Abstract. Metric spaces which are images of linear graphs under local expansions are investigated. (54E40)

Let $X$ and $Y$ be metric spaces with metrics $d_X$ and $d_Y$ respectively. A mapping $f: X \to Y$ of $X$ onto $Y$ is said to be a local expansion if it is continuous and if for every point $x$ of $X$ there exists an open neighborhood $U$ of $x$ and a constant $M > 1$ such that

$$(1) \quad M \cdot d_X(x',x'') \leq d_Y(f(x'),f(x'')) \quad \text{for every } x',x'' \in U.$$

It is known (see [2], Proposition 1 (v)) that if the space $X$ is compact, then the inverse image $f^{-1}(y)$ of every point $y \in Y$ is finite. However, a stronger result can be proved. Namely we have

**Proposition 1.** For every compact space $X$ and for every local expansion $f: X \to Y$ of $X$ onto a metric space $Y$ there is a natural number $m$ such that

$$(2) \quad \text{card } f^{-1}(y) \leq m \quad \text{for each point } y \in Y.$$

Indeed, for every point $x \in X$ take an open neighborhood $U_x$ as in the definition of the local expansion $f$. Thus $\{U_x : x \in X\}$ is an open cover of $X$, and by compactness of $X$ it contains a finite subcover $\{U_{x_i} : i=1,2,\ldots,n\}$. If the conclusion does not hold, then there is a point $y \in Y$ such that $\text{card } f^{-1}(y) > n$. Thus some two distinct points $x'$ and $x''$ of $f^{-1}(y)$ must be in the same $U_{x_i}$, which contradicts (1) because $f(x') = f(x'') = y$.

Note that the number $m$ heavily depends on $f$: if $X = Y = \{z : |z| = 1\}$ (where $z$ is a complex number) and if $f_n: X \to Y$ is given by $f_n(z) = z^n$ (for natural $n$), then we have $\text{card } f_n^{-1}(z) = n$ for each $z \in X$.

By a free arc in a space $X$ we mean an arc $ab$ such that $ab \setminus \{a,b\}$ is an open subset of $X$. A connected set is said to be a graph pro-
vided it is the union of a finite set of points, called vertices, and of a finite number of free arcs, called edges, so that the two end points of each edge are vertices. It is known that a metric continuum is a graph if and only if every its point is of some finite order and almost all its points are of order less than or equal to 2, and that every graph admits an equivalent convex metric. Thus all graphs considered here are assumed to be equipped with a convex metric. We denote by $E(X)$ the set of all end points and by $R(X)$ the set of all ramification points of a graph $X$. Note that for every graph $X$ these sets are finite.

Easy examples show that an image of a graph under a local expansion need not be a graph (see [2], Example 1). However, we have

**Proposition 2.** Every image of a graph under a local expansion is the union of a finite number of arcs.

**Proof.** Let $d$ be a (convex) metric on a graph $X$. For every $x \in X$ let $U_x$ be an open neighborhood as in the definition of the local expansion $f: X \to Y$. Thus the family $\{U_x: x \in X\}$ is a covering of $X$, and by compactness of $X$ it contains a finite subcovering $\mathcal{C} = \{U_{x_i}: i=1,2,\ldots,n\}$. Hence there is the Lebesgue coefficient of $\mathcal{C}$, i.e., such a positive number $c$ that each two points $y$ and $z$ of $X$ with $d(y,z) < c$ belong to the same element of $\mathcal{C}$ (see [4], §41, VI, Corollaries 4c and 4d, p. 23 and 24). Take a finite set $V$ in $X$ containing $E(X)$ and $R(X)$ and such that every point of $X$ is at a distance less than $c/2$ from some point of $V$ (see [3], §21, VIII, Theorem 1, p. 215 and [4], §41, VI, Theorem 1, p. 20). Consider $V$ as the set of vertices of $X$ and let $\{E_j: j=1,\ldots,m\}$ be the family of edges of $X$ with respect to $V$. Thus $X = \bigcup\{E_j: j=1,\ldots,m\}$. So for every $E_j$ there is $U_{x_j} \in \mathcal{C}$ such that $E_j \subseteq U_{x_j}$. By Proposition 1 (ii) of [2] for every $j$ the partial mapping $f|E_j$ is a homeomorphism. Hence $Y = \bigcup\{f(E_j): j=1,\ldots,m\}$ is the union of $m$ arcs $f(E_j)$.

Therefore we conclude from [5], p. 179 that every image of a graph under a local expansion is regular in the sense of theory of order (see [4], §51, I, p. 275) which means that each point has arbitrarily small neighborhoods whose boundaries are finite sets. The union of finitely many (even of two) arcs may contain a point of order $\omega$: see Menger's example in [5], p. 179 and 180. An easy modification of this example leads to a continuum which is also the union of two arcs containing a point of order $\omega$ and which admits a local expansion from $[0,1]$ onto itself. So we have
PROPOSITION 3. There exists a regular curve which contains a point of order $\omega$ and which is an image of the closed unit interval under a local expansion.

One can ask whether every regular curve, or every regular curve with bounded order of its points is an image of some graph under a local expansion. The answer is negative because if a metric space $X$ is mapped onto a metric space $Y$ under a local expansion $f$, then for each arc $ab \subset X$ no point of $ab \setminus \{a, b\}$ is mapped on an end point of $f(ab)$ (see [2], Proposition 1 (iv)), and hence for each end point $y$ of $Y$ the set $f^{-1}(y)$ is composed of end points of $X$ only. So we have

PROPOSITION 4. If a metric space $Y$ is an image of a metric space $X$ under a local expansion, then card $E(Y) \leq$ card $E(X)$; in particular, if $X$ is a graph, then the set $E(Y)$ of end points of the curve $Y$ is finite.

The problem below forms a program of further investigations.

PROBLEM. Given a graph $X$, characterize the class of all its images under local expansions, i.e., characterize (internally) all spaces $Y$ for which there is a local expansion from $X$ onto $Y$.

The authors do not have such characterizations even in case of the simplest graphs $X$: an arc or a circle. But if the class of locally expansive images of $X$ is restricted to graphs only, then we have

PROPOSITION 5. A graph is an image of an arc (of a circle) under a local expansion if and only if it has at most two (it has no) end points.

In fact, one way is a consequence of Proposition 4. To construct a local expansion onto a graph $Y$ we first choose a chain (a circular chain) containing all edges of $Y$ and ending at end points of $Y$ (if any). The existence of such a chain can be proved likely as in the proof of Lemma 2 of [1]. Next we divide the arc (the circle) into subarcs, the number of which is equal to the number of links in the chain, and we apply a proper standard mapping ($\S3$ of [1]) in a manner as it is done in [1], Theorem 1.

References

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