SOME GENERALIZATIONS OF HOMOGENEITY OF SPACES

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ABSTRACT. The concept of a homogeneous topological space is generalized by considering not only homeomorphisms, but also certain other classes of mappings. Some problems involving the new concepts are formulated.

Bibliography: 26 titles.

A topological space $X$ is said to be homogeneous if for any two points in it there is a homeomorphism of $X$ onto itself that carries one of the points into the other. Generalizing this notion, David P. Bellamy (in a conversation with the author) replaced a homeomorphism by an arbitrary continuous mapping "onto" and obtained the concept of a space that is homogenous with respect to continuity. In the present article this concept is generalized by considering not only homeomorphisms and continuous mappings, but also certain other classes of mappings. Some problems involving the concepts introduced are formulated. Our main goal is to lay the groundwork for studying several of the problems given below. This article also contains some new results.

DEFINITION 1. A space $X$ is said to be homogeneous with respect to a class $\mathcal{M}$ of mappings (written $X \in H(\mathcal{M})$) if for any two points $p$ and $q$ in $X$ there exists a continuous mapping $f$ of $X$ onto itself such that $f \in \mathcal{M}$ and $f(p) = q$.

If the $\mathcal{M}$ in Definition 1 is taken to be the class of homeomorphisms of $X$ onto itself, we obtain the familiar concept of homogeneity for a space $X$. If $\mathcal{M}$ is taken to be the class of continuous mappings of $X$ onto $X$, Bellamy's concept of homogeneity with respect to continuity is obtained.

DEFINITION 2. A space $X$ is said to be bihomogeneous with respect to a class $\mathcal{M}$ of mappings if for any two points $p$ and $q$ in $X$ there exists a continuous mapping $f$ of $X$ onto $X$ such that $f \in \mathcal{M}$, $f(p) = q$, and $f(q) = p$.

DEFINITION 3. A space $X$ is said to be argument-homogeneous with respect to a class $\mathcal{M}$ of mappings at a point $q \in X$ (written $X \in H(\mathcal{M})A_q$) if for any $p \in X$ there exists a continuous mapping $f$ of $X$ onto $X$ such that $f \in \mathcal{M}$ and $f(p) = q$.

DEFINITION 4. A space $X$ is said to be value-homogeneous with respect to a class $\mathcal{M}$ of mappings at a point $p \in X$ (written $X \in H(\mathcal{M})V_p$) if for any $q \in X$ there exists a continuous mapping $f$ of $X$ onto $X$ such that $f \in \mathcal{M}$ and $f(p) = q$.

The following proposition follows at once from the definitions.
PROPOSITION 1. Suppose that a space $X$ is given along with a class $\mathcal{M}$ of continuous mappings of $X$ onto itself. The following three conditions are equivalent:

(a) $X \in H(\mathcal{M})$.

(b) $X \in H(\mathcal{M}) A_x$ for any $q \in X$.

(c) $X \in H(\mathcal{M}) V_x$ for any $p \in X$.

PROBLEM. For a given space $X$, a given point $x \in X$, and a given class $\mathcal{M}$ of continuous mappings of $X$ onto itself the general problem is to find necessary and (or) sufficient conditions for $X$ to be in $H(\mathcal{M}) A_x$ or in $H(\mathcal{M}) V_x$.

DEFINITION 5. A class $\mathcal{S}$ of spaces is said to be homogeneous with respect to a class of mappings $\mathcal{M}$ if $\mathcal{S} \subset H(\mathcal{M})$, i.e., any element of $\mathcal{S}$ is homogeneous with respect to $\mathcal{M}$.

PROBLEM 2. For a given class $\mathcal{S}$ of spaces and a given class $\mathcal{M}$ of mappings the general problem is to find necessary and (or) sufficient conditions under which $\mathcal{S}$ is homogeneous with respect to $\mathcal{M}$.

Let us now consider some examples and some special classes of mappings and spaces.

