An example of a monostratiform \( \lambda \)-dendroid

by

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A metric compact continuum is said to be a \textit{dendroid} if it is hereditarily unicoherent and arcwise connected. It follows that it is hereditarily decomposable (see [2], (47), p. 239). A hereditarily unicoherent and hereditarily decomposable continuum is called a \textit{\( \lambda \)-dendroid}. Note that every subcontinuum of a \( \lambda \)-dendroid is also a \( \lambda \)-dendroid.

It is proved in [3], Corollary 2, p. 29, that for every \( \lambda \)-dendroid \( X \) there exists a unique decomposition \( D \) of \( X \) (called the \textit{canonical decomposition}):

\[
X = \bigcup \{S_d : d \in \Lambda(X)\}
\]

such that

(i) \( D \) is upper semicontinuous,

(ii) the elements \( S_d \) of \( D \) are continua,

(iii) the hyperspace \( \Lambda(X) \) of \( D \) is a dendroid,

(iv) \( D \) is the finest possible decomposition among all decompositions satisfying (i), (ii) and (iii).

The elements \( S_d \) of \( D \) are called \textit{strata} of \( X \). The question arises whether there exists a \( \lambda \)-dendroid \( X \) with trivial canonical decomposition, i.e. such that \( X \) has only one stratum.

The purpose of this paper is to give the affirmative answer to the above question.

Call a \( \lambda \)-dendroid to be \textit{monostratiform} if it consists of only one stratum. Thus the hyperspace of the canonical decomposition of a monostratiform \( \lambda \)-dendroid is a point. It follows from [3], Theorem 7, p. 29 that:

(1) A \( \lambda \)-dendroid \( X \) is monostratiform if and only if every monotone mapping onto a dendroid is trivial, i.e. the whole \( X \) goes onto a point.

(See also [4], Corollaries 1 and 2, p. 933).

Construction. The description of the example is based upon the description of Lelek's example of a dendroid with 1-dimensional set of end points (see [9], § 9, p. 314).
To describe the monostratiform $\lambda$-dendroid $X$ which we are going to construct, the following geometrical procedure is needed:

By an oriented triangle $T$ we mean a triangle (i.e. a 2-cell) in which an ordering $\prec$ of vertices is distinguished. If $a$, $b$, $c$ are vertices of $T$ and this ordering is just $a \prec b \prec c$, then we write $T = T(abc)$.

Let $T(abc)$ be a fixed oriented triangle lying in an Euclidean plane with the ordinary metric $\rho$, and let $a_i$, where $i = 1, 2, \ldots$, be points such that for all $k = 1, 2, \ldots$

\begin{align*}
(2) \quad & a_{2k-1} \in ac \quad \text{and} \quad a_{2k} \in bc \\
(3) \quad & \rho(a, a_{2k-1}) = \rho(a, c)/2k \quad \text{and} \quad \rho(b, a_{2k}) = \rho(b, c)/2k.
\end{align*}

Denote the centre of the straight segment $a_i a_{i+1}$ by $b_i$, where $i = 1, 2, \ldots$ and let for every $k = 1, 2, \ldots$ points $d_1^k, d_2^k, d_3^k, d_4^k$ be centres of straight segments $a_{2k-1} b_{2k-1}, b_{2k-1} a_{2k}, a_{2k} b_{2k}$ and $b_{2k} a_{2k+1}$ correspondingly. Thus, for every natural $k$, points $d_1^k$ and $d_2^k$ lie in the side $a_{2k-1} a_{2k}$ as well as points $d_3^k$ and $d_4^k$ lie in the side $a_{2k} a_{2k+1}$ of the triangle with vertices $a_{2k-1}, a_{2k}$ and $a_{2k+1}$. Divide each of two straight segments $d_1^k d_4^k$ and $d_2^k d_3^k$ into three equal parts and define points $c_{4k-3}, c_{4k-2}, c_{4k-1}, c_{4k}$ of this division as follows:

\begin{align*}
(4) \quad & c_{4k-3} \in d_1^k d_4^k \quad \text{and} \quad \rho(c_{4k-3}, d_1^k) = \rho(d_1^k, d_4^k)/3, \\
& c_{4k-2} \in d_2^k d_3^k \quad \text{and} \quad \rho(c_{4k-2}, d_2^k) = \rho(d_2^k, d_3^k)/3, \\
& c_{4k-1} \in d_3^k d_4^k \quad \text{and} \quad \rho(c_{4k-1}, d_3^k) = \rho(d_3^k, d_4^k)/3, \\
& c_{4k} \in d_1^k d_2^k \quad \text{and} \quad \rho(c_{4k}, d_1^k) = \rho(d_1^k, d_2^k)/3.
\end{align*}

