ON THE SET OF INTERIORITY OF A MAPPING

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Abstract. The set of points of a topological space $X$ at which a mapping $f: X \to Y$ is interior is investigated in the paper. Conditions are found under which this set is non-empty and $G_δ$.

Let a mapping $f: X \to Y$ from a topological space $X$ onto a topological space $Y$ be given. The set $\text{Int } f$ of points of $X$ at which $f$ is interior is investigated in this paper. After a preliminary part we study some conditions under which this set is not empty. The next part of the paper concerns the valuation of the Borel class of $\text{Int } f$. Further, some operations performed on mappings (as the composite, the restriction, the product etc.) are examined with regard to the set of interiority points of the mappings. Finally some relations to similar known concepts are discussed.

Theorems and propositions proved in the paper are formulated in such a way that all assumptions made on spaces and mappings are essential, which is shown by a number of examples. Some unsolved problems are stated.

All concepts introduced and investigated in the paper are related to openness of a mapping. Let us recall that some other ideas connected with openness are widely studied in [1] and in [5], where large lists of references are inserted.

Given a subset $A$ of a topological space $X$, we denote by $\overline{A}$ the closure of $A$, by $\text{Int } A$ the interior of $A$, i.e., $\text{Int } A = X \setminus \overline{X \setminus A}$ and by $\text{Fr } A$ the frontier of $A$, i.e., $\text{Fr } A = \overline{A} \cap \overline{X \setminus A}$. Recall that $A$ is said to be dense if $\overline{A} = X$, it is said to be boundary if its complement is dense, i.e., if $\overline{X \setminus A} = X$, and it is said to be nowhere dense if its closure is a boundary set, i.e., if $\overline{X \setminus \overline{A}} = X$. A mapping $f: X \to Y$ is not assumed to be continuous in general.

1. Let two topological spaces, $X$ and $Y$ be given. A mapping $f: X \to Y$ of $X$ onto $Y$ is said to be interior at a point $x_0 \in X$ provided that, for every open set $U$ in $X$ containing $x_0$, the point $f(x_0)$ is an interior point of $f(U)$ (see [11], p. 149).


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STATEMENT 1. Let $\mathcal{A}$ and $\mathcal{B}$ be bases for topological spaces $X$ and $Y$ respectively. The following are equivalent for a mapping $f : X \to Y$ and for a point $x_0 \in X$:

- $f$ is interior at $x_0$;  
  \begin{equation}
  (1)
  \end{equation}
  for every $A \in \mathcal{A}$ containing $x_0$ the point $f(x_0)$ is an interior point of $f(A)$;  
  \begin{equation}
  (2)
  \end{equation}
  for every $A \in \mathcal{A}$ containing $x_0$ there exists an element $B$ of $\mathcal{B}$ such that $f(x_0) \in B \subset f(A)$;  
  \begin{equation}
  (3)
  \end{equation}
  for every subset $S$ of $X$, if $x_0 \in \text{Int } S$, then $f(x_0) \in \text{Int } f(S)$.

Indeed, implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ are immediate.

Note that interiority of a mapping $f : X \to Y$ at a point $x_0 \in X$ neither implies nor is implied by continuity of $f$ at $x_0$. One can see it by the following

Example 1. There exist a mapping $f : R \to R$ of the real line $R$ with the natural topology onto itself, and two points $x_1, x_2$ of $R$ such that $f$ is interior but not continuous at $x_1$ and it is continuous but not interior at $x_2$.

In fact, the mapping $f$ defined by

$$
  f(x) = \begin{cases} 
  x \sin \frac{1}{x} & \text{for } x < 0, \\
  0 & \text{for } x = 0, \\
  1 & \text{for } x > 0,
  \end{cases}
$$

and the points $x_1 = 0$ and $x_2 = 1$ have required properties.

Given two topological spaces $X$ and $Y$, let $f : X \to Y$ be an arbitrary mapping of $X$ onto $Y$. Consider the set of all points of $X$ at which $f$ is interior, and denote this set by $\text{Int } f$:

$$
  \text{Int } f = \{ x \in X : f \text{ is interior at } x \}.
$$

In other words, we have

Definition 1. A point $x \in X$ is in $\text{Int } f$ provided that for every its open neighborhood $U$ in $X$ we have $f(x) \in \text{Int } f(U)$, i.e., there exists an open neighborhood $V$ of $f(x)$ in $Y$ that is contained in $f(U)$.

Recall that a mapping $f : X \to Y$ is said to be open provided that for every open subset $U$ of $X$ its image $f(U)$ is an open subset of $Y$. Therefore $f$ is open if and only if it is interior at each point of its domain, whence we have

STATEMENT 2. A mapping $f : X \to Y$ is open if and only if $\text{Int } f = X$. 

Let us come back to an arbitrary mapping \( f : X \to Y \) and consider points \( x \in X \setminus \text{Int} f \). For every such point \( x \) there exists an open neighborhood \( U \) with \( f(x) \in Y \setminus \text{Int} f(U) \). Then two cases are possible. First, if the image of this particular neighborhood \( U \) is a boundary set, i.e., if \( \text{Int} f(U) = \emptyset \). Denote the set of all such points \( x \) by \( \text{Bd} f \). So we have

**Definition 2.** A point \( x \in X \) is in \( \text{Bd} f \) provided that there exists an open neighborhood \( U \) of \( x \) whose image \( f(U) \) is a boundary set of \( Y \), i.e., such that \( \text{Int} f(U) = \emptyset \).

Remark that if \( U \) is the open neighborhood of the point \( x \) as in Definition 2, then for every open subset \( U_0 \) of \( U \) we have \( \text{Int} f(U_0) = \emptyset \), whence we conclude that \( \text{Bd} f \) is an open subset of \( X \).

The second case which is possible for a point \( x \) being out of \( \text{Int} f \) is that this point is out of \( \text{Bd} f \), too. Therefore, since \( x \in X \setminus \text{Bd} f \) it has no open neighborhood whose image is a boundary subset of \( Y \), i.e., for every open neighborhood \( U' \) of \( x \) we have \( \text{Int} f(U') \neq \emptyset \); and since \( x \in X \setminus \text{Int} f \), there is an open neighborhood \( U \) of \( x \) with \( f(x) \in Y \setminus \text{Int} f(U) \), i.e., with \( f(x) \in \text{Fr} f(U) \). Denote the set of all such points \( x \) of \( X \) by \( \text{Fr} f \). Thus we have

**Definition 3.** A point \( x \in X \) is in \( \text{Fr} f \) provided that for every neighborhood \( U' \) of \( x \) the inequality \( \text{Int} f(U') \neq \emptyset \) holds and that there exists a neighborhood \( U \) of \( x \) such that \( f(x) \in \text{Fr} f(U) \).

Therefore the following statement holds true:

**STATEMENT 3.** For every mapping \( f : X \to Y \)
the sets \( \text{Int} f \), \( \text{Bd} f \) and \( \text{Fr} f \) are pairwise disjoint; \( \text{Int} f \cup \text{Bd} f \cup \text{Fr} f = X \); \( \text{Bd} f \) is an open subset of \( X \).

To illustrate the introduced concepts let us consider the following example which we shall also use in the further part of the paper.

**Example 2.** There exists a continuous mapping \( f \) of the unit closed interval \([0, 1]\) onto itself such that \( \text{Int} f \) is an uncountable boundary neither closed nor open \( G_\delta \)-set, the set \( \text{Fr} f \) is countable and dense in itself, and \( \text{Int} f \cup \text{Fr} f \) is the Cantor ternary set.

