CONFLUENT AND RELATED MAPPINGS ON ARC-LIKE CONTINUA—AN APPLICATION TO HOMOGENEITY

J.J. CHARATONIK and T. MAĆKOWIAK

Institute of Mathematics, University of Wrocław, 50-384 Wrocław, Poland

Received 8 November 1984
Revised 9 December 1985

It is shown that the pseudo-arc is the only nondegenerate arc-like continuum which is homogeneous with respect to the class of confluent (monotone, open) mappings. Further a concept of a pseudo-end point of an arc-like continuum is introduced and is applied to obtain another characterization of the pseudo-arc in terms of homogeneity with respect to a class of mappings.

AMS (MOS) Subj. Class.: Primary 54C10, 54F20; Secondary 54F50, 54F65

arc-like hereditarily indecomposable
end confluent mapping
homogeneous monotone mapping
pseudo-arc open mapping

1. Introduction

In 1959 Bing [5] proved that the only nondegenerate arc-like homogeneous continuum is the pseudo-arc. The result has been extended to nondegenerate arc-like continua which are homogeneous with respect to various classes of mappings (as open, confluent, or all continuous) under some additional conditions (see [7] and [8]). These conditions are deleted in this paper and some characterizations are obtained of the pseudo-arc in terms of homogeneity with respect to classes of monotone, open and confluent mappings (Theorem 3.9). One implication in this theorem is essentially due to Bing (the homogeneity of the pseudo-arc, [5]), while the other one is a corollary to a more general result concerning the structure of continua that are not hereditarily indecomposable and are homogeneous with respect to the class of confluent mappings (Theorem 3.4). In a further part of the paper the concept of a pseudo-end point in an arc-like continuum is extracted from the same paper [5] of Bing, and it is discussed together with a concept of an end point (again due to Bing [4]).

It is shown that open mappings and monotone mappings preserve pseudo-ends, while for confluent mappings such a preservation remains an open problem.

The main result of this part of the paper is Corollary 4.8 saying that a nondegenerate arc-like continuum is the pseudo-arc if and only if it is homogeneous with respect to an admissible class of mappings that preserve ends and pseudo-ends (a class $M$ of mappings is admissible if it contains homeomorphisms and if for each mapping from $M$ its composite with a homeomorphism is in $M$, too). Some open problems are posed in the paper.

2. Preliminaria

All spaces considered in this paper are assumed to be metric and all mappings are continuous. A continuum means a compact connected space.

A mapping $f: X \to Y$ from a space $X$ onto a space $Y$ is said to be

(a) **monotone**, provided that the inverse image of each subcontinuum of $Y$ is connected;

(b) **open**, if for each open subset of $X$ its image under $f$ is an open subset of $Y$;

(c) **confluent**, if for each subcontinuum $Q$ of $Y$ each component of $f^{-1}(Q)$ is mapped onto the whole $Q$ under $f$ [6, p. 213];

(d) **weakly confluent**, if for each subcontinuum $Q$ of $Y$ there exists a component of $f^{-1}(Q)$ which is mapped onto the whole $Q$ under $f$ [20, p. 98];

(e) **confluent relative to a point $p \in X$**, if for each subcontinuum $Q$ of $Y$ such that $f(p) \in Q$ the component of $f^{-1}(Q)$ containing the point $p$ is mapped onto the whole of $Q$ under $f$ [12, p. 2];

(f) **confluent at a point $q \in Y$**, if for each subcontinuum $Q$ of $Y$ such that $q \in Q$ each component of $f^{-1}(Q)$ is mapped onto the whole $Q$ under $f$ [12, p. 2].

The following facts are well known (see [6, sections V and VI, p. 214] and [12, (4), p. 2].

2.1. **Fact.** Each monotone and each open mapping of a continuum is confluent.

2.2. **Fact.** A mapping $f$ is confluent at a point $q \in Y$ if and only if it is confluent relative to each point of $f^{-1}(q)$.