Each homogeneous space is homogeneous with respect to any class of mappings containing the homeomorphisms. The closed unit interval $[0,1]$ on the real line is bihomogeneous with respect to continuity (this follows from Assertion 1 below), but it is not homogeneous. Since any monotone mapping of an arc onto itself carries the endpoints into the endpoints (see [26], (1.1), p. 165), an arc is not homogeneous with respect to the class of monotone mappings. Moreover, any monotone mapping of an irreducible continuum carries a point of irreducibility into a point of irreducibility ([18], §48, I, Theorem 3, p. 192); thus, if an irreducible continuum is homogeneous with respect to monotone mappings, then each point of it is a point of irreducibility. Since this condition characterizes indecomposable continua ([18], §48, VI, Theorem 7', p. 213), we have

PROPOSITION 2. If an irreducible continuum is homogeneous with respect to monotone mappings, then it is indecomposable.

Since each chainable continuum is irreducible (see, for example, [18], §48, X, 1, p. 224), we at once get

COROLLARY 1. If a chainable continuum is homogeneous with respect to monotone mappings, then it is indecomposable.

The above arguments can be modified as follows. A point $p$ in a continuum $X$ is called an endpoint if, for any two subcontinua $A, B \subset X$ containing $p$, either $A \subset B$ or $B \subset A$ (see [3], 5(B), p. 660). A continuous mapping $f: X \to Y$ of a topological space $X$ into $Y$ is said to be confluent ([6], p. 213) if for any subcontinuum $Q$ in $Y$ each component $C$ of the preimage $f^{-1}(Q)$ is mapped by $f$ onto $Q$, i.e., $f(C) = Q$. Recall that monotone mappings of continua are confluent, just as open mappings are ([6], §2, V and VI, p. 214). We mention the following simple result.

LEMMA 1. A confluent mapping $f$ of a continuum $X$ carries each endpoint of $X$ into an endpoint of the image $f(X)$. 
Indeed, let \( p \) be an endpoint of \( X \) and let \( Q_1 \) and \( Q_2 \) be two subcontinua of \( f(X) \) containing \( f(p) \). Let \( C_1 \) and \( C_2 \) be the components of \( f^{-1}(Q_1) \) and \( f^{-1}(Q_2) \) containing \( p \). Then either \( C_1 \subset C_2 \) or \( C_2 \subset C_1 \). Since \( f(C_1) = Q_1 \) and \( f(C_2) = Q_2 \) (because \( f \) is confluent), we see that one of the subcontinua \( Q_1 \) and \( Q_2 \) must be contained in the other.

For chainable continua the definition of an endpoint given above is equivalent to the following definition ([3], S(C), p. 660, and Theorems 12 and 13, p. 661). A point \( p \) in a chainable continuum \( X \) is said to be an endpoint of \( X \) if for any positive number \( \epsilon \) there exists in \( X \) an \( \epsilon \)-chain such that only the first link in this chain contains \( p \) (see also [24], p. 259, and compare Lemma 1 with Corollary 1.2 in [24], p. 260). There are examples of chainable continua without endpoints (see [3], Example 7, p. 662).

The proof of the next lemma is actually contained in the proof of a theorem in [8] and uses some ideas in [24].

**Lemma 2.** Each nondegenerate chainable continuum that is homogeneous with respect to open mappings contains an endpoint.

Using this lemma and known properties of chainable continua, we can find some necessary and sufficient conditions under which a chainable continuum is a pseudo-arc. These conditions are formulated in terms of homogeneity with respect to various classes of mappings. The proof of these characterizations is taken from [9] and is presented here only for completeness.

**Proposition 3.** Let \( X \) be a nondegenerate chainable continuum. Then the following conditions are equivalent.

1. \( X \) is homogeneous with respect to open mappings.
2. \( X \) is homogeneous with respect to confluent mappings, and there is an endpoint in \( X \).
3. \( X \) is homogeneous with respect to continuous mappings, \( X \) contains an endpoint, and \( X \) is hereditarily indecomposable.
4. \( X \) is a pseudo-arc.

**Proof.** (1) \( \Rightarrow \) (2). If \( X \) is homogeneous with respect to open mappings, then it is homogeneous also with respect to confluent mappings, since each open mapping of a compact space is confluent ([6], §2, VI, p. 214). Next, Lemma 2 shows that \( X \) has an endpoint.

(2) \( \Rightarrow \) (4). If \( X \) is homogeneous with respect to confluent mappings and has an endpoint, then each point of \( X \) must be an endpoint, by Lemma 1. This property characterizes a pseudo-arc ([3], Theorem 16, p. 662).