Now, for every $k = 1, 2, \ldots$, take four oriented triangles

\begin{align*}
T_{4k-3} &= T(a_{2k-1} b_{2k-1} c_{4k-3}), \\
T_{4k-2} &= T(a_{2k} b_{2k-1} c_{4k-2}), \\
T_{4k-1} &= T(a_{2k} b_{2k} c_{4k-1}), \\
T_{4k} &= T(a_{2k+1} b_{2k} c_{4k})
\end{align*}

lying inside the triangle with vertices $a_{2k-1}, a_{2k}$ and $a_{2k+1}$.

We denote by $\mathcal{C}(abc)$ the sequence $\{T_i\}_{i=1,2,\ldots}$ of oriented triangles defined above (see Fig. 1). Therefore for any two triangles $T_i, T_j \in \mathcal{C}(abc)$ we have

\begin{align*}
(5) \quad & T_{2i-1} \cap T_{2i} = b_i \quad \text{and} \quad T_{2i} \cap T_{2i+1} = a_{i+1} \quad \text{for} \quad i = 1, 2, \ldots \\
(6) \quad & T_i \cap T_j = \emptyset \quad \text{for} \quad |i-j| > 1.
\end{align*}

Let $\delta(S)$ denote the diameter of a set $S$. Now we shall prove that

\begin{align*}
(7) \quad & \delta(T_i) \leq \frac{3}{2} \delta(T(abc)) \quad \text{for} \quad i = 1, 2, \ldots
\end{align*}
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Fig. 1

With this in view let us denote by \( q \) the centre of the segment \( ab \) and consider two trapezia: \( A \) with vertices \( a, a_1, b_1, q \) and \( B \) with vertices \( b, a_2, b_1, q \). The diameter of \( A \) is the maximum of the six numbers: four of them are lengths of the sides of \( A \), the other two are lengths of its diagonals. Put

\[
\delta(T(abc)) = \delta_0. \tag{8}
\]

To estimate \( \delta(A) \) let us observe that we have following inequalities for the sides of \( A \).

\[
\varrho(a, a_1) = \frac{1}{2} \varrho(a, c) \leq \frac{1}{2} \delta_0,
\]

\[
\varrho(a_1, b_1) = \frac{1}{2} \varrho(a_1, a_2) = \frac{1}{2} \varrho(a, b) \leq \frac{1}{2} \delta_0,
\]

\[
\varrho(b_1, q) = \varrho(b_1, c) = \frac{1}{2} \varrho(c, q) \leq \frac{1}{2} \delta_0,
\]

\[
\varrho(q, a) = \frac{1}{2} \varrho(a, b) \leq \frac{1}{2} \delta_0.
\]

For the diagonals of \( A \) we obtain

\[
\varrho(a_1, q) = \frac{1}{2} \varrho(b, c) \leq \frac{1}{2} \delta_0,
\]

\[
\varrho(a, b_1) \leq \varrho(a, a_1) + \varrho(a_1, b_1) \leq \frac{1}{2} \delta_0 + \frac{1}{2} \delta_0 = \frac{3}{2} \delta_0.
\]
Therefore \( \delta(A) \leq \frac{3}{4} \delta_0 \). In the same manner we can prove that \( \delta(B) \leq \frac{3}{4} \delta_0 \). Further, note by construction that for \( i = 1, 2, \ldots \) we have either \( T_i \subseteq A \) or \( T_i \subseteq B \), which leads to \( \delta(T_i) \leq \frac{3}{4} \delta_0 \), and (7) is proved by (8).

If \( A \) is a family of sets, let \( A^* \) denote the union of all members of \( A \).