A space $X$ is said to be homogeneous with respect to the class $M$ of mappings if for every two points $p$ and $q$ of $X$ there exists a surjection $f$ of $X$ onto itself such that $f \in M$ and $f(p) = q$. If $M$ is class of homeomorphisms then we get the well known notion of a homogeneous space. A space which is homogeneous with respect to the class of confluent (monotone, open) mappings is called confluentely (monotonously, openly) homogeneous.

A continuum is said to be **decomposable** provided that it is the union of two proper subcontinua. Otherwise it is called **indecomposable**. It is said to be **hereditarily indecomposable** provided that each of its subcontinua is indecomposable. The first known example of such a continuum is due to Knaster [16]. A point $p$ of a continuum
X is called an end point of X if for each two subcontinua of X both containing p, one of them contains the other [4, pp. 660 and 661].

The following fact is immediate.

2.3. Fact. A continuum is hereditarily indecomposable if and only if each of its points is an end point of it.

We denote by \( E(X) \) the set of all end points of a continuum X. A mapping \( f: X \to Y \) between X and Y is said to preserve ends provided that for each end point of X its image under f is an end point of Y, i.e., \( f(E(X)) \subseteq E(Y) \).

A continuum is said to be arc-like (or chainable, or snake-like) if for each positive number \( \epsilon \) there exists an \( \epsilon \)-chain covering it (see [2, p. 729] and [3, pp. 43 and 44]).

An hereditarily indecomposable arc-like nondegenerate continuum is a pseudo-arc [14, 2, 19, 20]. It is known to be unique up to homeomorphism [3, Theorem 1, p. 44] and homogeneous [2, Theorem 13, p. 740]. For some other results and new proofs of basic properties of the pseudo-arc see [24].

Given a continuum X, we denote by \( C(X) \) the collection of all subcontinua of X and, for a fixed point p in X, by \( C(p, X) \)—a subcollection of \( C(X) \) composed of all subcontinua of X which contain the point p. The collection \( C(X) \) can be metrized by the Hausdorff metric dist (see [18, section 42, II, p. 47] for the definition).

The symbol \( B(c, r) \) denotes an open ball with center c and radius r in a space X. Further, \( \text{cl } A \) means the closure of a set A.

3. Confluent mappings and a characterization of the pseudo-arc

The main result of this section says that the pseudo-arc is the only nondegenerate arc-like continuum which is conflually homogeneous. This is a generalization of Bing's result [5] and two results of the first author [7, Theorem, p. 901; and, 8, Theorem, p. 270], and answers Problem 2 of [9, p. 26]. However, the result is a corollary to a more general theorem concerning conflually homogeneous continua which are not hereditarily indecomposable.

We have the following result.

3.1. Proposition. For a mapping \( f: X \to Y \) from a continuum X onto Y the following conditions are equivalent:

(i) \( f \) is confluent;
(ii) for each point \( y \in Y \) the mapping \( f \) is confluent at \( y \);
(iii) for each point \( y \in Y \setminus E(Y) \) the mapping \( f \) is confluent at \( y \).

Proof. Indeed, the implications from (i) to (ii) and from (ii) to (iii) are obvious.

To show (iii) implies (i), consider a nondegenerate continuum Q of Y. If \( Q \setminus E(Y) \neq \emptyset \), then by (iii) each component of \( f^{-1}(Q) \) is mapped onto the whole of Q under \( f \).
If \( Q \setminus E(Y) = \emptyset \) (i.e., if \( Q \subset E(Y) \)) suppose there is a component \( C_0 \) of \( f^{-1}(Q) \) with \( Q \setminus f(C_0) \neq \emptyset \). Then taking an open set \( U \) about \( C_0 \) such that \( Q \setminus f(\overline{U}) \neq \emptyset \) and a component \( C \) of \( \overline{U} \) containing \( C_0 \) we have \( Q \setminus f(C) \neq \emptyset \neq f(C) \setminus Q \). Thus if \( y \in Q \cap f(C) \), then \( y \in Y \setminus E(Y) \), a contradiction. So we have \( f(C_0) = Q \) for each component \( C_0 \) of \( f^{-1}(Q) \), and (i) follows. The proof is complete. \( \square \)

Since no point \( y \) of a hereditarily indecomposable continuum \( Y \) is in \( Y \setminus E(Y) \) by Fact 2.3, we see that condition (iii) of Proposition 3.1 is satisfied for each continuous mapping \( f \), and therefore Proposition 3.1 leads to the following corollary which is due to Cook [15, Theorem 4, p. 243].