In fact, let \( C \) be the Cantor ternary set in the unit interval \( I = [0, 1] \), i.e., the set of all numbers \( x \) of the form

\[
 x = t_1/3 + t_2/9 + \ldots + t_n/3^n + \ldots ,
\]

where \( t_n \) takes one of the values 0 or 2. Let \( E \subset C \) be the (countable) set of those numbers \( x \) for which the digit \( t_n = 0 \) or 2 occurs in \( x \) a finite (but positive) number of times only. In other words, points of \( E \) are just end points of (open) intervals which are components of the set \( I \setminus C \). Thus \( \{0, 1\} \subset C \setminus E \).
Let \( \varphi : C \to I \) be a mapping defined by
\[
\varphi(x) = \frac{1}{2} (t_1/2 + t_2/4 + \ldots + t_n/2^n + \ldots),
\]
where \( x \) is represented in the form \( (8) \). Thus \( \varphi \) has the same value at both end points of each interval of \( I \setminus C \). We take this value as a constant value of a new function \( f : I \to I \) in this interval; otherwise, i.e., for \( x \in C \), we put \( f(x) = \varphi(x) \). Thus \( f \) is well-known continuous mapping of \( I \) onto \( I \) (so-called step function, see e.g. [7], p. 150 and 151). The following equalities are easy to see directly from the definitions:

\[
\begin{align*}
\text{Int} f &= C \setminus E. \\
\text{Bd} f &= I \setminus C, \\
\text{Fr} f &= E.
\end{align*}
\]

**Example 3.** There are continuous mappings \( f_1, f_2 \) and \( f_3 \) such that (a) \( \text{Int} f_1 = X \), (b) \( \text{Bd} f_2 = X \) and (c) \( \text{Fr} f_3 = X \).

Indeed, the identity mapping \( f_1 \) of the closed unit interval \( X = [0, 1] \) with the natural topology onto itself is obviously open, whence we have (a) by Statement 2. Further, the identity mapping \( f_2 \) of the closed unit interval \( X = [0, 1] \) with the discrete topology onto the closed unit interval \( Y = [0, 1] \) with the natural topology satisfies (b). Finally, for the identity mapping \( f_3 \) of the Sorgenfrey line \( X \) (see [4], Example 1.2.2, p. 39) onto the real line \( Y \) with the natural topology condition (c) is satisfied.

2. From among the three subsets of \( X \) mentioned above, we are mainly interested in the structure of \( \text{Int} f \). By virtue of (5) and (6), since \( \text{Bd} f \) is open (7), the structure of \( \text{Fr} f \) is in some sense determined by one of \( \text{Int} f \).

First of all, one can ask under what circumstances the set \( \text{Int} f \) is non-empty. Below we prove that is so provided that the mapping is continuous and the spaces are metric compact. To show this we need some preliminary results.

The following two statements are easy consequences of the definitions. Recall that every regular space is assumed to be \( T_1 \) ([4], p. 60).

**STATEMENT 4.** Let \( f : X \to Y \) be an arbitrary mapping. If \( x \in X \setminus \text{Int} f \), then there exists an open neighborhood \( U \) of \( x \) such that for every subset \( W \) of \( U \) we have \( f(x) \in Y \setminus \text{Int} f(W) \).

**STATEMENT 5.** Let \( X \) be a regular space, and let \( f : X \to Y \) be an arbitrary mapping. If \( x \in X \setminus \text{Int} f \), then there exists an open neighborhood \( U \) of \( x \) such that \( f(x) \in Y \setminus \text{Int} f(\overline{U}) \).
PROPOSITION 1. Let \( f: X \to Y \) be a continuous mapping of a complete metric space \( X \) onto a topological space \( Y \). Then the union \( \text{Bd} f \cup \text{Int} f \) is a dense subset of \( X \).

Proof. If not, then there exist a point \( x_0 \) of \( X \) and its neighborhood \( G_0 \) such that \( x_0 \in G_0 \subset X \setminus (\text{Bd} f \cup \text{Int} f) \). Choose such an open neighborhood \( U_0 \) of \( x_0 \) that \( U_0 \subset G_0 \) and \( \text{diam} U_0 < 1 \). Put \( G_1 = U_0 \cap f^{-1}(\text{Int} f(U_0)) \). Since \( f \) is continuous and \( x_0 \in X \setminus \text{Bd} f \), we conclude that \( G_1 \) is a non-empty open subset of \( X \). Further we have \( G_1 \subset U_0 \) and \( \text{diam} G_1 < 1 \). Take a point \( x_1 \in G_1 \) and its open neighborhood \( U_1 \) such that \( U_1 \subset G_1 \) and \( \text{diam} U_1 < \frac{1}{2} \). Put \( G_2 = U_1 \cap f^{-1}(\text{Int} f(U_1)) \). Since \( G_1 \subset U_0 \subset G_0 \subset X \setminus \text{Bd} f \), we have \( x_1 \in X \setminus \text{Bd} f \), whence by continuity of \( f \) we see that \( G_2 \) is non-empty and open. Moreover \( \text{diam} G_2 < \frac{1}{2} \) and \( G_2 \subset G_1 \). Inductively we define a sequence of points \( x_n \) and two sequences of non-empty open sets \( U_n \) and \( G_n \) such that for every \( n = 1, 2, 3, \ldots \) we have \( x_n \in U_n \subset U_n \subset G_n \) and \( \text{diam} G_n < 1/n \) and \( G_{n+1} = U_n \cap f^{-1}(\text{Int} f(U_n)) \). Since the space \( X \) is complete, it follows from the Cantor theorem ([7], § 34, II, p. 413) that there exists a point \( x \in X \) such that \( \{x\} = = \cap \{G_n : n = 1, 2, 3, \ldots\} \). Observe that the family \( \{G_n : n = 1, 2, 3, \ldots\} \) is a local base at the point \( x \), and that by [7], § 3, III, (13), p. 15 we have \( f(G_n) = f(U_{n-1} \cap f^{-1}(\text{Int} f(U_{n-1}))) = f(U_{n-1}) \cap \text{Int} f(U_{n-1}) = = \text{Int} f(U_{n-1}) \), so \( f(G_n) \) is an open subset of \( Y \), whence \( x \in \text{Int} f \). But \( x \in G_0 \subset X \setminus (\text{Bd} f \cup \text{Int} f) \), a contradiction.

By Statement 3 we can reformulate Proposition 1 in the following form:

COROLLARY 1. Let \( f: X \to Y \) be a continuous mapping of a complete metric space \( X \) onto a topological space \( Y \). Then \( \text{Fr} f \) is a boundary subset of \( X \).

Since every locally compact metric space is completely metrizable (see [4], 3.9. G, p. 257 and Theorem 4.3. 26, p. 343), we have

COROLLARY 2. Let \( f: X \to Y \) be a continuous mapping of a locally compact metric space \( X \) onto a topological space \( Y \). If \( \text{Bd} f \) is not a dense subset of \( X \), then \( \text{Int} f \neq \emptyset \).

Note that the reverse implication to the one mentioned in Corollary 2 is not true. Namely, Example 2 shows that for the step function \( f: I \to I \) we have \( \text{Int} f = C \setminus E \neq \emptyset \) and \( \text{Bd} f = I \setminus C \) is a dense subset of \( I \).

Metrizability of the space \( X \) is an essential assumption in Proposition 1. Namely we have
Example 4. There exists a compact non-metrizable space $X$ (which therefore is not second-countable) and a continuous mapping $f : X \to S^1$ of $X$ onto the unit circumference $S^1$ with the natural topology such that $\partial f = \emptyset$ and $\text{Int} f = \emptyset$.