So we see that
\[
\mathcal{C}^*(abc) = \bigcup_{i=1}^{\infty} T_i ,
\]
which is a connected set by (5). By construction
\[
(9) \quad \mathcal{L}_T T_i = ab = \overline{\mathcal{C}^*(abc)} \setminus \mathcal{C}^*(abc) ,
\]
whence we conclude that
\[
(10) \quad ab \cup \mathcal{C}^*(abc) \text{ is a continuum.}
\]

A point \( p \) of a connected set \( A \) is said to be a separating point of \( A \) if \( A \setminus \{p\} \) is not connected. So we see by (5) and (6) that
\[
(11) \quad \text{every point } a_i \text{ for } i > 1 \text{ as well as every point } b_i \text{ is a separating point of the continuum } ab \cup \mathcal{C}^*(abc).\]

Now we shall define for every \( n = 1, 2, \ldots \) a countable family \( \mathcal{S}_n \) of straight line segments and a countable family \( \mathcal{C}_n \) of oriented triangles.

Namely we put
\[
(12) \quad \mathcal{S}_1 = \{ab\}, \quad \mathcal{C}_1 = \mathcal{C}(abc)
\]
and
\[
(13) \quad \mathcal{S}_{n+1} = \mathcal{S}_n \cup \{a'b'|T(a'b'c') \in \mathcal{C}_n\},
\]
\[
(14) \quad \mathcal{C}_{n+1} = \bigcup \{\mathcal{C}(a'b'c')|T(a'b'c') \in \mathcal{C}_n\}.
\]

We also agree that \( \mathcal{C}_0 = \{T(abc)\} \).

Observe that for an arbitrary sequence of oriented triangles \( T^n \in \mathcal{C}_n \) we have by (7) and (14)
\[
(15) \quad \text{if } T^n \in \mathcal{C}_n, \text{ then } \lim_{n \to \infty} \delta(T^n) = 0.\]

The following properties of the above families \( \mathcal{S}_n \) and \( \mathcal{C}_n \) are readily seen:
\[
(16) \quad \mathcal{S}^*_n \text{ is a continuum},
\]
\[
(17) \quad \mathcal{S}^*_n \cup \mathcal{C}^*_n \text{ is a continuum},
\]
\[
(18) \quad \mathcal{S}^*_{n+1} \cup \mathcal{C}^*_{n+1} \subseteq \mathcal{S}^*_n \cup \mathcal{C}^*_n.
\]

The \( \lambda \)-dendroid \( X \) is defined by the formula
\[
(19) \quad X = \bigcap_{n=1}^{\infty} (\mathcal{S}^*_n \cup \mathcal{C}^*_n) .
\]
It is a continuum as the common part of the decreasing sequence of continua by (17) and (18).

Properties. It follows from (19) by construction that every triangle $T \in \mathcal{C}_n$ for any natural $n$ contains a homeomorphic image of $X$:

(20) if $T \in \mathcal{C}_n$ for some $n$, then $X \cap T$ is homeomorphic to $X$.

Further, we see—also by construction—that

(21) for every point $x$ of $X$ and for every neighbourhood $U$ of $x$ there is a natural $n$, sufficiently great, such that $U$ contains some triangle $T \in \mathcal{C}_n$.

Note that if a continuum $K$ is the common part of a decreasing sequence of continua $K_n$, then every separating point of any $K_n$ is a separating point of $K$. Since the common vertex of every two successive triangles $T_i, T_{i+1} \in \mathcal{C}_i$ is a separating point of the continuum $S_i^* \cup \mathcal{C}_i = ab \cup \mathcal{C}(abc)$ by (11), hence it is a separating point of $X$. Thus (20) implies that

(22) if a point $s$ is a common vertex of two triangles $T', T'' \in \mathcal{C}_n$ ($n = 1, 2, \ldots$), then $s$ is a separating point of $X$.

Let $S$ denote the set of all such points $s$. This means that $s$ is in $S$ if and only if $s$ is a common vertex of some two triangles $T'$ and $T''$ belonging to $\mathcal{C}_n$ for any natural $n$. Hence

(23) every point of $S$ is a separating point of $X$

by (22), and we conclude from (20) and (21) that $S$ is dense in $X$:

(24) $\overline{S} = X$.

The families $\mathcal{C}_n$ being countable for each $n$, the set $S$ is countable. We shall show below that no point of $X \setminus S$ separates $X$, i.e. that $S$ consists of all separating points of $X$.