3.2. Corollary (H. Cook). Each mapping of a continuum onto a hereditarily indecomposable continuum is confluent.

Recall the following known fact (see [10, Lemma, p. 172]).

3.3. Fact. Confluent mappings preserve ends.

We use the following notation. If \( K, L \in C(X) \) let
\[
g(K, L) = \min\{\text{dist}(K, K \cup L), \text{dist}(L, K \cup L)\}.
\] (3.1)

Note that
\[
a \text{ point } p \text{ of a continuum } X \text{ is an end point of } X
\]

if and only if for each two subcontinua \( K, L \in C(p, X) \)

we have \( g(K, L) = 0 \). (3.2)

3.4. Theorem. If a continuum \( X \) is homogeneous with respect to the class of confluent mappings and \( X \) is not hereditarily indecomposable, then there is a positive number \( \varepsilon \) such that for each point \( x \) in \( X \) there are continua \( K \) and \( L \) in \( C(x, X) \) with \( g(K, L) \geq \varepsilon \).

Proof. Put
\[
F_n = \{ x \in X : \text{there are } K, L \in C(x, X) \text{ such that } g(K, L) \geq 1/n \}.
\]

Obviously
\[
F_n \text{ is closed and } F_n \subseteq F_{n+1} \text{ for } n \in \{1, 2, \ldots \}.
\] (3.3)

Note that
\[
\text{no point of } X \text{ is its end point.} \quad (3.4)
\]

In fact, since \( X \) is not hereditarily indecomposable, by Fact 2.3 there is a point \( x_0 \in X \setminus E(X) \). For an arbitrary point \( x \) of \( X \) there exists a confluent surjection \( f: X \to X \) such that \( f(x) = x_0 \). Since \( f \) preserves ends by Fact 3.3, we see that \( x \) is not an end point of \( X \). So (3.4) holds. Now (3.2) and (3.4) imply that for each point
\[ x \in X \text{ there are subcontinua } K, L \in C(x, X) \text{ such that } g(K, L) > 0, \text{ whence we conclude that each point of } X \text{ is in } F_n \text{ for some } n, \text{ and therefore} \]

\[ X = \bigcup_{n=1}^{\infty} F_n. \tag{3.5} \]

The Baire Category Theorem implies by (3.3) and (3.5) that there is an integer \( n \) such that the interior of \( F_n \) in \( X \) is nonempty. Let \( U \) be an open set in \( X \) such that \( U \subset F_n \), and fix a point \( y \in U \). Let \( x \) be an arbitrary point of \( X \) and consider a confluent surjection \( f: X \to X \) such that \( f(x) = y \). By continuity of \( f \) we find a positive number \( \delta \) such that \( \text{cl}(f(B(x, \delta))) \subset U \). Take a point \( p \in B(x, \delta) \) and note that \( f(p) \in U \subset F_n \). Since \( f(p) \) is not an end point of \( X \) by (3.4), the mapping \( f \) is confluent relative to \( p \) by Fact 2.2 and Proposition 3.1, and thus we conclude that there are continua \( K_p, L_p \in C(p, X) \) such that

\[ g(f(K_p), f(L_p)) \geq 1/n, \tag{3.6} \]

according to the definition of \( F_n \). We claim that

there is a positive number \( \eta \) such that \( g(K_p, L_p) \geq \eta \)