Indeed, let $X$ be the two arrows space, i.e., let $X = C_0 \cup C_1 \subset \mathbb{R}^2$, where $C_0 = \{(x, 0) : 0 < x < 1\}$ and $C_1 = \{(x, 1) : 0 < x < 1\}$, with the topology on $X$ generated by the base consisting of sets of the form

$$\{(x, j) \in X : x_0 - 1/k < x < x_0 \text{ and } j = 0, 1\} \cup \{(x_0, 0)\},$$

where $0 < x_0 < 1$ and $k = 1, 2, 3, \ldots$, and of sets of the form

$$\{(x, j) \in X : x_0 < x < x_0 + 1/k \text{ and } j = 0, 1\} \cup \{(x_0, 1)\},$$

where $0 < x_0 < 1$ and $k = 1, 2, 3, \ldots$ (cf. [4], 3.10. C, p. 270). The subspaces $C_0$ and $C_1$ of the space $X$ are homeomorphic to the Sorgenfrey line ([4], 3.10. C (a), p. 270) which is not metrizable (because it is separable ([4], 1.3.9, p. 44) but not second-countable ([4], 1.2.2, p. 39), which is impossible for metric spaces ([4], 4.1.16, p. 319)), whence non-metrizability of $X$ follows. Further, $X$ is not second-countable (because this property is hereditary, [4], p. 96) but it is a compact space (see [4], 3.10. C (b), p. 270).

Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and define a mapping $f : X \to S^1$ by $f((x, j)) = \exp(2\pi ix)$ for $(x, j) \in X$. Thus $f$ is a surjection. It is easy to verify that the inverse images of open sets in $S^1$ are open in $X$, so $f$ is continuous. Note further that for every basic neighborhood $U$ of a point $x_0 \in X$ we have $\text{Int} f(U) \neq \emptyset$ and $f(x_0) \in \partial f(U)$, whence we conclude $\partial f = X$, which leads to $\text{Int} f = \emptyset = \partial d f$ by (5) and (6) of Statement 3.

Completeness of $X$ is also an essential assumption in Proposition 1. It can be seen by

Example 5. There exists a metric space $M$ which is not complete and not locally compact, and a continuous mapping $g : M \to Y$ of $M$ onto a dense subset $Y$ of the unit square such that $\partial g = \emptyset$ and $\text{Int} g = \emptyset$.

To construct the example let us recall the well known mapping $f$ of Peano from the closed unit interval $I = [0, 1]$ onto the unit square $I^2$, due to Sierpiński and described e.g. in [9], p. 251 or - in a more detailed way - in [3], p. 367–370. For $n = 0, 1, 2, \ldots$ the set

$$F_n = \{0, 9^{-n}, 2 \cdot 9^{-n}, \ldots, (9^n - 1) \cdot 9^{-n}, 1\}$$

divides $I$ into $9^n$ equal segments, every of which is mapped under $f$ onto one of $9^n$ equal squares, into which the unit square $I^2$ has been divided. Let $\partial f I^2$ be the frontier of $I^2$ in the plane $\mathbb{R}^2$, i.e., $\partial f I^2 = I \times \{0, 1\} \cup \{0, 1\} \times I$. Note that for every $n = 0, 1, 2, \ldots$ each
element of the set $f(F_n) \setminus \operatorname{Fr} I^2$ is a common vertex of four squares properly chosen from among $9^n$ of them, mentioned above. Put $F = \bigcup_{n=0}^{\infty} F_n$ and define 

$$M = F \setminus f^{-1}(\operatorname{Fr} I^2).$$

Thus $M$ evidently is a countable dense subset of $F$, which is in turn countable and dense in $I$. Therefore considering $M$ as a topological space with the relative topology taken from $I$ we see that $M$ is neither complete nor locally compact. Put $Y = f(M) \subset I^2 \setminus \operatorname{Fr} I^2$ and $g = f | M$. Thus 

$$g : M \to Y$$

is continuous and onto. Take $x \in M$ and let $U'$ be an open (in $M$) neighborhood of $x$. Thus there exists a closed interval $[x, x + 9^{-n}]$ for a sufficiently large $n$, such that $[x, x + 9^{-n}] \cap M \subset U'$. Since $f([x, x + 9^{-n}])$ is a square, so its interior (in $I^2$) is non-empty, we conclude that $\operatorname{Int} g(U') \neq \emptyset$ (in the topology in $Y$). Further, let $m$ be a natural number such that $x \in F_m$. Then putting $U = (x - 9^{-m}, x + 9^{-m}) \cap M$ we see that $U$ is an open neighborhood of $x$, and, since $x$ is the common point of two intervals: $[x - 9^{-m}, x]$ and $[x, x + 9^{-m}]$ every of which is mapped under $f$ onto a square having $f(x)$ as its vertex, and since $f(x)$ is the common vertex of some four squares of the same size, we conclude that $f(x) \in \operatorname{Fr} f([x - 9^{-m}, x + 9^{-m}])$, whence it follows that $g(x) \in \operatorname{Fr} g(U)$. Therefore $x \in \operatorname{Fr} g$, i. e., we have proved that $\operatorname{Fr} g = M$, which implies the equalities $\operatorname{Int} g = \emptyset$ and $\operatorname{Bd} g = \emptyset$ by (5) and (6) of Statement 3.

Note that the mapping $g$ of Example 5 has properties mentioned in Example 3 (c), and, in addition, the domain of $g$ is a metric space.

By a modification of the mapping $g$ mentioned in Example 5 we can show that the hypothesis of continuity of the mapping cannot be omitted in Proposition 1. Namely we have

**Example 6.** There exists a non-continuous mapping $h$ from the unit closed interval $I$ onto a dense subset $Y'$ of the unit square such that $\operatorname{Bd} h = \emptyset$ and $\operatorname{Int} h = \emptyset$.

Indeed, defining the mapping $h$ by $h(x) = g(x)$ if $x \in M$ and $h(x) = p = \left(\frac{1}{3}, \frac{1}{3}\right)$ if $x \in I \setminus M$, and putting $Y' = Y \cup \{p\}$, where $g$ and $Y$ have the same meaning as in Example 5, we have $h : I \to Y'$. One can verify in a similar manner as it was done in Example 5 that the mapping $h$ has the required properties.

Observe that the same Examples 4, 5 and 6 show that metrizability and local compactness of $X$ and continuity of $f$ are essential assumptions in Corollary 2, respectively. The hypothesis concerning $\operatorname{Bd} f$ is also necessary in Corollary 2. Namely Example 3 (b) shows a con-
THEOREM 1. Let \( f : X \to Y \) be a continuous mapping of a locally compact second-countable space \( X \) onto a Hausdorff space \( Y \). Then \( f(X \setminus \text{Int} f) \) is of the first category in \( Y \).

Proof. If \( \text{Int} f = X \), then the conclusion obviously holds. So assume \( \text{Int} f \neq X \) and let \( x \in X \setminus \text{Int} f \). Thus by definition of \( \text{Int} f \) there exists an open neighborhood \( U \) of \( x \) such that

\[
f(x) \in f(U) \setminus \text{Int} f(U).
\]

Since \( X \) is locally compact and second-countable, there exists a countable base \( \mathcal{B} \) in \( X \) consisting of relatively compact open sets (i.e. of open sets whose closures are compact, see [2], 6.3, p. 238). Note that every locally compact space is completely regular ([4], 3.3.1, p. 196), thus regular ([4], p. 61). Hence there is a set \( V \in \mathcal{B} \) such that \( x \in V \subseteq \overline{V} \subseteq U \) (cf. [4], 1.5.5, p. 60), so we have \( f(x) \in f(\overline{V}) \setminus \text{Int} f(\overline{V}) \) by (12). Therefore the following inclusion holds

\[
f(X \setminus \text{Int} f) \subset \bigcup \{ f(V) \setminus \text{Int} f(V) : V \in \mathcal{B} \}.
\]

Since \( \overline{V} \) is compact, its image \( f(\overline{V}) \) is compact and thus it is a closed subset of \( Y \), so \( f(\overline{V}) \setminus \text{Int} f(\overline{V}) \) is also closed. Further, this set has empty interior. Since the base \( \mathcal{B} \) is countable, the union of the right hand of inclusion (13) is of the first category in \( Y \). So the conclusion follows.