It follows from definitions (12)-(14) of the families $S_n$ and $\mathcal{C}_n$ and from (19) that $S_n^* \subseteq X$ for every $n$. Thereby

(25) $\bigcup_{n=1}^{\infty} S_n^* \subseteq X$.

According to (12) and (13) the union $\bigcup_{n=1}^{\infty} S_n^*$ consists of the side $ab$ of $T(abc)$ and of the sides $a'b'$ of oriented triangles $T(a'b'c') \in \mathcal{C}_n$, $n = 1, 2, \ldots$. Since a common vertex of any two oriented triangles $T'' = T(a'b'c')$ and $T''' = T(a''b''c'')$ both belonging to the same $\mathcal{C}_n$ is
just the common end point of the sides \(a'b'\) and \(a''b''\) of these triangles, hence, by definition of \(S\), we have

\[ \mathcal{S} \subseteq \bigcup_{n=1}^{\infty} S_n \tag{26} \]

which implies by (24) and (25) that

\[ X = \bigcup_{n=1}^{\infty} S_n^* \tag{27} \]

Note that in every triangle \(T(a'b'c') \in \mathcal{T}_n\) its side \(a'b'\) is a continuum of convergence of the sequence of sides \(a''b''\) of triangles \(T(a''b''c'') \in \mathcal{T}_{n+1}\) by (9) and (20). So, no interior point of the side \(a'b'\) separates \(X\). Thus

\[ \text{if a separating point of } X \text{ is in } \bigcup_{n=1}^{\infty} S_n^*, \text{ it is in } S. \tag{28} \]

Now put

\[ E = \bigcup_{n=1}^{\infty} S_n^* \setminus \bigcup_{n=1}^{\infty} S_n^* \tag{29} \]

i.e. by (27)

\[ E = X \setminus \bigcup_{n=1}^{\infty} S_n^*. \tag{30} \]

Let \(t\) be a point of \(E\). Thus \(t\) is a common point of some decreasing sequence of triangles \(T^n\) such that \(T^n \in \mathcal{T}_n\) and \(t \in \text{Int } T^n\) for every \(n\):

\[ t = \bigcap_{n=1}^{\infty} T^n. \tag{31} \]

Now let \(x \neq t\) be a point of \(X\), and let \(\varepsilon\) be a positive number, less than the distance from \(x\) to \(t\). Therefore by (15) there is a natural \(m\) such that \(t \in \text{Int } T^m, x \in X \setminus T^m\) and \(\delta(T^m) < \varepsilon\). The sequence of triangles \(T^n\) to which \(t\) belongs being decreasing, we have \(t \in T^{m+1} \subseteq T^m\) and \(T^{m+1} \in \mathcal{T}_{m+1}\). So \(T^{m+1}\) must be a term of the sequence \(\mathcal{T}(a'b'c')\) of triangles, where \(T^m = T(a'b'c')\). Let \(T_i^{m+1}\) \((i = 1, 2, \ldots)\) denote the \(i\)th term of this sequence, where indices \(i\) are placed in the same manner like it was done by (4) for the sequence \(\mathcal{T}(abc)\). If \(j\) denote the index of \(T^{m+1}\) in the sequence \(\mathcal{T}(a'b'c')\), i.e. if

\[ T_i^{m+1} = T_j^{m+1}, \]

then the common vertex \(s\) of \(T_j^{m+1}\) and \(T_{j-1}^{m+1}\) is a separating point of \(X\) according to (22) because both these triangles belong to \(\mathcal{T}_{m+1}\). We see that \(s\) separates \(X\) into two sets \(M\) and \(N\) such that \(x \in M\) and \(t \in N\):

\[ X \setminus \{s\} = M \cup N. \]
But $N \cup (s) \subset \bigcup_{i=1}^{j} T_{i}^{m+1} \subset T^{m}$, whence $\delta(N) < \varepsilon$. It shows that $t$ is an end point of $X$ in the sense of Menger-Urysohn, i.e. that

$$\text{ord}_t X = 1$$

(see e.g. [8], § 46, I, p. 200), whence we conclude that

(32) $X$ is locally connected at $t$

by [8], § 46, IV, 1, p. 209. Obviously $X$ is not locally connected at any point of the union $\bigcup_{n=1}^{\infty} S_{n}^{*}$, whence

(33) $E$ is the set of all points at which $X$ is locally connected.

