Generalized Homogeneity
and Some Characterizations of Solenoids

by

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Presented by C. RYLL-NARDZEWSKI on November 9, 1982

Summary. It is proved that a circle-like continuum is a solenoid if and only if it is homogeneous with respect to the class of open mappings and each of its subcontinua is an arc. Some related results are shown and several questions are raised.

A space \( X \) is said to be homogeneous with respect to the class \( M \) of mappings if for every two points \( p \) and \( q \) of \( X \) there exists a continuous surjection \( f \) of \( X \) onto itself such that \( f \in M \) and \( f(p) = q \). If \( M \) is the class of homeomorphisms, then we get the well-known notion of a homogeneous space.

A continuum means a compact connected metric space. We use the concept of an arc-like (circle-like) continuum in the sense introduced by Bing in [2] and discussed by many authors (see e.g. [6] for exact definitions). A point \( p \) of a continuum \( X \) is called an end point of \( X \) provided that if \( K \) and \( L \) are two subcontinua of \( X \) both containing \( p \), then either \( K \subset L \) or \( L \subset K \). A continuum is said to be indecomposable if it is not the union of two proper subcontinua. It is hereditarily indecomposable if each of its subcontinua is indecomposable. A pseudo-arc is a non-degenerate, hereditarily indecomposable, arc-like continuum.

A mapping \( f : X \to Y \) is open if it maps open sets in \( X \) onto open sets in \( Y \). It is called confluent if for every subcontinuum \( Q \) of \( Y \) every component of \( f^{-1}(Q) \) is mapped onto the whole \( Q \) under \( f \).

In [4], p. 270, the author proved the following theorem, giving some characterizations of the pseudo-arc via homogeneity with respect to various classes of mappings (the most essential result contained in that theorem, namely homogeneity of the pseudo-arc, is due to Bing—see [1], Theorem 13, p. 740).

\textbf{Theorem A.} Let a non-degenerate continuum \( X \) be arc-like. Then the following conditions are equivalent:
(i) $X$ is homogeneous;
(ii) $X$ is homogeneous with respect to open mappings;
(iii) $X$ is homogeneous with respect to confluent mappings and there is an end point in $X$;
(iv) $X$ is homogeneous with respect to continuous mappings, there is an end point in $X$, and $X$ is hereditarily indecomposable;
(v) $X$ is the pseudo-arc.

By a solenoid we mean the inverse limit of a sequence of unit circles \{\(z \in \mathbb{R}^2 : |z| = 1\)\} with bonding mappings having the form \(z \to z^n\) for some natural $n$. The aim of this paper is to give a characterization of solenoids in the class of circle-like continua, which is similar to one of pseudo-arcs obtained in Theorem A for the class of arc-like continua. The main theorem characterizes solenoids using homogeneity with respect to open mappings. However, the author was not able to get a characterization of this sort in terms either of confluent or of continuous mappings. The proof of the characterization obtained heavily depends on the property of Kelley and essentially exploits the characterization of solenoids due to Krupski ([7], Theorem 1, p. 379).

The lemma below has been formulated in [4], p. 270.

**Lemma.** The image of an end point of a continuum $X$ under a confluent mapping $f$ from $X$ onto $Y$ is an end point of $Y$.

Indeed, let $K$ and $L$ be subcontinua of $Y$ with $f(x) \in K \cap L$, where $x$ is an end point of $X$, and let $C$ and $D$ be components of $f^{-1}(K)$ and $f^{-1}(L)$, respectively, with $x \in C \cap D$. Then either $C \subseteq D$ or $D \subseteq C$ by the definition of the end point $x$, and $f(C) = K$ and $f(D) = L$ by confluency of $f$. Thus either $K \subseteq L$ or $L \subseteq K$.

Consider the following properties (A) and (B) a continuum can possess:

(A) every proper subcontinuum of the continuum is an arc;
(B) every point $x$ of the continuum belongs to an arc in the continuum, and the end points of the arc both are different from $x$.

**Proposition.** If a non-degenerate continuum $X$ is homogeneous with respect to the class of confluent mappings and if it has property (A), then it has property (B).

Indeed, if not, then there exists a point $x \in X$ that is an end point of each arc in $X$ containing $x$. Since each proper subcontinuum of $X$ is an arc by (A), we conclude that $x$ is an end point of $X$. Thus homogeneity of $X$ with respect to the class of confluent mappings implies by the lemma that each point of $X$ is an end point of $X$. However, any point that lies on an arc and is different from end points of the arc cannot be an end point of $X$, and the contradiction follows.

Observe the following easy implication.
If a non-degenerate continuum $X$ is homogeneous, and if it contains an arc, then it has property (B).

**Question 1.** Can homogeneity of $X$ be relaxed to homogeneity with respect to the class of open mappings in (\*)?

We say that a continuum $X$ (with a metric $d$) has the property of Kelley provided that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each two points $a$ and $b$ of $X$ satisfying $d(a, b) < \delta$ and for each subcontinuum $A$ of $X$ containing the point $a$ there exists a subcontinuum $B$ of $X$ containing the point $b$ and satisfying $\text{dist}(A, B) < \varepsilon$ (here dist denotes the Hausdorff distance).

**Theorem.** Let a continuum $X$ be circle-like. Then the following statements I–V are equivalent:

I. $X$ is homogeneous, and $X$ contains an arc;
II. $X$ is homogeneous, and every proper subcontinuum of $X$ is an arc;
III. $X$ is homogeneous with respect to the class of open mappings, and every proper subcontinuum of $X$ is an arc;
IV. $X$ has the property of Kelley, and each point $x \in X$ belongs to an arc with end points different from $x$;
V. $X$ is a solenoid.

**Proof.** We have the following circle of implications. The implication from II to III is obvious. A statement has been proved in [5] that if a continuum is homogeneous with respect to the class of open mappings, then it has the property of Kelley. Hence III implies IV by the proposition (using the well-known fact that open mappings of compact spaces are confluent, [9], (7.5), p. 148). The implication from IV to V is proved in [7], Theorem 1, p. 379; and one from V to II is well known (cf. [6], Theorem 2, p. 434). Thus conditions II thru V are equivalent. Further, the implication from II to I is trivial, and finally I implies IV since homogeneous continua have the property of Kelley (see [8], Theorem 2.5, p. 293) and since (\*) holds. So the proof is complete.

Note that the equivalence of conditions I and V is due to Bing (see [3], Theorem 9, p. 228 for the implication from I to V; the other way is a well-known result).

Comparison of conditions I and II with condition III leads to the formulation of the following condition:

VI. $X$ is homogeneous with respect to the class of open mappings, and $X$ contains an arc.

The implication from III to VI is obvious. The inverse is unknown. So we have

**Question 2.** Let a circle-like continuum $X$ satisfy condition VI. Is then $X$ a solenoid?
Remark that an affirmative answer to Question 1 leads to the implication from VI to IV, whence a positive answer to Question 2 would follow.

Hagopian has proved ([6], Theorem 1, p. 429) that if a continuum \( X \) satisfies condition II of the theorem, then \( X \) is circle-like.

**Question 3.** Does condition III of the Theorem imply circle-likeness of the continuum \( X \)?

**REFERENCES**


Я. Е. Харатоник, Обобщённая однородность и некоторые характеристики соленоидов

Доказывается, что окружно-подобный континуум является соленоидом тогда и только тогда, когда он однородный относительно класса открытых отображений, и каждый его собственный подконтинуум является дугой. Кроме этого рассматриваются некоторые результаты связанные с этой характеристикой и ставятся нерешённые проблемы.