(x_i) \} \) \((i = 1, 2, \ldots, n)\) is mapped homeomorphically under \( f \) onto the union \( \bigcup \{ f(x_i) \} \) \((i = 1, 2, \ldots, n)\), whence we conclude that the number \( m \) of edges incident to \( f(x) \) must be greater than or equal to \( n \).

**Proposition 6.** Let \( f: X \to X \) be a local expansion on a linear graph \( X \). If a point \( e \in X \) is an end point of \( X \), then also \( f(e) \) is an end point of \( X \).

Indeed, recall that the set \( F \) of all end points of \( X \) is finite, and put \( \bar{k} = \mbox{card} F \). Suppose, on the contrary, that there is a point \( e \in F \) with \( e \in X \cap F \). Since the mapping \( f \) is a surjection, there is a set of \( k \) points of \( X \), say \( x_1, x_2, \ldots, x_k \), with the property that \( f(x_1), f(x_2), \ldots, f(x_k) \) are all end points of \( X \). Thus at least one of \( x_1, x_2, \ldots, x_k \) is not an end point of \( X \); call it \( x \). This means that \( X \) is of order greater than \( 1 \) at \( x \) and of order 1 at \( f(x) \), contrary to Proposition 5.

Let \( X \) be a simple linear graph with a convex metric \( g \), and let an arc \( A = ab \in X \) be given. Consider the sequence of all vertices of \( X \) lying in \( A \), ordered from \( a \) to \( b \):

\[ a = v_1 < v_2 < \ldots < v_n = b \]

and put

\[ \lambda(A) = g(a, v_1) + \sum_{i=1}^{n-1} g(v_i, v_{i+1}) + g(v_n, b). \]

It is evident that if we take, instead of \( \{v_i\} \), some other finite sequence of points, say \( \{p_i\} \), lying in \( A \), ordered from \( a \) to \( b \):

\[ a = p_0 < p_1 < p_2 < \ldots < p_k = b \]

and such that the consecutive points \( p_1, p_1+1 \) lie closely enough, e. g.,

\[ g(p_1, p_1+1) < \frac{1}{3} \min \{ g(a, v_1), g(v_1, v_2), \ldots, g(v_{n-1}, v_n), g(v_n, b) \} \]

for every \( i = 0, 1, \ldots, k-1 \), then we have

\[ \lambda(A) = \sum_{i=0}^{k-1} g(p_i, p_{i+1}). \]

**Proposition 7.** Let \( f: X \to X \) be a local expansion of a simple linear graph \( X \) (with a convex metric) onto itself. If an arc \( A \subseteq X \) is mapped onto an arc \( f(A) \) under \( f \), then

\[ \lambda(A) \leq \lambda(f(A)). \]

**Proof.** Every point \( x \in A \) has an open and connected neighborhood \( U_x \) with the property as in the definition of the local expansion \( f \). Furthermore, we can take neighborhoods \( U_x \) sufficiently small, e. g. such that the diameter of each \( U_x \) is less
than the minimum of the lengths of edges of the graph $X$. The family $\{U_x \mid x \in A\}$ is an open covering of $A$. Take a finite subcovering of $A$ such that no its element is contained in another, and let $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ be a sequence of all the subcovering ordered in such a way that if $a$ and $b$ are end points of $A$, then $a \in U_{x_i}, b \in U_{x_j} \text{ and } U_{x_i} \cap U_{x_j} \neq \emptyset$ for every $i = 1, 2, \ldots, k-1$. Let $M_i > 1$, where $i = 1, 2, \ldots, k$, be the constants mentioned in the definition of the local expansion, i.e. such that if $y, z \in U_{x_i}$, then (see (1))

$$q(f(y), f(z)) \geq M_i \cdot q(y, z).$$

Put $M = \min(M_1, M_2, \ldots, M_k)$. Thus $M > 1$. Choose points $p_i \in U_{x_i} \cap U_{x_{i+1}}$ for $i = 1, 2, \ldots, k-1$ and define $p_0 = a$ and $p_k = b$. The sets $U_{x_i}$ being connected by construction, we have $p_i p_{i+1} \subseteq U_{x_{i+1}}$, for every $i = 0, 1, \ldots, k-1$, and, by (42)

$$q(f(p_i), f(p_{i+1})) \geq M_i \cdot q(p_i, p_{i+1}).$$

Further, we see that points $f(p_i)$ lie in the arc $f(A)$ in the same order as points $p_i$ lie in $A$. The neighborhoods $U_{x_i}$ are small enough so that condition (39) holds and thus (40) can be applied. Therefore by (43) we have

$$\lambda(A) = \sum_{i=0}^{k-1} q(p_i, p_{i+1}) < M \cdot \sum_{i=0}^{k-1} q(p_i, p_{i+1}) \leq \sum_{i=0}^{k-1} M_i \cdot q(p_i, p_{i+1}) = \sum_{i=0}^{k-1} q(f(p_i), f(p_{i+1})) = \lambda(f(A)),
$$

the last equality being an easy consequence of the uniform continuity of $f$ so that formula (40) is applicable to the image $f(A)$. Thus (41) follows.

§ 6. Existence of local expansions—necessity. We are ready now to prove our second main result.

**Theorem 2.** Let a simple linear graph $X$ metricized by a convex metric be given. If there exists a local expansion $f$: $X \rightarrow X$ of $X$ onto itself, then there is a point $c \in X$ of the maximal order in $X$, i.e., satisfying (18), such that for every component $X \setminus \{c\}$ its closure contains a simple closed curve.