Further, it is readily seen that no point of $X \setminus E$ is an end point of $X$. So by (32) we have

(34) $E$ is the set of all end points of $X$, whence

(35) $E$ is the set of all end points of $X$, whence

$$\text{dim} E = 0$$

by [8], § 46, V, 2, p. 217.

It is immediately seen by (20) that every triangle $T \in C_{n}$ for some $n$ contains a point of $E$. Thus (21) implies that $E$ is dense in $X$.

Remark that (30) gives

$$E = \bigcap_{n=1}^{\infty} (X \setminus S_{n}^{*}) ,$$

whence we see by (16) that $E$ is a $G_{\delta}$-set.

Let us come back now to separating points of $X$. Since no end point of a continuum is a separating point (see [8], § 46, V, 1, p. 217), hence the set of all separating points of $X$ and the set $E$ are disjoint by (35).

It implies by (30) that all separating points of $X$ are in the union $\bigcup_{n=1}^{\infty} S_{n}^{*}$, therefore by (23) and (28) the set of all separating points of $X$ is just $S$.

So we have the following statement concerning separating points of $X$:

(37) The set of all separating points of $X$ is dense and countable. It consists of common vertices of any two triangles $T', T''$, both being in the same $C_{n}$, where $n = 1, 2, ...$

To prove the hereditary decomposability of $X$ consider two different points $x$ and $y$ of $X$. Since they are different, hence the sequence of triangles

$$T(abc) = T^{0} \cup T^{1} \cup T^{2} \cup ... \cup T^{m} \cup ...$$
such that $x$ and $y$ are both in each $T^n$ and $T^n \in \mathcal{C}_n$ for $n = 1, 2, ...$ has the last element, say $T^m$. Observe that there is no irreducible continuum from $x$ to $y$ which intersects the complementary of $T^m$: every such a continuum must lie entirely in $T^m$. The set $X \cap T^m$ being homeomorphic to $X$ by (20), we may assume without loss of generality that $m = 0$, i.e. that no triangle of the sequence $\mathcal{C}(abc)$ contains both $x$ and $y$.

If $x$ and $y$ are both in $ab$, then the unique irreducible continuum from $x$ to $y$ is an arc.

If $x$ is in $ab$ and $y$ is not, then $y$ belongs to some triangle $T_k \in \mathcal{C}(abc)$ and we have countably many points, common vertices of every two successive triangles $T_i, T_{i+1}$ of $\mathcal{C}(abc)$ for $i > k$, which separate $x$ from $y$. Thus an irreducible continuum joining $x$ and $y$ is separated by these points.

Finally if neither $x$ nor $y$ is in $ab$, then they are in different triangles of $\mathcal{C}(abc)$. Let $x \in T_i$, $y \in T_k$ where $T_i, T_k \in \mathcal{C}(abc)$. We may assume $j < k$ (the opposite case, $j > k$, is quite similar). Thus every common vertex of two successive triangles $T_i$ and $T_{i+1}$, where $i = j, j+1, ..., k-1$, is a separating point of $X$ and separates $x$ from $y$, whence an irreducible continuum from $x$ to $y$ must be separated by such a point.

Therefore we conclude that every irreducible subcontinuum of $X$ is separated by a point, whence the hereditary decomposability of $X$ follows (see [8], § 43, V, 1, p. 145).

Before proving the hereditary unicoherence of $X$, firstly we recall some known notions and a theorem concerning continua lying in the plane $E^2$, and secondly, we observe some properties of the construction in $T(abc)$.

A continuum $C \subset E^2$ is said to cut (or to be a cutting of) $E^2$ between points $a$ and $b$ provided that $a, b \in E^2 \setminus C$ and every continuum which contains $a$ and $b$ intersects $C$. It is called an irreducible cutting of $E^2$ between $a$ and $b$ provided that it is a cutting of $E^2$ between $a$ and $b$ and no proper its subcontinuum cuts $E^2$ between these points. A continuum $C \subset E^2$ is said to cut (or to be a cutting of) $E^2$ if there exist two points such that $C$ cuts $E^2$ between them.

The following theorem is known:

(38) A hereditarily decomposable plane continuum is hereditarily unicoherent if and only if it does not cut the plane.