(4) \( \Rightarrow \) (1). The implication follows from the fact that a pseudo-arch is homogeneous ([11], Theorem 13, p. 740, Theorem 10, p. 737, and [2], Theorem 1, p. 44), and each homeomorphism is obviously an open mapping.

(4) \( \Rightarrow \) (3). A pseudo-arc has the properties listed in (3), since it is hereditarily indecomposable by definition (cf. also [1], Theorem 10, p. 737), homogeneous ([11], Theorem 13, p. 740), and being nondegenerate, has an endpoint ([3], Theorem 16, p. 662).
(3) $\Rightarrow$ (2). It is known that any continuous mapping of a continuum onto a hereditarily indecomposable continuum is confluent ([11], Theorem 5, p. 243). Therefore, if $X$ is hereditarily indecomposable, then any mapping of $X$ onto itself is confluent; hence the implication.

The question arises as to whether the assumption that the continuum has an endpoint is essential in proving the implication (2) $\Rightarrow$ (4). In other words, we have

**Problem 3. Does there exist a nondegenerate chainable continuum that is homogeneous with respect to confluent or monotone mappings and is not a pseudo-arc?**

Since each homogeneous (i.e., homogeneous with respect to homeomorphisms) nondegenerate chainable continuum is a pseudo-arc ([4], the theorem on p. 345), this problem can be reformulated as the question as to whether there is a chainable continuum that is homogeneous with respect to confluent mappings, but is not homogeneous with respect to homeomorphisms.

We return to the consideration of an arc. It follows from Proposition 3 that an arc is not homogeneous with respect to confluent mappings. However, it is easy to see that an arc is homogeneous with respect to the class of all continuous mappings, and it is not hard to find certain smaller classes of mappings having the property that an arc is homogeneous with respect to them. For example, let us say that a mapping $f: X \rightarrow Y$ of a continuum $X$ onto $Y$ is submonotone if $X$ contains a subcontinuum $C$ such that the partial mapping $f|_C$ is monotone and carries $C$ onto the whole of $Y$. If a subcontinuum $Q \subseteq Y$ is given, then $(f|_C)^{-1}(Q) = C \cap f^{-1}(Q)$ is a continuum (because $f$ is submonotone) which is clearly mapping by $f|_C$ (hence also by $f$) onto $Q$; thus, every submonotone mapping of a continuum is weakly confluent (see [20], p. 98, for the definition of this concept). Recall that the degree of a mapping $f: X \rightarrow Y$ is defined as the maximum of the (finite) cardinalities of the preimages $f^{-1}(y)$ of points $y \in Y$ (cf. [26], p. 199). The following result can be verified simply by going through the different possible cases.

**Proposition 4.** An arc is homogeneous with respect to the class of submonotone mappings of degree 2.

We note that this proposition cannot be generalized to chainable continua (see the proof below for the curve $\sin 1/x$).

Suppose that now that $X$ is an arbitrary metric continuum, and let $f$ be an open mapping of $X$. It is known that the Menger-Urysohn order of a point in $X$ does not increase under $f$ (see, for example, [26], Corollary 7.31, p. 147). Thus, if $X$ contains two points of different Menger-Urysohn order, then $X$ cannot be homogeneous with respect to the open mappings. In other words, we have

**Proposition 5.** If a metric continuum $X$ is homogeneous with respect to the class of open mappings, then all the points in $X$ have the same Menger-Urysohn order.

For example, a simple closed curve is such a continuum. From Theorem 8" in [18] (§51, V, p. 192) and Proposition 5 we deduce

**Corollary 2.** A metric continuum $X$ is a simple closed curve if and only if all the points in $X$ have finite Menger-Urysohn order and $X$ is homogeneous with respect to the open mappings.
The condition formulated in Proposition 5 is necessary but not sufficient for a continuum to be homogeneous with respect to the class of open mappings. This can be seen from the example of a Cantor fan (i.e., a cone over the Cantor set) that has at all points Menger-Urysohn order equal to the continuum but is not even homogeneous with respect to the confluent mappings. Indeed, from Theorem 12 in [7] (p. 32) we have

**Proposition 6. No fan is homogeneous with respect to the confluent mappings.**

We now describe the construction of a class of metric continua \( X \) each containing a point \( p \) such that \( X \) is not in \( H(\mathcal{M})_p \), where \( \mathcal{M} \) is the class of weakly confluent mappings ([20], p. 98). Namely, we show that each weakly confluent continuous mapping of \( X \) onto itself leaves the point \( p \) fixed. Thus, no element of the constructed class is homogeneous with respect to the weakly confluent mappings. In particular, we obtain an example of a smooth dendroid that is not homogeneous with respect to this class.