Obviously we have \( Y \setminus f(\text{Int} f) \subset f(X \setminus \text{Int} f) \), whence by Theorem 1 it follows

COROLLARY 3. Let \( f : X \to Y \) be a continuous mapping of a locally compact second-countable space \( X \) onto a Hausdorff space \( Y \). Then the complement of \( f(\text{Int} f) \) is a set of the first category in \( Y \).

We now show that the hypotheses of Theorem 1 (or of Corollary 3) do not suffice to prove that \( \text{Int} f \) is not empty. Before we construct the required example, we recall that a space \( Y \) is said to be a Baire space provided that the union of each countable family of closed boundary sets in \( Y \) is boundary (cf. [2], p. 249). Now we pass to the example.

Example 7. There exist a non-compact locally compact second-countable metric space \( X \) and a continuous mapping \( f : X \to Y \) of \( X \) onto a Hausdorff space \( Y \) (which is not a Baire space) such that \( \text{Int} f = \emptyset \).

Indeed, let \( X = [0, \infty) \) be the closed real half line with the natural topology, let \([0, 1] \times [0, 1]\) be the Euclidean unit square and
let $D$ denote the set of dyadic rationals of the closed unit segment $[0, 1]$. Set $D$ into the following sequence:

$$0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$$

and let $d_n$ denote the $(n + 1)$-th term of this sequence ($n = 0, 1, 2, \ldots$).

First, define a mapping $f$ from the set $N \subset X$ of natural numbers into the unit square $[0, 1] \times [0, 1]$ putting for every $n = 0, 1, 2, \ldots$

$$f(4n) = (d_{2n}, 0), \quad f(4n + 1) = (d_{2n}, 1),$$

$$f(4n + 2) = (d_{2n+1}, 1), \quad f(4n + 3) = (d_{2n+1}, 0).$$

Second, extend $f$ defining it on the whole half line $X$ piecewise linearly, i.e., if $x \in [n, n + 1)$ for some $n = 0, 1, 2, \ldots$, then we put $f(x) = (x - n) [f(n + 1) - f(n)] + f(n)$, where arithmetic operations are considered as in the algebraic structure of the Euclidean plane. Put $Y = f(X) = D \times I \cup I \times \{0, 1\}$. Thus $f : X \to Y$ is well-defined, continuous and onto. It is easy to verify that for every bounded subset of $X$ its image under $f$ is a nowhere dense subset of $Y$. Therefore we have $\text{Bd} f = X$, whence $\text{Int} f = \emptyset$ follows by (5) and (6) of Statement 3. It can be observed directly from the definition that $Y$ is not a Baire space.

The last fact can also be deduced from the following corollary which is an immediate consequence of Corollary 3 and of the definition of the Baire space. Recall that a subset of a space is said to be residual ([3], p. 160) if its complement is of the first category.

**COROLLARY 4.** Let $f : X \to Y$ be a continuous mapping of a locally compact second-countable space $X$ onto a Hausdorff space $Y$ which is a Baire space. Then the set $f(\text{Int} f)$ is residual, whence it is a dense set of the second category, and therefore $\text{Int} f$ is not empty.

A particular but in fact very important case of Corollary 4 is when the spaces under consideration are compact. We have

**COROLLARY 5.** Let $f : X \to Y$ be a continuous mapping of a compact metric space $X$ onto a Hausdorff space $Y$. Then $f(\text{Int} f)$ is residual, whence it is a dense set of the second category, and thus $\text{Int} f$ is not empty.

Indeed, the space $X$ satisfies hypotheses of Corollary 4. Further, the assumptions made on the spaces imply that the space $Y$ is compact and metrizable ([8], § 41, III, Theorem 1, p. 11 and VI, Theorem 3, p. 21), therefore complete ([8], § 41, VI, Theorem 1, p. 20), which implies that it is a Baire space (cf. [7], § 34, IV, Theorem, p. 414). So the conclusion follows by Corollary 4.
Now we shall analyze the essentiality of hypotheses of Theorem 1 and of Corollaries 3, 4 and 5. We show that no one of the assumptions made on the spaces and on the mapping can be omitted if the conclusion holds.

Local compactness of $X$ is essential in Theorem 1 and in Corollary 3 because of the following

**Example 8.** There exist topological spaces $X$ and $Y$, and a mapping $f : X \to Y$ of $X$ onto $Y$ such that

- $X$ is Hausdorff, \hspace{1cm} (14)
- $X$ is second-countable, \hspace{1cm} (15)
- $X$ is Baire, \hspace{1cm} (16)
- $X$ is not regular, \hspace{1cm} (17)
- $X$ is not locally compact, \hspace{1cm} (18)
- $Y$ is the real line with the natural topology, \hspace{1cm} (19)
- $f$ is continuous, \hspace{1cm} (20)
- $\text{Int} f$ is countable, \hspace{1cm} (21)
- $\text{Int} f$ is closed, \hspace{1cm} (22)
- $\text{Int} f$ is not a $G_δ$-set, \hspace{1cm} (23)
- $f(\text{Int} f)$ is not of the first category in $Y$, \hspace{1cm} (24)
- $f(\text{Int} f)$ is not of the second category in $Y$. \hspace{1cm} (25)

The example of the space $X$ is patterned after Example 1.5.7 of [3], p. 61. Let $X$ be the set of real numbers and let $Q \subset X$ be the set of rationals. For every $x \in X$ and for every positive integer $i$ let $U_i(x) = \{y \in X : x - 1/i < y < x + 1/i\}$ and

$$B(x) = \begin{cases} \{U_i(x) : i = 1, 2, 3, \ldots\} & \text{if } x \in Q, \\ \{U_i(x) : i = 1, 2, 3, \ldots\} & \text{if } x \in X \setminus Q. \end{cases}$$

One can easily verify that the collection $\{B(x) : x \in X\}$ has properties (BP1) — (BP4) of [4], p. 28 and 59; hence the space $X$ with the topology generated by the neighborhood system $\{B(x) : x \in X\}$ is a Hausdorff space (14) by Proposition 1.5.2 of [4], p. 59. Note that the set $Q$ is closed in $X$. To show (17), fix a point $p \in X \setminus Q$. For any open sets $U_1$ and $U_2$ such that $p \in U_1$ and $Q \subset U_2$ we have $U_1 \cap U_2 \neq \emptyset$. Hence $X$ is not regular. Since every locally compact space is regular (see [4], Theorem 3.3.1, p. 196 and p. 61), hence (18) follows from (17). The reader can check that the family $\cup B(x) \cup B(x + p) : x \in Q$ composed of all sets $U_i(x)$ and $U_i(x + p) \setminus Q$ for $x \in Q$ and $i = 1, 2, 3, \ldots$ (where $p \in X \setminus Q$ is a fixed point) forms a countable base for the space $X$. Thus $X$ is second-countable, i.e. (15) holds. To prove (16) observe that the set $X \setminus Q$ is dense in $X,$
and that it is homeomorphic to the set $\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers with the topology induced by the natural topology of the real line $\mathbb{R}$ (namely the identity is the required homeomorphism). Further, $\mathbb{R}\setminus\mathbb{Q}$ is Čech complete ([4], p. 252) and thus a Baire space ([4], 3.9.3, p. 253). Therefore we see that the space $X$ contains a dense Baire subspace $X\setminus\mathbb{Q}$, which implies by Theorem 1.15 of [6], p. 12 that $X$ itself is a Baire space.