Indeed, every cutting of $E^2$ between two points contains an irreducible cutting between these points (a theorem due to S. Mazurkiewicz; see [7], Theorem I, p. 133). So if we suppose that a hereditarily decomposable and hereditarily unicoherent continuum $K$ cuts $E^2$ between $a$ and $b$, then we conclude that $K$ must contain an irreducible cutting $L$ of $E^2$ between $a$ and $b$ which is decomposable and unicoherent. Therefore for
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an arbitrary decomposition of \( L \) into two its proper subcontinua \( L_1 \) and \( L_2 \) we see that neither \( L_1 \) nor \( L_2 \) cuts \( E^2 \) between \( a \) and \( b \), and that \( L_1 \cap L_2 \) is a continuum; hence the union \( L_1 \cup L_2 = L \) cannot cut \( E^2 \) between \( a \) and \( b \) according to a theorem due to Z. Janiszewski ([6], Theorem A, p. 48; see also [7], (ii), p. 136), and we get a contradiction.

Invertedly, if a hereditarily decomposable plane continuum \( K \) is not hereditarily unicoherent, then it contains a subcontinuum \( M \) that can be decomposed into two subcontinua \( M_1 \) and \( M_2 \) the intersection of which is not a continuum. Thus \( M_1 \cup M_2 = M \) is a cutting of \( E^2 \) (see [6], Theorem B, p. 55; also [7], (iii), p. 136), and so is \( K \) because \( K \) is 1-dimensional.

Observe now the following two properties of the construction in the triangle \( T(abc) \).

(39) For every point \( x_0 \in T(abc) \setminus \overline{G}(abc) \) there exists an arc \( x_0 p_0 \) such that \( p_0 \in ac \cup bc \) and \( x_0 p_0 \subset T(abc) \setminus \overline{G}(abc) \).

In fact, if \( x_0 \in T(a_{j-1}a_ja_{j+1}) \subset T(abc) \), then we take as \( p_0 \) an arbitrary point of \( a_{j-1}a_ja_{j+1} \setminus (a_{j-1}) \setminus (a_{j+1}) \) (see the figure) and we see that such an arc \( x_0 p_0 \) does exist.

Similarly, if \( T(a'b'c') \in \gamma(abc) \), then

(40) For every point \( p_1 \in (a'c' \cup b'c') \setminus \overline{\gamma}(abc) \) there exists an arc \( p_1 p_0 \) such that \( p_0 \in ac \cup bc \) and \( p_1 p_0 \subset T(abc) \setminus \overline{G}(abc) \).

Indeed, if \( p_1 \in a'c' \cup b'c' \subset T(a'b'c') \subset T(a_{j-1}a_ja_{j+1}) \), then we take as \( p_0 \) an arbitrary point of \( a_{j-1}a_ja_{j+1} \setminus (a_{j-1}) \setminus (a_{j+1}) \) and we can easy find such an arc \( p_1 p_0 \) lying entirely in \( T(a_{j-1}a_ja_{j+1}) \).

Now, in order to prove the hereditary unicoherence of the continuum \( X \), it is sufficient to prove, according to (38), that \( X \) does not cut the plane, i.e. that for any two points \( x_1 \) and \( x_2 \) in \( E^2 \setminus X \) there is a continuum joining \( x_1 \) and \( x_2 \) and having no common point with \( X \). Let \( y \) be a point of \( E^2 \setminus T(abc) \). Observe that if \( K_1 \) and \( K_2 \) are continua such that \( x_i \in K_i \), \( y \in K_i \) and \( K_i \cap X = \emptyset \) for \( i = 1 \) and 2, then their union \( K_1 \cup K_2 \) is a continuum \( K \) with properties \( x_1, x_2 \in K \) and \( K \cap X = \emptyset \). Therefore it is enough to show that a point \( x \in E^2 \setminus X \) can be joined with \( y \) by a continuum which does not intersect \( X \). The existence of such a continuum is obvious if \( x \in E^2 \setminus T(abc) \). In the opposite case, if \( x \in T(abc) \setminus X \), we see that \( x \) is not in \( \bigcup_{n=1}^{\infty} S_n \) by (25), whence, the sequence of sets \( \gamma_n \) being decreasing by (14), we conclude by (19) that there exists a natural \( m \) with property

(41) \( x \in \gamma_m \setminus \gamma_{m+1} \).

