A Characterization of Cyclic Graphs

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Summary. It is proved that a graph $Y$ is cyclic if and only if for each natural number $n$ there exists a graph $X_n$ and a local homeomorphism of degree $n$ from $X_n$ onto $Y$.

The aim of this paper is to characterize cyclic graphs via local homeomorphisms.

By a graph we mean a one-dimensional connected polyhedron. A graph which contains a simple closed curve is called cyclic; otherwise it is said to be acyclic. An arc $A$ with end points $a$ and $b$ contained in a continuum $X$ is called free provided that $A\setminus\{a, b\}$ is an open subset of $X$. A continuous mapping $f:X\to Y$ of a topological space $X$ onto a topological space $Y$ is said to be a local homeomorphism provided that every point $x\in X$ has an open neighborhood $U$ such that the partial mapping $f|U:U\to f(U)$ is a homeomorphism and $f(U)$ is an open subset of $Y$ ([2], p. 199). Given a mapping $f:X\to Y$, we define a mapping $k$ from $Y$ to cardinal numbers (called the degree of $f$) putting $k(y) = \text{card}\, f^{-1}(y)$ for $y\in Y$. It is known that if $f:X\to Y$ is a local homeomorphism of a graph $X$ onto $Y$, then $Y$ is a graph (since $f$ is open, see [2], p. 199 and Theorem 1.1, p. 182) whence it follows that the degree $k$ is constant on $Y$ ([2], Theorem 6.1, p. 199).

A well-known result of Whyburn says that each local homeomorphism $f:X\to Y$ from a continuum $X$ onto a dendritic $Y$ is a homeomorphism (see [2], Corollary, p. 199). Applying this result to graphs we conclude that each local homeomorphism from a continuum onto an acyclic graph is a homeomorphism. Taking the unit circle $S = \{z: |z| = 1\}$ and defining $f:S\to S$ by $f(z) = z^n$ for $n = 1, 2, 3, ...$ we can see not only that the acyclicity is an essential assumption in this result, but moreover that for every natural number $n$ there exists a local homeomorphism of degree $n$ of a graph onto a circle (this characterizes a simple closed curve among graphs, see [1]). We generalize the last statement from a circle to arbitrary graphs containing a simple closed curve. Namely we have
THEOREM. For each cyclic (planar) graph \( Y \) and for each natural number \( n = 1, 2, 3, \ldots \) there exist a cyclic (planar) graph \( X \) and a local homeomorphism of degree \( n \) from \( X \) onto \( Y \).

Proof. Let a cyclic graph \( Y \) be given, and consider a simple closed curve in \( Y \) as the union of a free arc \( A \) with end points \( a \) and \( b \) and of the other arc \( B \), i.e. such that \( A \cap B = \{ a, b \} \). Further take \( p, c, d \in A \setminus \{ a, b \} \) such that \( c \in ap \) and \( d \in bp \), where \( ap \cup pb = A \). Let \( ac, \ \text{and} \ \ bd \) denote the corresponding subarcs in \( A \), and put \( Y' = (X \setminus A) \cup ac \cup bd = X \setminus cd \). Thus \( Y' \) is a graph contained in \( Y \). Now let a mapping \( g: Y' \to Y \) be defined from \( Y' \) onto \( Y \) in such a way that \( g(X \setminus A) \) is the identity, while \( g|ac: ac \to ap \) and \( g|bd: bd \to bp \) are homeomorphisms with \( g(a) = a \), \( g(b) = b \) and \( g(c) = g(d) = p \). Thus \( g \) is continuous by its definition, each point of \( Y \setminus \{ p \} \) has a one-point inverse, while the inverse image of \( p \) is the two-point set \( \{ c, d \} \). Fix an arbitrary natural number \( n \) and consider \( n \) copies \( X_1, X_2, \ldots, X_n \) of the graph \( Y' \) in which a point \( x_i \in X_i \) denotes the copy of a point \( x \in Y' \) for every \( i \in \{ 1, 2, \ldots, n \} \). Form the union \( X = \bigcup \{ X_i: i = 1, 2, \ldots, n \} \) such that, if \( n = 2 \), then \( X_1 \cap X_2 = \{ c_1, d_2 \} = \{ c_2, d_1 \} \), and if \( n > 2 \), then \( X_i \cap X_{i+1} = \{ d_i \} = \{ c_{i+1} \} \) for every \( i = 1, 2, \ldots, n-1 \), and \( X_1 \cap X_n = \{ d_n \} = \{ c_1 \} \), and \( X_i \cap X_j = \emptyset \) for all other pairs of different indices \( i, j \). Therefore the set \( X \) is a graph. For every \( i = 1, 2, \ldots, n \) denote by \( c_i, d_i \) the arc from \( c_i \) to \( d_i \) in \( X_i \) being the \( i \)-th copy of the arc \( ca \cup B \cup bd = A \cup B \cup cd \) from \( c \) to \( d \) in \( Y \). Then the union \( c_1 d_1 \cup c_2 d_2 \cup \ldots \cup c_n d_n \) (where \( d_1 = c_2, d_2 = c_3, \ldots, d_{n-1} = c_n, d_n = c_1 \)) is a simple closed curve contained in \( X \), so the graph \( X \) is cyclic. It is evident that, in case of \( Y \) is planar, the whole construction can be made in such a way that the resulting space \( X \) is contained in a plane, too.