Let $f$ denote the identity mapping from $X$ onto the real line $Y$ equipped with the natural topology. Then the family of the sets $U_i(x)$ for $x \in \mathbb{Q}$ and $i = 1, 2, 3, \ldots$ is a base for $Y$; the inverse images of these sets under $f$ (i.e. the same sets $f^{-1}(U_i(x)) = U_i(x)$) are open in $X$, hence $f$ is continuous. So (19) and (20) is shown. By Definition 1 we have $\text{Int} f = \mathbb{Q}$, whence (21) and (22) follow. Now we prove (23). Indeed, given a set $G$ open in $X$ and containing $\mathbb{Q}$, there exists, for each point $x \in \mathbb{Q}$, an index $i_x$ such that $U_{i_x}(x) \subseteq G$, whence it follows that $\bigcup \{U_{i_x}(x) : x \in \mathbb{Q}\} \subseteq G$. So every set $G$ open in $X$ and containing $\mathbb{Q}$ contains a set $G' = \bigcup \{U_{i_x}(x) : x \in \mathbb{Q}\}$ which is open with respect to the natural topology of the set $X$ of real numbers, and which still contains $\mathbb{Q}$. If $Q$ would be the intersection of a countable family of such sets, say $G_n$ (where $n = 1, 2, 3, \ldots$) we would have $Q = \bigcap_{n=1}^{\infty} G_n \supset \bigcap_{n=1}^{\infty} G'_n \supset \mathbb{Q}$, whence $Q = \bigcap_{n=1}^{\infty} G'_n$, i.e. $Q$ would be a $G_\delta$-set with respect to the natural topology on the real line, which is a contradiction (see e.g. [9], Remark 3, p. 187). So (23) is established.

Since $\text{Int} f = \mathbb{Q}$ and $f$ is the identity, we have $f(X \setminus \text{Int} f) = Y \setminus \mathbb{Q}$ which is the set of all irrational numbers of the real line (with the natural topology); it follows from the Baire theorem applied to $Y$ that $X \setminus \mathbb{Q}$ is not of the first category, i.e., (24) follows. (25) is a consequence of (24). Therefore all required properties of the example have been established.

Example 4 shows that the second-countability assumption of $X$ cannot be omitted in Theorem 1 and in Corollaries 3 and 4. The same example indicates that metrizability of $X$ is necessary in Corollary 5. Compactness of $X$ in Corollary 5 is not only essential, but even it cannot be relaxed to local compactness as it can be seen by Example 7. The hypothesis that the space $Y$ is Hausdorff cannot be replaced by one of being a $T_1$-space in Theorem 1 as well as in Corollaries 3, 4 and 5. Namely we have

**Example 9.** There exists a continuous mapping $f$ from the closed unit interval with the natural topology onto a $T_1$, non-Hausdorff Baire space $Y$ such that $\text{Int} f = \emptyset$.

Indeed, let $Y$ be the closed unit interval with the topology of finite complements, i.e., such that the family of the open sets in $Y$ consists of the empty set and of all complements of finite sets. Then $Y$ is $T_1$ but not Hausdorff (cf. [4], 1.2.6, p. 40 and p. 58). Since every proper closed subset of $Y$ is finite, hence the union of each countable
family of closed boundary sets in $Y$ is at most countable, so the union is a boundary set, and thus $Y$ is a Baire space. The identity mapping $f: [0, 1] \to Y$ has the required properties.

Further, Example 7 shows that the assumption that $Y$ is a Baire space is essential in Corollary 4.

Continuity of the mapping $f$ is necessary in Theorem 1 and in Corollaries 3 and 4. Namely we have

Example 10. There exists a non-continuous mapping $f: R \to R$ of the real line with the natural topology onto itself such that $\text{Int} f = \emptyset$.

Let $Q \subset R$ denote the set of rationals. To construct the example consider first a mapping $h$ from positive irrationals to irrationals defined by

$$h(x) = \begin{cases} 1 - 1/x & \text{for } x \in [0, 1] \setminus Q, \\ x - 1 & \text{for } x \in [1, \infty) \setminus Q. \end{cases}$$

Thus $h: R_+ \setminus Q \to R \setminus Q$ is a homeomorphism of the set of positive irrational numbers $R_+ \setminus Q$ onto the set of all irrationals (both sets with the topology induced by the natural topology of the real line $R$). Next define $f: R \to R$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in Q, \\ h(x) & \text{if } x \in R \setminus Q \text{ and } x > 0, \\ h(-x) + \pi & \text{if } x \in R \setminus Q \text{ and } x < 0, \end{cases}$$

where $\pi$ denotes a fixed irrational number. Since for every rational number $r$ the difference $r - \pi$ is irrational, hence it is equal to $h(x')$ for some $x' \in R_+ \setminus Q$; taking $x = -x'$ we have $x \in R \setminus Q$ and $x < 0$, and thus $f(x) = r$ by the definition of $f$. So $Q \subset f(R)$. Since $R \setminus Q \subset f(R)$ by the definitions of $h$ and of $f$, we conclude that $f$ is a surjection. It follows from Definition 1 that $\text{Int} f = \emptyset$.

It can be seen from Example 6 that continuity of $f$ is necessary in Corollary 3, too.

So, it has been shown that all hypotheses concerning the spaces as well as the mappings are essential in Proposition 1, Theorem 1 and Corollaries 1 thru 5.

However, it is tempting to conjecture that assumptions concerning the space $X$ in Theorem 1, namely local compactness, can be relaxed in some way. Example 8 shows that it is not the case if we take a Baire space instead of a locally compact one. Note that the space $X$ described in Example 8 is not metrizable (because every metrizable space is normal ([4], 4.1.13, p. 318, and p. 68) and hence regular, see (17)). But even if we assume that the space $X$ under consideration is homeomorphic to a complete metric space, we are still not able to show the conclusion of Theorem 1. This can be seen by the following
Example 11. There exists a continuous surjection $f : R \setminus Q \to R$ of the set of irrational numbers (with the relative topology induced by the natural topology of the real line $R$) onto $R$ such that $\text{Int} f = \emptyset$.

Let $h : R_+ \setminus Q \to R \setminus Q$ have the same meaning as in Example 10. Define $f$ by

$$f(x) = \begin{cases} h(x), & \text{if } x > 0, \\ h(-x) + \pi, & \text{if } x < 0, \end{cases}$$

where $\pi$ denotes a fixed irrational number. One can see in the same way as it was done in Example 10 that $f$ is a surjection. The continuity of $f$ follows from continuity of $h$. Note that $R \setminus Q$ is Čech-complete ([4], p. 252) and thus a Baire space ([4], 3.9.3, p. 253). Further, it follows from [4], Theorem 4.3.26, p. 343 that $X$ is completely metrizable. Nevertheless we have $\text{Int} f = \emptyset$ directly from the definitions.

3. Our further study of the set $\text{Int} f$ concerns the valuation of the Borel class of this set. To prove the main result of the present part of the paper, we need the following characterization of the complement of $\text{Int} f$.