Consider the finite sequence of triangles \( T^n \in \gamma_n \) such that

(42) \( x \in T_m \subset T_m-1 \subset \ldots \subset T_1 \subset T^0 = T(abc) \).
Putting \( T^m = T(\text{uvw}) \) we have \( \mathcal{C}(\text{uvw}) \subseteq \mathcal{C}_{m+1} \), whence \( \mathcal{C}^*(\text{uvw}) \subseteq \mathcal{C}_{m+1}^* \). Thus

\[
(43) \quad x \in T(\text{uvw}) \setminus \mathcal{C}^*(\text{uvw})
\]

by (42) and (41). The intersection \( X \cap T^m \) being homeomorphic to \( X \) by (20), we have similarly to (9)

\[
(44) \quad uv = \overline{\mathcal{C}^*(\text{uvw})} \setminus \mathcal{C}^*(\text{uvw}) .
\]

Since \( T^m \in \mathcal{C}_m \), hence we conclude from (13) that \( uv \subseteq \mathcal{S}_{m+1}^* \), thus \( uv \subseteq X \) by (25). But the point \( x \) is not in \( X \), so it is not in \( ur \), thereby (43) and (44) give

\[
x \in T(\text{uvw}) \setminus \overline{\mathcal{C}^*(\text{uvw})} .
\]

Applying (39) to the triangle \( T(\text{uvw}) \) in place of \( T(abc) \), which is possible by virtue of (20), we deduce that there exists an arc \( \text{xp}_m \) such that \( \text{p}_m \) lies in the boundary of \( T^m \) and \( \text{xp}_m \cap X = \emptyset \). Using (20) and applying (40) to triangles \( T^{m-1}, T^{m-2}, \ldots, T^0 \) we infer that there is a finite sequence of arcs \( \text{p}_m \text{p}_{m-1} \cap \text{p}_{m-1} \text{p}_{m-2}, \ldots, \text{p}_1 \text{p}_0 \) every of which is disjoint with \( X \) and such that \( \text{p}_i \) belongs to the boundary of \( T^i \) for \( i = 0, 1, \ldots, m-1 \). So the union \( \text{xp}_m \cap \text{p}_m \text{p}_{m-1} \cap \cdots \cap \text{p}_1 \text{p}_0 \) is a continuum which lies in \( T(\text{abc}) \setminus X \) and which joins the point \( x \) with the point \( \text{p}_0 \in ac \cap bc \). Thus if we join \( \text{p}_0 \) with \( y \) by an arc \( \text{p}_0 \text{y} \) such that \( \text{p}_0 \text{y} \cap T(\text{abc}) = (\text{p}_0) \), we obtain a continuum

\[
\text{xp}_m \cup \text{p}_m \text{p}_{m-1} \cup \cdots \cup \text{p}_1 \text{p}_0 \cup \text{p}_0 \text{y} \subseteq E^2 \setminus X,
\]

and therefore the proof of the hereditary unicoherence of \( X \) is finished.

Being hereditarily decomposable and hereditarily unicoherent, \( X \) is a \( \lambda \)-dendroid.

Now we shall prove the main property of the \( \lambda \)-dendroid \( X \), that \( X \) is monostratiform. Let

\[
\phi: X \to \Delta(X)
\]

be the canonical mapping of \( X \) onto the dendroid \( \Delta(X) \), i.e. such a mapping that

\[
\phi^{-1}(d) = S_d \quad \text{for} \quad d \in \Delta(X) ,
\]

where \( S_d \) are the elements of the canonical decomposition \( \mathcal{D} \) (see [3], p. 25). We shall show that \( \Delta(X) \) reduces to a point. To establish this it is sufficient to verify that \( S_n^* \) goes to a point under \( \phi \) for each \( n = 1, 2, \ldots \). In fact, if it is so, each \( S_n^* \) must go onto the same point, because the sequence of continua \( S_n^* \), \( n = 1, 2, \ldots \), is increasing by (13). It implies that the union \( \bigcup_{n=1}^{\infty} S_n^* \) is mapped onto the point, thus \( X \) is by (27).
So, we should demonstrate that

\[(45) \quad \phi(S_n^*) \text{ is a point for each } n = 1, 2, \ldots \]

Recall that if a continuum \( C \) is a tranche (in the sense of Kuratowski, see [8], § 43, IV, p. 139) of an irreducible subcontinuum of an arbitrary \( \lambda \)-dendroid and if \( f \) is a monotone mapping of this \( \lambda \)-dendroid onto a dendroid, then \( f(C) \) is a point (see [3], Theorem 5, p. 26).

