Let a class $M$ of continuous mappings be given. A space $X$ is said to be homogeneous with respect to $M$ from $p \in X$ to $q \in X$ provided there is a mapping $f$ of $X$ onto $X$ such that $f \in M$ and $f(p) = q$. If this condition holds for each pair $p, q \in X$, then $X$ is called homogeneous with respect to $M$. The aim of this paper is to present a sequence of results concerning homogeneity of the Sierpiński universal plane curve with respect to various classes of mappings as local homeomorphisms, elementary, simple, monotone, and open mappings. For the class of homeomorphisms the results have been obtained in [2]. We give here outlines of proofs only. The full proofs will be published elsewhere.

1. INTRODUCTION

All spaces considered in the paper are metric continua and all mappings are continuous. By the Sierpiński curve we mean any continuum homeomorphic to the well-known plane locally connected curve in the unit square that is known to be universal in the class of all plane curves. The Sierpiński curve is denoted by $S$. The union of all boundaries of com-
plementary domains of $S$ in the plane is denoted by $R$ and is called the rational part of $S$. The remainder part of the curve is called its irrational part.

The following result is proved in [2].

**Theorem K.** The Sierpiński curve is homogeneous (with respect to homeomorphisms) from $p$ to $q$ if and only if either $p, q \in R$ or $p, q \in S \setminus R$.

We extend this result considering several other classes of mappings, each of which contains homeomorphisms.

A mapping $f: X \to Y$ is called:

*open*, if images under $f$ of open subsets of $X$ are open in $Y$;

*elementary*, if there is an $\epsilon > 0$ such that for each two different points $x', x'' \in X$ the condition $f(x') = f(x'')$ implies $d(x', x'') \geq \epsilon$, where $d$ is the metric on $X$ (see [1], p. 84);

*simple*, if $\text{card } f^{-1}(y) < 2$ for each $y \in Y$ (see [1], p. 84);

*monotone*, if $f^{-1}(y)$ is connected for each $y \in Y$;

*a local homeomorphism*, if each point of $X$ has an open neighborhood $U$ such that $f|U$ is a homeomorphism of $U$ onto $f(U)$ and $f(U)$ is open in $Y$.

Note that $f$ is elementary if and only if each point of $X$ has an open neighborhood $U$ such that $f|U: U \to f(U)$ is a homeomorphism. Thus a mapping is a local homeomorphism if and only if it is elementary and open.

The following two lemmas will be used in the sequel.

**Lemma 1.** If $S_1, S_2$ are the Sierpiński curves and $R_1, R_2$ are their rational parts, then $S_1 \subseteq S_2$ implies $S_1 \setminus R_1 \subset S_2 \setminus R_2$.

**Lemma 2.** A point $p$ is in $S \setminus R$ if and only if for each $\epsilon > 0$ there is an open connected $\epsilon$-neighborhood about $p$ in $S$ whose boundary is a simple closed curve.
2. LOCAL HOMEOMORPHISMS

Proposition 1. If a surjection \( f: S \to S \) is elementary, then \( f(S \setminus R) \subset S \setminus R \).

Proof. Let \( U \) be an open connected neighborhood of a point \( p \in S \setminus R \) such that \( \bar{U} \) is homeomorphic to \( S \) and \( f \mid \bar{U} \) is a homeomorphism. Thus \( p \) is in the irrational part of \( \bar{U} \) and by Theorem K the point \( f(p) \) is in the irrational part of \( f(\bar{U}) \), the Sierpiński curve in \( S \). Applying Lemma 1 we conclude \( f(p) \in S \setminus R \).

Using Lemma 2 one can prove

Proposition 2. If a surjection \( f: S \to S \) is a local homeomorphism, then \( f(R) \subset R \).

Propositions 1 and 2 together with Theorem K imply

Theorem 1. The Sierpiński curve \( S \) is homogeneous with respect to local homeomorphisms from \( p \) to \( q \) if and only if either \( p, q \in R \) or \( p, q \in S \setminus R \).

Openness of \( f \) is essential in Proposition 2. Moreover, we have

Proposition 3. For each \( p \in R \) and \( q \in S \setminus R \) there is a simple elementary surjection \( f: S \to S \) with \( f(p) = q \).