for each point \( p \in B(x, \delta) \). \tag{3.7}

Indeed, otherwise we have a convergent sequence of points \( p_k \in B(x, \delta) \) such that \( g(K_{p_k}, L_{p_k}) \) tends to zero as \( k \) tends to infinity. Without loss of generality we can assume that the sequences of continua \( K_{p_k}, L_{p_k} \) converge to continua \( K \) and \( L \) respectively, for which we have \( g(K, L) = 0 \) by continuity of \( g \). Then either \( \text{dist}(K, K \cup L) = 0 \) or \( \text{dist}(L, K \cup L) = 0 \) by (3.1). If \( \text{dist}(K, K \cup L) = 0 \), then \( K = K \cup L \); thus \( f(K) = f(K) \cup f(L) \), whence \( g(f(K), f(L)) = 0 \), a contradiction to (3.6). The same for the other case. So (3.7) is established.

Therefore the ball \( B(x, \delta) \) is contained in \( F_m \) for some positive integer \( m \). Hence conditions (3.3) and (3.5) and compactness of \( X \) imply the equality \( X = F_s \) for some \( s \), which completes the proof. \( \square \)

3.5. Remark. Recall that the Sierpiński universal plane curve \( S \) (see e.g. [18, section 51, I, Example 5, p. 275]) is known to be monotonously homogeneous [11, Theorem 5]). As a locally connected continuum it is not hereditarily indecomposable, and therefore it can serve as a continuum to which Theorem 3.4 can be applied.

3.6. Corollary. Each confluenly homogeneous nondegenerate arc-like continuum is a pseudo-arc.

Proof. Let \( X \) be such a continuum, and suppose on the contrary that \( X \) is not hereditarily indecomposable. Let \( \varepsilon \) be a positive number satisfying the conclusion of Theorem 3.4, and let \( \{U_1, U_2, \ldots, U_n\} \) be an \( \varepsilon/3 \)-chain covering \( X \). Fix a point \( x \in U_1 \) and take continua \( K, L \in C(x, X) \) such that \( g(K, L) \geq \varepsilon \).
There are positive integers \( r \) and \( s \) such that \( K \subseteq U_1 \cup \cdots \cup U_r, \ L \subseteq U_1 \cup \cdots \cup U_s, \ K \cap U_i \neq \emptyset \) for \( i \leq r \) and \( L \cap U_i \neq \emptyset \) for \( i \leq s \). If \( r \leq s \), then \( \text{dist}(L, K \cup L) < \varepsilon \); if \( s \leq r \), then \( \text{dist}(K, K \cup L) < \varepsilon \). In both cases by the definition of \( g \) we obtain a contradiction with the inequality \( g(K, L) \geq \varepsilon \).

As a consequence of Corollary 3.6 and of Fact 2.1 we get the next two corollaries.

3.7. Corollary. Each monotonously homogeneous nondegenerate arc-like continuum is a pseudo-arc.

3.8. Corollary. Each openly homogeneous nondegenerate arc-like continuum is a pseudo-arc.

Bing has shown homogeneity of the pseudo-arc [2, Theorem 13, p. 740] (for a new proof see [24]), whence the converses to Corollaries 3.6, 3.7 and 3.8 follow. So we have the following characterizations of the pseudo-arc.

3.9. Theorem. Let a nondegenerate continuum \( X \) be arc-like. Then the following conditions are equivalent:

(i) \( X \) is homogeneous,

(ii) \( X \) is openly homogeneous,

(iii) \( X \) is monotonously homogeneous,

(iv) \( X \) is confluentely homogeneous,

(v) \( X \) is the pseudo-arc.

3.10. Remark. A natural class of mappings larger than that of the confluent ones (but not so large as the class of all continuous mappings) is the class of weakly confluent mappings. However, if we assume that the range space is an arc-like continuum, then each continuous mapping is weakly confluent [25, Theorem 4, p. 236]; i.e., the classes of all continuous mappings and of all weakly confluent ones coincide.