Define \( f: X \to Y \) putting \( f|X_i = g \) for every \( i = 1, 2, \ldots, n \). This definition is correct because every \( X_i \) is a copy of \( Y' \), so we can assume that \( g \) maps \( X_i \) onto \( Y \). It is obvious that, for each point \( y \in Y \) its inverse image \( f^{-1}(y) \) consists of \( n \) points exactly. Namely \( f^{-1}(p) = \{ d_1, d_2, \ldots, d_n \} = \{ c_2, c_3, \ldots, c_n, c_1 \} \), while \( f^{-1}(y) \) for \( y \neq p \) consists of some \( n \) points \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) with \( x^{(i)} \in X_i \) for \( i = 1, 2, \ldots, n \). So \( f \) is of degree \( n \). Further, it is quite easy to observe that each point of \( X \) has a small open neighborhood whose image under \( f \) is an open subset of \( Y \), and moreover such that \( f \) restricted to this neighborhood is a homeomorphism. Hence \( f \) is a local homeomorphism, and the proof is finished.

Consider a class of all graphs \( Y \) having the property that \( (*) \) for every natural number \( n \) there exist a graph \( X_n \) and a local homeomorphism \( f_n: X_n \to Y \) of degree \( n \) from \( X_n \) onto \( Y \).

The theorem above and the mentioned result of Whyburn ([2], Corollary, p. 199) imply

COROLLARY 1. A graph \( Y \) is cyclic if and only if \( Y \) has property \( (*) \).
Corollary 2. A graph $Y$ has the property that each local homeomorphism from a continuum onto $Y$ is a homeomorphism if and only if $Y$ is acyclic.

A method has been shown in the proof of the theorem of constructing a graph $X$ which can be mapped onto a given graph $Y$ under a local homeomorphism of a prescribed degree $n$. The structure of $X$ depends on the choice of a free arc $A$ in $Y$. Taking an arc $A' \neq A$ under consideration in place of $A$ it is possible to get, under some circumstances, another (i.e. topologically different) graph $X$, even if the same method of construction has been applied. So we have

Problem 1. Given a graph $Y$ and a natural number $n$, what is the cardinality of the largest class of topologically different graphs $X$ such that $Y$ is an image of $X$ under a local homeomorphism of degree $n$?

Whyburn has defined in [2], Example, p. 189, a planar graph $X$ and a local homeomorphism of degree 3 from $X$ onto one of the two well-known non-planar graphs of Kuratowski. In connection with this we have

Problem 2. For which non-planar graphs $Y$ and for which natural numbers $n$ there exist a planar graph $X$ and a local homeomorphism of degree $n$ from $X$ onto $Y$?

REFERENCES


Я. Е. Харатоник, С. Миклос, Характеризация циклических графов

Доказывается, что граф $Y$ является циклическим тогда и только тогда, когда для всякого натурального числа $n$ найдутся: граф $X_n$ и локальный гомеоморфизм степени $n$ с $X_n$ на $Y$. 