Let \( \varphi \) be a function assigning an ordinal number \( \varphi(X) \) to a continuum \( X \) in such a way that (1) \( \varphi(Y) \leq \varphi(X) \) if \( Y \subset X \) and (2) \( \varphi(f(X)) \leq \varphi(X) \) for any continuous mapping \( f \) (examples of such functions are known; for hereditarily unicoherent and hereditarily decomposable continua see [5], p. 190, [15], Theorem 1, and Theorem 3, and [22], p. 345). We take a sequence of continua \( X_n \) such that for each positive integer \( n \) we have (i) \( \varphi(X_n) = n \); (ii) \( \text{diam } X_n \leq 2^{-n} \); and (iii) \( X_n \cap Y_{n+1} \) is a singleton. Let \( p \) be the remainder in the one-point compactification of the union of all the \( X_n \), and let \( X = \{ p \} \cup \bigcup X_n \). Then \( X \) is a continuum. Let \( f: X \to X \) be a weakly confluent mapping of \( X \) onto \( X \). Assume that \( f(p) \neq p \). Then \( f^{-1}(p) \) is a closed subset of \( X \) not containing \( p \). Let \( U \) be an arbitrary neighborhood of \( p \) lying in \( X \setminus f^{-1}(p) \). Since \( \text{Lim } X_n = \{ p \} \) by construction, there exists a positive number \( k \) such that the continuum \( K = \{ p \} \cup \bigcup_{k=1}^\infty X_n \) lies in \( U \). The mapping \( f \) is weakly confluent; therefore, there exists a component \( C \) of the set \( f^{-1}(K) \) such that \( f(C) = K \). But then \( C \cap f^{-1}(p) \neq \emptyset \), whence \( C \subset \bigcup X_n \), and we get by (1) that \( \varphi(C) \leq \varphi(\bigcup X_n) = k \). Further, \( \varphi(K) = \varphi(f(C)) \leq \varphi(C) \) by (2); but this is impossible, because \( K \) contains a subcontinuum \( X_n \) with arbitrarily large number \( \varphi(X_n) = n \).

We now take the \( \varphi \) in the general construction considered above to be the degree \( \tau \) of nonlocal connectivity, and the \( X_n \) to be the \( n \)-harmonic fan \( F_{Hn} \) (see (43) on p. 200 of [5] for the definition), made smaller so that \( \text{diam } X_n = 2^{-n} \). We form the union of all the \( X_n \) in such a way that the only point in \( X_n \cap X_{n+1} \) is the vertex of \( X_{n+1} \) and the unique point of the derived set of \( n \)-th order of the set of endpoints of \( X_n \) (here an endpoint is understood to be an endpoint of every arc lying in the continuum). The continuum \( X \) so obtained appears as the continuum represented in [14], Figures 3-9, p. 113, with the exception that the successive fans are not simply harmonic but \( n \)-harmonic. Then \( X \) is a smooth dendroid (see [10], p. 298, for the definition). There is a conjecture that for this dendroid \( X \) the property \( f(p) = p \) holds not only for weakly confluent mappings, but also for arbitrary continuous mappings of \( X \) onto itself. Krupski ([16], Example 1) recently used analogous methods to construct a fan \( X \) with vertex \( p \) such that \( f(p) = p \) for any continuous mapping \( f \) of \( X \) onto itself.

We now turn our attention to the class of all continuous mappings.
Lemma 3. For any nondegenerate metric continuum $X$ and for any two distinct points $p$ and $q$ in $X$ there exists a continuous mapping $f_1: X \rightarrow [0,1] = f_1(X)$ such that $f_1(p) = 0$ and $f_1(q) = 1$.