Note that homogeneity with respect to the class of weakly confluent mappings (and therefore with respect to the class of all continuous mappings) cannot be related to properties listed in Theorem 3.9, because an arc (as a locally connected continuum) is continuously homogeneous [17, Theorem 1, p. 347]. But even if we restrict ourselves to continua which are not locally connected, one can find arc-like continua different from the pseudo-arc that are continuously homogeneous. Such as, for example, the (simplest) Knaster indecomposable continuum \( D \) (see e.g. [18, Example 1, Fig. 4, pp. 204 and 205].

A proof of its homogeneity with respect to the class of continuous mappings runs similarly to Bellamy's proof of the theorem saying that each nondegenerate indecomposable continuum can be mapped continuously onto \( D \) (see [1, Theorem, p. 305], cf. also the very end of [17], p. 355). Furthermore, there are hereditarily decomposable nondegenerate arc-like continua other than the arc which are continuously homogeneous.
The reader can verify that, for example, the irreducible continuum (due to Knaster) described in [18, Example 5, p. 191] has all of these properties. Thus the following problem seems to be natural.

3.11. **Problem.** Characterize all nondegenerate arc-like continua which are continuously homogeneous.

The reader can find some partial results concerning the above problem in [17] and [14].

3.12. **Remark.** S. Mazurkiewicz [21, p. 137] has shown that a simple closed curve is the only locally connected plane homogeneous continuum. This result cannot be generalized to confluent or monotone homogeneity, since the Sierpiński universal plane curve is monotonously homogeneous [11, Theorem 5]. However, the authors do not know if these results can be extended to homogeneity with respect to open mappings. So we have (cf. [11, Problem 4]):

3.13. **Problem.** Is the simple closed curve the only locally connected plane openly homogeneous continuum?

4. **Pseudo-ends and another characterization of the pseudo-arc**

In this section we assume that the continuum $X$ under consideration is arc-like. Assuming this, recall that a point $p$ is an end point of $X$ if and only if for each positive number $\varepsilon$ there is an $\varepsilon$-chain covering $X$ such that only an end link of the chain contains $p$ [4, p. 660, and Theorem 13, p. 661]. We introduce the following definition. A point $p$ of an arc-like continuum $X$ is said to be a *pseudo-end point* of $X$ provided that for each neighborhood $U$ of $p$ and for each positive number $\varepsilon$ there is an $\varepsilon$-chain covering $X$, one of whose end links lies in $U$. This concept is due to R. H. Bing [5, p. 346] and has already been discussed earlier (e.g. in [7, p. 901]). Obviously each end point of $X$ is a pseudo-end point of $X$, but not conversely; e.g. for the continuum of Example 7 of [4, p. 662] the origin is a pseudo-end point and is not an end point of this continuum. There are arc-like continua without end points (the same Example 7 of [4, p. 662], while there are none without pseudo-end points. Namely we have the following proposition, a proof of which is in fact contained in Bing's proof of the theorem of [5, p. 345].

4.1. **Proposition.** Each nondegenerate arc-like continuum has a pseudo-end point.

Indeed, let $X$ be such a continuum and, for each positive integer $n$, let $p_n$ be a point of $X$ such that a $1/n$-chain covers $X$ and an end link of this chain contains $p_n$. Some subsequence of the sequence $\{p_n\}$ converges to a point $p$. Then $p$ is a pseudo-end point of $X$. 
4.2. Proposition. The following three conditions are equivalent for an arc-like continuum $X$:

(i) each point of $X$ is its pseudo-end point;
(ii) the set of pseudo-end points of $X$ is dense;
(iii) the set of end points of $X$ is dense;