In fact, by Theorem K there is a homeomorphism \( h_1 \) of \( S \) onto \( S \) carrying \( p \) to \( \left( 0, \frac{1}{2} \right) \) (we consider \( S \) as the well-known curve in the unit square in the Euclidean plane). Next we define \( g: S \to S \) as an identification map which puts together the points \( (0, y) \) and \( (1, y) \) for \( y \in [0, 1] \), and observe \( g(S) \) is homeomorphic to \( S \) by the Whyburn characterization of \( S \), [3]. Furthermore, \( g\left(0, \frac{1}{2}\right) \) is in the irrational part of \( g(S) \). So again by Theorem K there is a homeomorphism \( h_2 \) of \( g(S) = S \) onto itself taking \( g\left(0, \frac{1}{2}\right) \) to \( q \), and it is enough to put \( f = h_2 g h_1 \).

As a consequence of Propositions 1 and 3 and of Theorem K we get
Theorem 2. The Sierpiński curve $S$ is homogeneous with respect to elementary (or simple elementary) mappings from $p$ to $q$ if and only if either $p \in R$ and $q$ is arbitrary, or $p, q \in S \setminus R$.

3. SIMPLE MAPPINGS

Proposition 4. For each $p \in S \setminus R$ and $q \in R$ there is a simple open surjection $f : S \rightarrow S$ with $f(p) = q$ and $f(R) \subset R$.

Really, let $g$ be an identification mapping on $S$ which identifies $(x, y)$ with $(y, x)$ (i.e. a folding of $S$ along the diagonal $D$ from $(0, 0)$ to $(1, 1)$ of the unit square). Thus $g(S)$ is homeomorphic to $S$ by Whyburn’s result [3]. Take a point $p_0 \in D \cap (S \setminus R)$ and note that $g(p_0)$ is in the rational part of $g(S)$. Therefore it is enough to put $f = h_2 gh_1$, where homeomorphisms $h_1$ and $h_2$ of $S$ onto itself carry $p$ to $p_0$ and $g(p_0)$ to $q$ respectively, according to Theorem K.

Proposition 4 and Theorem K imply

Corollary 1. If either $p \in S \setminus R$ and $q$ is arbitrary, or $p, q \in R$, then $S$ is homogeneous with respect to simple open mappings from $p$ to $q$.

Problem 1. Is the converse to Corollary 1 true?

Corollary 1 and Proposition 3 imply

Theorem 3. The Sierpiński curve is homogeneous with respect to simple mappings.

Problem 2. Characterize locally connected continua which are homogeneous with respect to simple mappings.

Note that the simple triod is not homogeneous with respect to simple mappings from its center to an end point.

4. MONOTONE MAPPINGS

Proposition 5. There is a monotone surjection $g : S \rightarrow S$ which carries a point $p_0 \in R$ to a point $q_0 \in S \setminus R$.
In fact, pick arbitrary \( p_0 \in R \) and let \( C \) be the boundary of the complementary domain of \( S \) in the plane to which \( p_0 \) belongs. Defining \( g \) as the shrinking of \( C \) to a point we see that \( g(S) \) is homeomorphic to \( S \) and that \( g(p_0) = q_0 \) is in the irrational part of \( g(S) \).

**Proposition 6.** There is a monotone surjection \( g: S \to S \) which carries a point \( p_0 \in S \setminus R \) to a point \( q_0 \in R \).

To see this, shrink to a point the straight line segment joining \( p_0 = (-\frac{1}{6}, \frac{1}{3}) \in S \setminus R \) with \( (\frac{1}{3}, \frac{1}{3}) \). The resulting space is again homeomorphic to \( S \), and the image of \( p_0 \) is in its rational part.

Now, if we compose the mappings \( g \) of Propositions 5 and 6 with homeomorphisms \( h_1 \) and \( h_2 \) of \( S \) onto itself (suitable chosen according to Theorem K, i.e. carrying \( p \) to \( p_0 \) and \( q_0 \) to \( q \), we conclude

**Theorem 4.** The Sierpiński curve is homogeneous with respect to monotone mappings.

**Problem 3.** Characterize locally connected continua which are homogeneous with respect to monotone mappings.

Again note that the simple triad is not in this class of continua: its center is a fixed point under arbitrary monotone surjection.

5. OPEN PROBLEMS ON OPEN MAPPINGS

One of such problems is Problem 1 above. We present two others. The first is a more general version of Problem 1 and it is also related to Proposition 2.

**Problem 4.** Is it true that for an arbitrary open surjection \( f: S \to S \) we have \( f(R) \subset R \)?

The second problem is not related to the Sierpiński curve in a direct manner: the author conjectures Problem 4 has an affirmative answer, so the Sierpiński curve seems not to be a good candidate to solve the next problem in the positive.
Problem 5. Does there exist a locally connected plane continuum which is homogeneous with respect to open mappings and which is different from a simple closed curve?

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