**Proposition 2.** Let $f : X \to Y$ be an arbitrary mapping of a topological space $X$ onto a topological space $Y$, and let $\mathcal{B}$ be an open base for $X$. Then

$$X \setminus \text{Int} f = \bigcup \{ U \cap f^{-1}(\text{Fr} f(U)) : U \in \mathcal{B} \}. \quad (26)$$

In fact, if a point $x$ of $X$ is not in $\text{Int} f$, then there exists an element $U$ of $\mathcal{B}$ containing $x$ and such that $f(x) \in Y \setminus \text{Int} f(U)$, i.e., $f(x) \in \text{Fr} f(U)$, whence we see that $x \in U \cap f^{-1}(\text{Fr} f(U))$. Conversely, if there is an open set $U$ such that $x \in U \cap f^{-1}(\text{Fr} f(U))$, then $x \in X \setminus \text{Int} f$ by the definition of $\text{Int} f$. Hence the proposition follows.

**Theorem 2.** If a mapping $f : X \to Y$ of a second-countable regular space $X$ onto a topological space $Y$ is continuous, then $\text{Int} f$ is a $G_\delta$-set.

**Proof.** Note that every second-countable regular space is metrizable ([4], 4.3.6, p. 335), whence it follows that every open subset of such space is an $F_\sigma$-set ([4], 4.1.12, p. 318). Let $\mathcal{B}$ be a countable open base for the space $X$. For an arbitrary element $U$ of $\mathcal{B}$ the set $\text{Fr} f(U)$ is closed, so $f^{-1}(\text{Fr} f(U))$ is a closed subset of $X$ by continuity of $f$. Thus the intersection $U \cap f^{-1}(\text{Fr} f(U))$ is an $F_\sigma$-set, whence we have by (26) that $X \setminus \text{Int} f$ is an $F_\sigma$-set, too, and the conclusion follows.

Note that the valuation of the Borel class of $\text{Int} f$ in Theorem 2 is the best possible, as Example 2 shows.
It follows from Example 8 that regularity of $X$ is an essential assumption in Theorem 2.

Second-countability of $X$ is also necessary in Theorem 2; this can be seen by the following

**Example 12.** There exist a regular, neither first- nor second-countable and not perfectly normal space $X$, and a continuous mapping $f : X \to Y$ of $X$ onto a topological space $Y$ such that $\text{Int} f$ is a one-point set that is not a $G_\delta$-set.

Let $W_0$ be the set of all countable ordinal numbers, and let $Q$ denote the set of all rational numbers lying in the halfclosed interval $[0, 1)$. In the set $W_0 \times Q$ consider the linear order $<$ defined by letting $(a_1, t_1) < (a_2, t_2)$ whenever $a_1 < a_2$ or $a_1 = a_2$ and $t_1 < t_2$ (lexicographic order). Adjoining the point $\omega_1$ to $W_0 \times Q$ and assuming that $x < \omega_1$ for all $x \in W_0 \times Q$ we obtain a linearly ordered set; we topologize it with the topology induced by the linear order $<$ and we denote by $Y$ the topological space just defined. Note that $Y$ is a rational part of the long segment ([4], 3:12.18, p. 297).

To get $X$, we retopologize $Y$. Namely put $X = W_0 \times Q \cup \{\omega_1\}$ and define a topology (i.e. the family of open subsets) in $X$ as the family consisting of all subsets of $X$ that do not contain the point $\omega_1$ and of all subsets of $X$ that have finite or countable complement. Then all one-point subsets of $X$, except for the set $\{\omega_1\}$, are open and closed, the set $\{\omega_1\}$ is closed but not open. Thus $X$ is a $T_1$-space, whence regularity of $X$ follows by [4], 1.5.5., p. 60. It can be shown in the same manner as in [4], Example 1.1.8, p.31 that $X$ is not second-countable. Further, it is known that the one-point set $\{\omega_1\}$ is not a $G_\delta$-set, so $X$ is not perfectly normal (see [4], p. 68) and not first-countable space ([4], p. 58).

It can be easily verified that the identity mapping $f : X \to Y$ is continuous, and that $\text{Int} f = \{\omega_1\}$.

The next statement enables us to verify that continuity of $f$ is also an essential assumption of Theorem 2.

**STATEMENT 6.** For every dense proper subset $H$ of the real line $R$ (with the natural topology) there exists a non-continuous mapping $f : R \to H$ of $R$ onto $H$ such that $\text{Int} f = H$.

In fact, take a point $p \in H$ and define $f$ by

$$f(x) = \begin{cases} x & \text{for } x \in H, \\ p & \text{for } x \in R \setminus H. \end{cases}$$

It is easy to see that the conclusion of the statement holds.

Taking for $H$ an arbitrary dense subset of $R$ which is not a $G_\delta$-set (e. g. the set of all rational numbers) we get a needed example showing that continuity of $f$ is necessary in Theorem 2.
Remark that the union of an open set and of a $G_\delta$-set is again a $G_\delta$-set. Thus Theorem 2 and Statement 3 imply the following

**COROLLARY 6.** If a mapping $f : X \to Y$ of a second-countable regular space $X$ onto a topological space $Y$ is continuous, then (a) $\text{Int } f$ is a $G_\delta$-set; (b) $\text{Bd } f$ is an open set; (c) $\text{Fr } f$ is an $F_\sigma$-set.

By Corollary 6 and Proposition 1 we have

**COROLLARY 7.** If a mapping $f : X \to Y$ of a second-countable completely metrizable space $X$ onto a topological space $Y$ is continuous, then $\text{Int } f \cup \text{Bd } f$ is a dense $G_\delta$-subset of $X$ (thus it is of the second category).

Let us come back for a moment to Example 2. For the step function $f : I \to I$ constructed there we have the set $\text{Int } f$ which is neither closed nor open, but it is a $G_\delta$-set according to Theorem 2. Thus a question is natural concerning finding conditions for the mapping $f$ which guarantee improvement of the valuation of the Borel class of $\text{Int } f$. More precisely, we have the following

**Problem 1.** Let $f : X \to Y$ be a continuous mapping of a second-countable regular space $X$ onto a topological space $Y$. Give necessary and sufficient conditions for the mapping $f$ under which the set $\text{Int } f$ is (a) closed, (b) open.

The next problem is also related to Theorem 2.

**Problem 2.** Let a second-countable regular space $X$ be given. Characterize these $G_\delta$-sets $A \subset X$ for which there exist continuous mappings $f : X \to Y$ from $X$ onto a topological space $Y$ such that $A = \text{Int } f$.

4. Now we discuss behaviour of the set $\text{Int } f$ of points of $X$ at which a mapping $f : X \to Y$ is interior in case when the mapping is related in some way to other mappings as their composite, or their product, or the restriction to a subspace. We begin with the restriction of a mapping.

**STATEMENT 7.** Let $f : X \to Y$ be an arbitrary mapping of a topological space $X$ onto a topological space $Y$. Let $A \subset X$ be a subspace of $X$ such that $f(A) = Y$, and let $f \mid A : A \to Y$ be the partial mapping. Then

$$\text{Int } (f \mid A) \subset A \cap \text{Int } f.$$  \hspace{1cm} (27)

Indeed, since $A$ is equipped with the relative topology, and since $f(A) = Y$, the inclusion follows directly from the definitions.

Note however that the assumption $f(A) = Y$ is necessary in the statement. In fact, if in Example 7 we take $A = [n, n + 1]$ for a fixed $n$, then $f(A)$ is a straight line segment which is a closed boundary
subset of \( Y \); but \( f \mid A : A \to f(A) \subset Y \) is a linear mapping according to the definition of \( f \); so \( f \mid A \) is open, and we have \( \text{Int} (f \mid A) = A \) by Statement 2, while \( \text{Int} f = \emptyset \).