Proof. The implication from (i) to (ii) is trivial. We shall show the one from (ii) to (iii) (cf. [5, p. 346]). Let a nonempty set $U \subset X$ be open, and take a pseudo-end point $p_0 \in U$. Thus there is an end link $E_1$ of a 1-chain covering $X$ such that $\text{cl} E_1 \subset U$. By (ii), the link $E_1$ contains a pseudo-end point $p_1$ of $X$. Then, by the definition of a pseudo-end point, there is an end link $E_2$ of a 1/2-chain covering $X$ such that $\text{cl} E_2 \subset E_1$, and $E_2$ contains a pseudo-end point $p_2$ of $X$. In turn we find both an end link $E_3$ of a 1/3-chain covering $X$ such that $\text{cl} E_3 \subset E_2$, and a pseudo-end point $p_3$ of $X$ in $E_3$. Similarly we define $E_4, E_5, \ldots$. Then the point $p$ which is in the intersection $\bigcap \{\text{cl} E_n : n \in \{1, 2, \ldots\}\}$ is an end point of $X$, and is contained in $U$. Thus (iii) follows. Finally (iii) implies (i) simply by the definitions of a pseudo-end and of an end point of $X$.  

We say that a mapping $f: X \to Y$ of an arc-like continuum $X$ onto an arc-like continuum $Y$ preserves pseudo-ends provided that for each pseudo-end point $p$ of $X$ its image $f(p)$ is a pseudo-end point in $Y$. Obviously homeomorphisms preserve pseudo-ends. Using exactly the same methods as Bing and Rosenholtz use in their proofs of the theorems saying that arc-likeness of continua is an invariant under monotone mappings [3, Proof of Theorem 3, p. 47] and under open ones [26, Proof of Theorem 1.0, p. 259] (see also the proof of the theorem in [7, p. 901]) we will show that monotone mappings preserve pseudo-ends. We hope that these results can be generalized to confluent mappings. Preservation of pseudo-ends will be applied to obtain a characterization of the pseudo-arc (Theorem 4.7 below).


Proof. The basic ideas of this proof come from Bing's proof of Theorem 3 of [2, p. 47]. Consider an arbitrary arc-like continuum $X$ having a point $p$ as its pseudo-end point (see Proposition 4.1). Let a monotone mapping $f: X \to Y$ be defined from $X$ onto a metric space $Y$. It is known, from the Bing result quoted above, that $Y$ is an arc-like continuum. Let $V$ be a neighborhood of the point $f(p)$ in $Y$, and let $\varepsilon$ be a positive number. We shall show that there is an $\varepsilon$-chain covering $Y$, one of whose end links lies in $V$. The mapping $f$ being uniformly continuous, there exists a positive number $\delta$ such that if the distance between two points of $X$ is less than $\delta$, then the distance between their images under $f$ is less than $\varepsilon/5$. Put $B = B(f(p), \varepsilon/2)$, the open $\varepsilon/2$-ball about $f(p)$ in $Y$. Let $\{C(1), C(2), \ldots, C(n)\}$ be a $\delta$-chain covering $X$ whose end link $C(1)$ lies entirely in the neighborhood $f^{-1}(B \cap V)$ of the pseudo-end point $p$. Let $n_1 = 1, n_2, n_3, \ldots, n_j = n$ be a strictly increasing sequence
of integers such that a point-inverse \( f^{-1}(y) \) intersects \( C(n_i) \) and \( C(n_{i+1}) \) but no point-inverse intersects \( C(n_i) \) and \( C(n_{i+1} + 1) \). For each pair \((i, j)\) of integers such that \( i < j \), put \( D(i, j) = \{ y \in Y : f^{-1}(y) \subset C(u) \cup \cdots \cup C(j) \} \). Thus every \( D(i, j) \) is an open subset of \( Y \) (because the set-valued mapping \( f^{-1} \) is upper semicontinuous; see [18, Theorem 1, p. 57]). Note that \( D(n_1, n_3) \), \( D(n_4, n_8) \), \( D(n_7, n_{11}) \), \ldots, \( D(n_{3k+1}, n_j) \) (where \( j - 5 \leq 3k + 1 < j - 3 \)) is an \( \varepsilon \)-chain covering \( Y \), the end link \( D(n_1, n_5) \) of which intersects \( Y \) and is contained in \( B \cap V \). The proof is complete.