Indeed, $f_1$ can be defined by the formula

$$f_1(x) = d(x, p)/[d(x, p) + d(x, q)]$$

for any $x \in X$, where $d$ is the metric on $X$.

Lemma 4. For any locally connected metric continuum $X$ and for any two distinct points $p$ and $q$ in $X$ there exists a continuous mapping $f_2: [0,1] \rightarrow X = f_2([0,1])$ such that $f_2(0) = q$ and $f_2(1) = p$.

Indeed, since the metric continuum $X$ is locally connected, there exists a continuous mapping $f'$ of the closed interval $[1, \frac{1}{2}]$ onto $X$. Since $X$ is arcwise connected, there exist arcs $q f'(t) = q$ and $f'(t) = p$, and we can consider an arbitrary homeomorphism $h_1$ of $[0,\frac{1}{2}]$ onto $f'(\frac{1}{2})$ such that $h_1(0) = q$ and an arbitrary homeomorphism $h_2$ of $[\frac{1}{2},1]$ onto $f'(\frac{1}{2})$ such that $h_2(1) = p$. In the case where $f'(\frac{1}{2}) = q$ or $f'(\frac{1}{2}) = p$ we define $h_1$ or $h_2$ as a constant mapping, i.e., $h_1([0,\frac{1}{2}]) = \{q\}$ or $h_2([\frac{1}{2},1]) = \{p\}$, respectively. Finally, the mapping $f_2$ defined as $h_1$ on $[0,\frac{1}{2}]$, $f'$ on $[\frac{1}{2},1]$, and $h_2$ on $[\frac{1}{2},1)$ is the desired mapping.

Assertion 1. Each locally connected metric continuum is bihomogeneous with respect to continuity.

Indeed, suppose that the metric continuum $X$ is locally connected. Without loss of generality we can confine ourselves to the nondegenerate case. If $p$ and $q$ are distinct points in $X$, then the mapping $f = f_2 f_1$ (where $f_1$ and $f_2$ are the mappings considered in Lemmas 3 and 4) maps $X$ continuously onto itself, and $f(p) = q$ and $f(q) = p$.

Problem 4. What classes $\mathcal{M}$ of mappings (according to Definition 2) can be substituted in place of the class of continuous mappings in Assertion 1?

We recall some known examples which relate to the questions discussed here. Cook constructed continua that admit only the identity mapping onto nondegenerate subcontinua (III, Theorem 8, p. 245; Theorems 10 and 11, p. 247; Theorem 13, p. 248); these continua are thus very far from being homogeneous with respect to continuity. We now consider a simple example of a continuum that is not homogeneous with respect to continuity.

The curve $\sin \frac{1}{x}$, i.e., the closure $\overline{S}$ of the set

$$S = \{(x,y): y = \sin \frac{1}{x}, \text{ where } 0 < x \leq 1\},$$

fails at all points to be argument-homogeneous with respect to continuity. Indeed, let $f: \overline{S} \rightarrow \overline{S}$ be a continuous mapping "onto". We first take $q \in S$ and $p \in \overline{S} \setminus S$. Then the equality $f(p) = q$ implies that $f(\overline{S} \setminus S) \subset S$, because $\overline{S} \setminus S$ is arcwise connected. Since $S$ is also arcwise connected, its image $f(S)$ must be contained either in $\overline{S} \setminus S$ or in $S$. The inclusion $f(S) \subset \overline{S} \setminus S$ is not possible, because the image $f(\overline{S}) = f(S) \cup f(\overline{S} \setminus S)$ is connected. Thus, $f(S) \subset S$, from which it follows that $f(\overline{S}) \subset S$, and $f$ is not a mapping onto $\overline{S}$. Now take $q \in \overline{S} \setminus S$ and $p \in S$. Then the equality $f(p) = q$ implies that $f(S) \subset \overline{S} \setminus S$, because $S$ is arcwise connected, and thus $f(\overline{S}) \subset \overline{S} \setminus S$, since $S$ is dense in $\overline{S}$. It can be verified similarly that the curve $\overline{S}$ fails at all its points.
to be value-homogeneous with respect to continuity. Thus, in particular, $S$ is not homogeneous with respect to continuity.