One can conjecture, under the hypotheses of Statement 7, that the inclusion (27) can be replaced by the equality

\[
\text{Int} (f \mid A) = A \cap \text{Int} f. \tag{28}
\]

However, it is not the case in general, as one can see by the following Example 13. There exist an open continuous mapping \( f : \mathbb{R} \to [0, \infty) \) from the real line onto the closed real half line (both with their natural topologies), and subsets \( A_1 \) and \( A_2 \) of \( \mathbb{R} \) such that \( f(A_i) = [0, \infty) \) and \( \text{Int} (f \mid A_i) \) is a proper subset of \( A_i \cap \text{Int} f \) for \( i = 1 \) and 2, \( A_1 \) is closed, \( A_2 \) has no isolated point, and \( f \mid A_2 \) is irreducible (i.e., no proper subset of \( A_2 \) is mapped onto the whole \([0, \infty))\).

Indeed, define \( f \) by \( f(x) = |x| \) for each \( x \in \mathbb{R} \) and put \( A_1 = (-\infty, 0] \cup \{1\} \) and \( A_2 = (-\infty, -1] \cup [0, 1) \). Then we have \( \text{Int} f = \mathbb{R} \), \( \text{Int} (f \mid A_1) = (-\infty, 0] \) and \( \text{Int} (f \mid A_2) = (-\infty, -1) \cup \{1\} \). In connection with Statement 7 and with the above example it is natural to ask under what conditions concerning the set \( A \) equality (28) holds in place of inclusion (27). The next statement, which easily follows from the definitions, partially answers this question.

**STATEMENT 8.** Let \( f : X \to Y \) be an arbitrary mapping of a topological space \( X \) onto a topological space \( Y \). Let \( A \subset X \) be an open subset of \( X \) such that \( f(A) \) is an open subset of \( Y \), and let \( f \mid A : A \to f(A) \subset Y \) be the partial mapping. Then equality (28) holds.

Observe that openness of \( A \) is essential in the statement by Example 13. Also openness of \( f(A) \) is necessary as it can be seen from Example 2 taking \( A = (1/3, 2/3) \); then \( f(A) = \{1/2\} \), \( f \mid A \) is an open mapping, so \( \text{Int} (f \mid A) = A \) by Statement 2; but the intersection in the right member of (28) is empty.

As an immediate consequence of Statement 8 we have

**COROLLARY 8.** Let \( X \) and \( Y \) be topological spaces and let a mapping \( f : X \to Y \) of \( X \) onto \( Y \) be continuous. Let \( V \) be an open subset of \( Y \), and let \( f \mid f^{-1}(V) : f^{-1}(V) \to V \) be the partial mapping. Then equality (28) holds with \( A = f^{-1}(V) \).

Since the set \( \text{Int} f \) is --- by definition --- the set of all such points \( x \in X \) at which the mapping \( f : X \to Y \) is interior, it is tempting to conjecture that the restriction of \( f \) to \( \text{Int} f \) is interior at each point of its domain, i.e., that (see Statement 2) \( f \mid \text{Int} f \) is an open mapping, which means that

\[
\text{Int} (f \mid \text{Int} f) = \text{Int} f. \tag{29}
\]

However, it is not so, because of the following...
Example 14. There exists a continuous mapping \( g : [0, 2] \to [0, 1] \) of the closed interval \([0, 2]\) onto the closed interval \([0, 1]\) for which \( \text{Int} (g \mid \text{Int} g) \) is a proper subset of \( \text{Int} g \).

Indeed, let \( f : I \to I \) be the step-function considered in Example 2, and put
\[
g(x) = \begin{cases} f(x), & \text{if } x \in [0, 1] \\ 2 - x, & \text{if } x \in (1, 2). \end{cases}
\]

Then \( g \) is obviously continuous, and we have \( \text{Int} g = (C \setminus E) \cup (1, 2] \) and \( g(\text{Int} g) = [0, 1] \). Observe that the images of sufficiently small neighborhoods of points of \([0, 1) \cap \text{Int} g = (C \setminus E) \setminus \{1\}\) under the partial mapping \( g \mid \text{Int} g \) have the empty interiors, while the other points of \( \text{Int} g \), i.e., the points of \([1, 2]\) are ones at which \( g \mid \text{Int} g \) is interior. So we have \( \text{Int} (g \mid \text{Int} g) = [1, 2] \).

In the light of the above example one can ask the following question which seems to be rather natural: given two topological spaces \( X \) and \( Y \), characterize these (continuous) surjections \( f : X \to Y \) for which equality (29) holds.

We consider now the composite of two mappings.

**STATEMENT 9.** Let topological spaces \( X, Y \) and \( Z \) be given. If mappings \( f : X \to Y \) and \( g : Y \to Z \) are surjective (i.e., onto), then
\[
f^{-1} (\text{Int} g) \cap \text{Int} f \subset \text{Int} (gf). \tag{30}
\]

If, in addition, the mapping \( f \) is continuous, then
\[
\text{Int} (gf) \subset f^{-1} (\text{Int} g). \tag{31}
\]

**Proof.** Take a point \( x \in f^{-1} (\text{Int} g) \cap \text{Int} f \) and its neighborhood \( U \). Since \( x \in \text{Int} f \), the set \( f(U) \) is a neighborhood of the point \( f(x) \) in \( Y \), and since \( f(x) \in \text{Int} g \), we see that the set \( g(f(U)) \) is a neighborhood of the point \( g(f(x)) \) in \( Z \), whence \( g(f(x)) \in \text{Int} g(f(U)) \), and therefore (30) is proved.

To show (31) take a point \( x \in \text{Int}(gf) \) and a neighborhood \( V \) of the point \( f(x) \) in \( Y \). By continuity of \( f \) there exists a neighborhood \( U \) of \( x \) in \( X \) such that \( f(U) \subset V \). Since \( x \in \text{Int}(gf) \), we have \( g(f(x)) \in \text{Int} g(f(U)) \), and therefore (31) holds, and therefore the proof is complete.

Observe that no one of the two inclusions (30) and (31) can be replaced by the corresponding equality. In fact, take a continuous surjection \( f : X \to Y \) with \( \text{Int} f = \emptyset \) and a constant mapping \( g : Y \to Z \) of \( Y \) onto a one-point space \( Z \). Then the intersection in the left member of (30) is empty, while the right one is equal to the whole \( X \), so we do not have the equality in (30). To see the same for (31), take a continuous surjection \( f : X \to Y \) with \( \text{Int} f \neq X \), put \( Z = Y \) and define \( g : Y \to Y \) be the identity. Then \( \text{Int} (gf) = \text{Int} f \neq X \), while \( f^{-1} (\text{Int} g) = f^{-1}(Y) = X \).
To see that continuity of \( f \) is necessary for (31), consider the following

**Example 15.** There are topological spaces \( X, Y \) and \( Z \) and surjective mappings \( f : X \to Y \) and \( g : Y \to Z \) such that \( g \) is continuous while \( f \) is not, \( \text{Int} \, g = \emptyset \) and \( \text{Int} \, (gf) \neq \emptyset \).

Take as \( X \) the real line \( \mathbb{R} \) (with the natural topology), as \( Y \) the set of real numbers with the discrete topology and let \( f : X \to Y \) be the identity. Further, consider as the space \( Z \) the set \( \mathbb{Q} \) of all rationals with its natural topology (inherited from the real line), and define \( g : Y \to Z \) by

\[
g(y) = \begin{cases} y, & \text{if } y \in \mathbb{Q}, \\ 0, & \text{if } y \in Y \setminus \mathbb{Q}. \end{cases}
\]

Observe that the composite \( gf : \mathbb{R} \to \mathbb{Q} \) is just the mapping mentioned in Statement 6 (with \( H = \mathbb{Q} \)), so we have \( \text{Int} \, (gf) = \mathbb{Q} \subset \mathbb{R} = X \). The other conditions are obvious.