Arguing similarly and repeating ideas of the Rosenholtz proof of Theorem 1.0 of [26, p. 259] (cf. also the proof of the Theorem in [7, pp. 901 and 902] we obtain:

4.4. **Proposition.** *Each open mapping between arc-like continua preserves pseudo-ends.*

4.5. **Remark.** The authors conjecture that Propositions 4.3 and 4.4 can be generalized to confluent mappings (cf. Fact 2.1). So we have

4.6. **Problem.** Do confluent mappings between arc-like continua preserve pseudo-ends?

Observe however that our proofs of preservation of the notion of a pseudo-end under monotone and open mappings (Propositions 4.3 and 4.4 above) depend on the corresponding proofs of preservation of arc-likeness of continua. Obviously we do not have any proof showing that arc-likeness of continua is an invariant under confluent mappings (this is a well-known problem due to Lelek, [19, Problem 4, p. 94] that can be applied to our purposes. But it seems to us that Problem 4.6 can be solved independently of that of Lelek, and that it is even easier than that one, because we may assume that the range space is an arc-like continuum.

In the remainder of this section, we are occupied with a class \( M_1 \) of mappings between arc-like continua that preserve both ends and pseudo-ends. The next theorem gives a partial solution (in a sense) to a problem asked in the last part of [7, p. 902].

4.7. **Theorem.** *If an arc-like continuum \( X \) is homogeneous with respect to a class \( M_1 \) of mappings of \( X \) onto itself that preserve both ends and pseudo-ends, then \( X \) is the pseudo-arc.*

**Proof.** By Proposition 4.1, the continuum \( X \) has a pseudo-end point. Thus by homogeneity of \( X \) with respect to \( M_1 \), each point of \( X \) is a pseudo-end point; i.e. condition (i) of Proposition 4.2 is satisfied. Now Proposition 4.2 implies that its condition (iii) holds, whence we conclude that \( X \) has an end point. Again by homogeneity of \( X \) with respect to \( M_1 \) we see each point of \( X \) is an end point, so \( X \) is the pseudo-arc by Theorem 16 of [4, p. 662]. \( \square \)
Recall that a class $M$ of mappings is said to be admissible [11] if it contains all homeomorphisms and if for each mapping in $M$ its composite with a homeomorphism is also in $M$. Note that all classes of mappings considered in this paper (as open, monotone, and continuous) are admissible.

Since the pseudo-arc is homogeneous (with respect to homeomorphisms; see [2, Theorem 13, p. 740] and since homeomorphisms preserve both ends and pseudo-ends, the converse to Theorem 4.7 holds true and therefore we have the following characterization of the pseudo-arc.

4.8. Corollary. A nondegenerate arc-like continuum is a pseudo-arc if and only if it is homogeneous with respect to an admissible class of mappings of the continuum onto itself that preserve both ends and pseudo-ends.

4.9. Remark. Since monotone mappings and open mappings between arc-like continua preserve ends (Facts 3.3 and 2.1) and pseudo-ends (Propositions 4.3 and 4.4) we conclude that Corollaries 3.7 and 3.8 are consequences of Theorem 4.7.

For the same reason, an affirmative answer to Problem 4.6 would imply Corollary 3.6 as a consequence of Theorem 4.7.

4.10. Remark. Observe two assumptions on the class $M_1$ of mappings from arc-like continua onto themselves that occur in Theorem 4.7:
(a) each mapping in $M_1$ preserves ends, and
(b) each mapping in $M_1$ preserves pseudo-ends.

The authors do not know if each of the two assumptions is essential in the theorem.

4.11. Remark. In a forthcoming paper [13] the authors give a structural characterization of continua (not necessarily arc-like) having a dense set of their end points. It is proved there that a continuum having this property either is indecomposable or is the union of two proper indecomposable subcontinua with connected intersection, each having a dense set of its end points lying out of the composant containing the intersection, and such that the intersection is an end point of a quotient space obtained by shrinking the intersection to a point in each of the two subcontinua.

References