This example shows that homogeneity with respect to continuity is not invariant under continuous mappings: $S$ is a continuous image of the Cantor set, which is homogeneous (being a topological group; for another proof see [14], Exercises 2–17, p. 100); $S$ is also a continuous image of a pseudo-arc (see [13], Theorem 4.1, p. 389, and [19], Corollary 3, p. 276; compare [21], Theorem 3, p. 184, and Theorem 5 and footnote (2) on p. 188), which is homogeneous with respect to homeomorphisms (see [1]. Theorem 13, p. 740; compare [23], p. 57). The following problem seems natural in the light of the given examples.

**Problem 5.** Find the classes $\mathcal{F}$ of mappings for which the following is valid: If $X$ is a continuum homogeneous with respect to continuity and $f$ is a mapping in $\mathcal{F}$ defined on $X$, then $f(X)$ is homogeneous with respect to continuity.

For a given space $X$ let $N(X)$ denote the set of all points in $X$ at which $X$ is not locally connected.

**Proposition 7.** Suppose that the continuum $X$ is such that $N(X) \neq \emptyset \neq X \setminus N(X)$, and suppose that each component of $N(X)$ is an arcwise connected component of $X$. If $X$ is homogeneous with respect to continuity, then the number of components of $N(X)$ is infinite.

Indeed, let $C$ be a component of $N(X)$, and let $p \in C$ and $q \in X \setminus N(X)$. Since $X$ is homogeneous with respect to continuity, there exists a continuous mapping $f: X \to X$ of $X$ onto itself such that $f(p) = q$. This implies that $f(C) \subseteq X \setminus N(X)$. Indeed, if this is not so, i.e., $N(X) \cap f(C) \neq \emptyset$, then there is a component $D$ of $N(X)$ such that $D \cap f(C) \neq \emptyset$, and, since the sets $D$ and $f(C)$ are both arcwise connected, we get that $D \cup f(C)$ is arcwise connected. But $D$ is an arcwise connected component of $X$; hence $f(C) \subseteq D$, and so $q = f(p) \in f(C) \subseteq D \subseteq N(X)$, which contradicts the choice of $q$. Since $N(f(N(X))) \subseteq f(N(X))$ for any compact metric space $X$ and any continuous mapping $f$ (see [12], 3, p. 28), we have that $N(X) \subseteq f(N(X))$ for every continuous mapping $f$ of $X$ onto itself. Thus, in our case $N(X) \subseteq f(N(X) \setminus C) \cup f(C)$, and so $N(X) \subseteq f(N(X) \setminus C)$. Assume that $N(X)$ has finitely many (say, $n$) components. Then $f(N(X) \setminus C)$ is the union of at most $n - 1$ disjoint arcwise connected sets, and this union contains the set $N(X)$, which consists of $n$ arcwise connected components of $X$, a contradiction.

**Proposition 8.** Suppose that the continuum $X$ contains an arcwise connected component $A$ that is a proper dense subset of $X$. If $X$ is homogeneous with respect to continuity, then $A$ is a boundary set.

Indeed, take $p \in A$ and $q \in X \setminus A$. There exists a continuous mapping $f$ of $X$ onto itself such that $f(p) = q$. Then $f(A) \cap (X \setminus A) \neq \emptyset$, and so $f(A) \subseteq X \setminus A$, because $f(A)$ is arcwise connected and $A$ is an arcwise connected component of $X$. Since $A$ is a dense subset of $X$, we conclude that $f(A)$ is also dense and, consequently, $X \setminus A$ is dense in $X$, i.e., $A$ is a boundary set.

**Corollary 3.** Suppose that the continuum $X$ contains an arcwise connected component $A$ that is a proper dense subset of $X$, and suppose that a point in $A$ is a point of
irreducibility of \( X \). If \( X \) is homogeneous with respect to continuity, then it is indecomposable.

Indeed, we apply Proposition 8 and Urysohn’s theorem (see [18], §48, VI, Theorem 9, p. 214; cf. [25], p. 226).

Some results on continua homogeneous with respect to continuity were recently obtained in [17], in particular, conditions implying that certain continua are not homogeneous with respect to the class of continuous mappings.

**BIBLIOGRAPHY**


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