The following corollaries are immediate consequences of Statements 9 and 2.

**COROLLARY 9.** Let topological spaces \( X, Y \) and \( Z \) be given, and let mappings \( f : X \to Y \) and \( g : Y \to Z \) be surjections. If \( f \) is continuous and open, then

\[
f^{-1}(\text{Int} \, g) = \text{Int} \, (gf).
\]

**COROLLARY 10.** Let topological spaces \( X, Y \) and \( Z \) be given, and let mappings \( f : X \to Y \) and \( g : Y \to Z \) be surjections. If \( g \) is open, then

\[
\text{Int} \, f \subset \text{Int} \, gf.
\]

To see that openness of \( f \) is essential in Corollary 9 put \( X = Y = Z = I \), define \( f \) as the step function (see Example 2) and \( g \) as the identity. Then \( f^{-1}(\text{Int} \, g) = I \), while \( \text{Int} \, gf = C \setminus E \) according to (9). Taking the same mappings in the opposite order, i.e., \( f \) as the identity and \( g \) as the step function, one can see that openness of \( g \) is essential in Corollary 10.

The next result is related to the concept of the Cartesian product of mappings (see [4], p. 109 for the definition). Since this result is an easy consequence of the definitions, its proof is omitted.

**STATEMENT 10.** Let an arbitrary family of surjective mappings \( f_t : X_t \to Y_t \) (where \( t \in T \)) be given, and let \( f = \prod_{t \in T} f_t : \prod_{t \in T} X_t \to \prod_{t \in T} Y_t \) be the Cartesian product of mappings of this family. Then

\[
\text{Int} \, f = \prod_{t \in T} \text{Int} \, f_t.
\]
The authors do not discuss possible relations between interiority of a mapping and some other operations on topological spaces as taking of the disjoint union, of the inverse limit and so on. The study of such and related problems is left for the future.

5. In the end of the paper we shortly discuss connections between interiority of a mapping at a point and some related concepts.

Kuratowski in [7], § 13, XIV, p. 116 defines a continuous mapping \( f : X \to Y \) to be open at a point \( y_0 \in Y \) if for every open subset \( U \) of \( X \) we have that \( y_0 \in f(U) \) implies \( y_0 \in \text{Int} f(U) \). To compare interiority at a point with openness at a point, we delete continuity of \( f \) from the latter one, and we accept the following definition.

A mapping \( f : X \to Y \) of \( X \) into \( Y \) is said to be open at a point \( y_0 \in f(X) \subset Y \) provided that for every open set \( U \subset X \) if \( y_0 \in f(U) \), then \( y_0 \in \text{Int} f(U) \).

Similarly to Statement 1 we have

**STATEMENT 11.** Let \( A \) and \( \mathcal{B} \) be bases for topological spaces \( X \) and \( Y \) respectively. The following are equivalent for a mapping \( f : X \to Y \) and for a point \( y_0 \in f(X) \subset Y 

1. \( f \) is open at \( y_0 \);
2. for every \( A \in A \), if \( y_0 \in f(A) \), then \( y_0 \in \text{Int} f(A) \);
3. for every \( A \in A \) there exists \( B \in \mathcal{B} \) such that if \( y_0 \in f(A) \), then \( y_0 \in B \subset f(A) \);
4. for every subset \( S \) of \( X \), if \( y_0 \in f(\text{Int} S) \), then \( y_0 \in \text{Int} f(S) \).

Indeed, the circle of implications can be easily verified.

The next statement is a straight consequence of definitions.

**STATEMENT 12.** A mapping \( f : X \to Y \) is open at a point \( y_0 \in f(X) \subset Y \) if and only if it is interior at each point of \( f^{-1}(y_0) \subset X \).

Note that interiority of a mapping \( f \) at a point \( x_0 \in X \) does not imply openness of \( f \) at \( f(x_0) \). For example, the mapping \( f : R \to R \) defined by \( f(x) = x(x - 1)^2 + 2 \) for \( x \in R \) is interior at \( x_0 = 0 \) but it is not open at \( f(x_0) = 2 \) because it has a local minimum equal to 2 at the point \( x_1 = 1 \), so it is not interior at \( x_1 \in f^{-1}(f(x_0)) \).

Applying a general notion of localization of a mapping (see [10], Chapter 4, C, p.18) to the class of open mappings we get the following definition. A mapping \( f : X \to Y \) is called locally open at a point \( x_0 \in X \) provided that there exists a closed neighborhood \( V \) of \( x_0 \) such that \( f(V) \) is a closed neighborhood of \( f(x_0) \) and the partial mapping \( f \mid V \) is open.

We have the following
STATEMENT 13. If a mapping \( f : X \to Y \) is locally open at a point \( x_0 \in X \), then it is interior at \( x_0 \).

Proof. Let \( V \) be a subset of \( X \) such that
\[
x_0 \in \text{Int} \ V \subset V = \overline{V},
\]
\[
f(x_0) \in \text{Int} f(V) \subset f(V) = f(\overline{V}),
\]
\[
f|_V : V \to f(V) \text{ is open},
\]
and let \( U \) be an open subset of \( X \) with \( x_0 \in U \). Thus \( U \cap \text{Int} V \) contains \( x_0 \) by (36), whence \( f(x_0) \in f(U \cap \text{Int} V) \). But \( U \cap \text{Int} V \) is an open subset of \( V \), and by (38) we conclude that \( f(U \cap \text{Int} V) \) is an open subset of \( f(V) \), i.e., that there is an open subset \( G \) of \( Y \) such that \( f(U \cap \text{Int} V) = G \cap f(V) \). Therefore the set \( G \cap \text{Int} f(V) \) is open in \( Y \) and it contains \( f(x_0) \) by (37). Furthermore, this set is contained in \( f(U) \) and, being open in \( Y \), it is contained in \( \text{Int} f(U) \). Hence \( f(x_0) \in \text{Int} f(U) \), and so \( f \) is interior at \( x_0 \).

The inverse implication is not true: interiority of a mapping at a point does not imply its local openness at this point even for continuous mappings from reals to reals. It can be seen by an example of a mapping \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 
  x \sin \frac{1}{x} & \text{for } x \neq 0, \\
  0 & \text{for } x = 0,
\end{cases}
\]
that is continuous and interior at \( x_0 = 0 \) but not locally open at this point: for no neighborhood \( V \) of \( x_0 \) the partial mapping \( f|_V \) is open. Further, observe that the same mapping is open at the point \( y_0 = 0 \), and hence openness of a mapping at \( f(x_0) \) does not imply its local openness at \( x_0 \). However, we have the following statement, which is a direct consequence of Statements 13 and 12.

STATEMENT 14. If a mapping \( f : X \to Y \) is locally open at each point of \( f^{-1}(y_0) \), then it is open at \( y_0 \).

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On the set of interiority of a mapping


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O SKUPU NUTARNJIH TOČAKA PRESLIKAVANJA

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Sadržaj

Neka je $f : X \rightarrow Y$ preslikavanje na (koje nije nužno neprekidno). U članku se istražuje skup nutarnjih točaka $\text{Int} f \subseteq X$ preslikavanja $f$. Nađeni su uvjeti pod kojima je taj skup neprazan i $G_δ$. 