**Proof.** Suppose the contrary. This means that for every point $c$ of the maximal order in $X$ there is a component of $X \setminus \{c\}$ such that its closure $T(c)$ contains no simple closed curve (in other words, $T(c)$ is a tree). Let $P$ denote the set of all points $c$ in $X$ which are of the maximal order $n = \text{ord}_X c$ in $X$. Thus $\text{ord}_X c = n$ for every $c \in P$. It follows from Proposition 5 that $f(P) \subseteq P$. For each point $c \in P$ take the open neighborhood $U$ of $c$ in the definition of the local expansion $f$ and, for each index $i = 1, 2, \ldots, n$ choose exactly one point $a_i \in U \cap E_i$, where $E_i$ is an edge of $X$ incident to $c$. We distinguish two kinds of arcs $a_i \in E_i$. Namely, an arc $a_i$ is said to be of the first kind if it is contained (together with the edge $E_i$) in a tree $T(c)$; it is said to be of the second kind if it is contained (also together with the edge $E_i$ containing $a_i$) in the closure $K(c)$ of a component of $X \setminus \{c\}$ such that $K(c)$ contains a simple closed curve. Given a point $c \in P$, let $n_1(c)$ or $n_2(c)$ denote the number of arcs $a_i$ which are of the first or of the second kind respectively. Thus,

$$n_1(c) + n_2(c) = n$$

for every point $c \in P$.

Let us recall that, by Proposition 2, every arc $a_i$ is mapped homeomorphically onto an arc $f(c) f(a_i)$. We claim that

(45) if $a_i$ is of the second kind, then its image $f(a_i)$ is of the second kind, too.

Indeed, if not, then consider two cases. If $a_i$ lies on a simple closed curve $S$, then, since $f(a_i)$ is contained in some tree $T(f(c))$, the image $f(S)$ has a nondegenerate intersection with $T(f(c))$. Thus $f(S) \cap T(f(c))$ is a tree as a subcontinuum of $T(f(c))$, and hence $f(S)$ contains an end point of itself, contrary to Proposition 3. If $a_i$ is contained in no simple closed curve, then, since it is of the second kind, there is an arc $\delta b$ such that

$$\delta b \subseteq E_i \subseteq \delta b \cup S' \subseteq K(c)$$

and $\delta b \cap S' = \{b\}$, where $K(c)$ is the closure of the component of $X \setminus \{c\}$ containing $a_i$, and $S'$ is a simple closed curve contained in $K(c)$. Since no point of $\delta b \cap \{b\}$ is mapped to an end point of $f(\delta b)$ by Proposition 4, and since $f(\delta b)$ is a subcontinuum of a tree $T(f(c))$, hence $f(\delta b)$ is an arc with $f(\delta b) \neq f(c)$. Thus $f(S')$ has a nondegenerate intersection with $T(f(c))$, which implies, as in the previous case, a contradiction with Proposition 3. So claim (45) is proved.

The restricted mapping $f\mid U$ being a homeomorphism by Proposition 1, we conclude from (45) that

$$n_2(c) \leq n_2(f(c)).$$

Let us take a subset $Q$ of $P$ composed of all points $c \in P$ for which the number $n_2(c)$ is maximal. Since $\text{ord}_X c = n$, both (44) and (46) imply that $n_2(c) = n_2(f(c))$ for every point $c \in Q$, whence we conclude at once that

(47) if $c \in Q$ and if an arc $a_i$ is of the first kind, then its image $f(a_i)$ is of the first kind, too.

Finally take a subset $R$ of $Q$ composed of all points $c \in Q \subseteq P$ with the property that for all end points $e$ of all trees $T(c)$ the distance $\delta(f(e), e)$ is maximal. In other words, a point $c \in Q$ is in the set $R$ if and only if there is a tree $T(c)$ and an end point $e \in T(c)$ such that

$$\delta(f(e), e) = \delta(e, e)$$

for all points $c' \in Q$, all trees $T(c')$ and all end points $e'$ of $T(c')$.

Consider an arc $ce$ with $c \in R$, with an end point $e \in T(c)$ and with the maximal distance $\delta(c, e)$. Then the arc $ce$ of $X$ is of the first kind, and we see by (47) that its image $f(ce)$ is also of the first kind. Since $f(e)$ is an end point of $X$ by Proposition 6, and since the image $f(ce)$ does not contain end points of itself except of $f(c)$ and
$f(e)$ by Proposition 4, we see that $f(ce)$ is an arc contained in a tree $T(f(c))$ being the closure of some component of $X \setminus \{c\}$. So the arcs $ce$ and $f(ce)$ are contained in the trees $T(c)$ and $T(f(c))$ respectively, and thereby we conclude from the convexity of the metric $q$ that

\begin{equation}
q(ce) = \lambda(ce) \quad \text{and} \quad q(f(c), f(e)) = \lambda(f(ce)).
\end{equation}

Further, $f(c)$ is a point $c'$ of $Q$ and $f(e)$ is an end point $e'$ of $T(c')$, and therefore we have $\lambda(f(ce)) \leq \lambda(ce)$ by (49) and (48), contrary to Proposition 7. The contradiction completes the proof.

References


INSTITUTE OF MATHEMATICS OF THE WROCLAW UNIVERSITY

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