Firstly, observe that \( S_1^* = ab \) is a tranche of the irreducible continuum \( S_1^* \). Thus \( \phi(S_1^*) \) is a point by the above reason. Denote this point by \( d \):

\[ \phi(S_1^*) = d. \]

Secondly, assume that

\[(46) \quad \phi(S_n^*) = d \]

for some fixed \( n \). The family \( \mathcal{C}_{n-1} \) of oriented triangles being obviously countable, let

\[ T^1, T^2, \ldots, T^k, \ldots, \text{ where } T^k \in \mathcal{C}_{n-1} \]

be an arbitrary sequence of all its elements. Thus

\[(47) \quad \mathcal{C}_n = \bigcup_{k=1}^{\infty} \mathcal{C}(T^k) \]

by (14) with \( n \) instead of \( n+1 \), where \( \mathcal{C}(T^k) \) means the same as \( \mathcal{C}(pqr) \) if \( T^k = T(pqr) \). Denote by \( \mathcal{R}_k \) the family of all straight segments \( a'b' \) such that \( T(a'b'c') \in \mathcal{C}(T^k) \). Therefore (47) implies that

\[ \{a'b' \mid T(a'b'c') \in \mathcal{C}_n\} = \bigcup_{k=1}^{\infty} \mathcal{R}_k. \]

whence

\[(48) \quad S_{n+1}^* = S_n^* \cup \bigcup_{k=1}^{\infty} \mathcal{R}_k \]

by (13). Consider now oriented triangles \( T_i^k \), terms of the sequence \( \mathcal{C}(T^k) \), where indices \( i \) are defined in the same way as is was done by (4) for triangles \( T_i \) of the sequence \( \mathcal{C}(abc) \). Let \( A_i \) denote the side \( a''b'' \) of the oriented triangle \( T_i^k = T(a''b''c'') \). Hence by definition of \( \mathcal{R}_k \) we have

\[(49) \quad \mathcal{R}_k^* = \bigcup_{i=1} A_i. \]

Applying (5) to the sequence \( \mathcal{C}(T^k) \) we see that

\[(50) \quad A_i \text{ and } A_{i+1} \text{ have an end point in common.} \]

Observe that every segment \( A_i \) is a tranche of an irreducible continuum. Namely the set

\[ A_i \cup \bigcup \{uv \mid T(uvw) \in \mathcal{C}(T_i^k)\} \subset S_{n+2}^* \]
is an irreducible continuum, homeomorphic to the closure of the graph of the function \( y = \sin(1/x) \) for \( 0 < x \leq 1 \), and it has \( A_0 \) as the only tranche different from a point. By the above argument \( \phi(A_i) \) is a point. Call it \( d'_i \):

\[
\phi(A_i) = d'_i .
\]

If \( \phi(A_{i+1}) = d'_{i+1} \), then \( d'_i = d'_{i+1} \) by (50), and we may omit the indices \( i \) and write

\[
\phi(A_i) = d' \quad \text{for } i = 1, 2, \ldots
\]

Thus by (49)

\[
\phi(\mathcal{R}_k^*) = d' .
\]

But \( \mathcal{R}_k^* \) is a continuum as well as \( S_n^* \) is, and we have

\[
S_n^* \cap \mathcal{R}_k^* = pq ,
\]

where \( T(pqr) = T^k \), which implies by (46) that \( d \) and \( d' \) coincide. So we conclude that

\[
\phi(S_{n+1}^*) = d
\]

by (48). Therefore (45) is established.

**Remarks.** Some modifications of the above construction lead to various kinds of monostratiform \( \lambda \)-dendroids, e.g. to an example of a monostratiform \( \lambda \)-dendroid without separating points. Since \( X \) is a plane continuum no subcontinuum of which separates the plane, it is tree-like (see [1], the definition on p. 653 and Theorem 6, p. 656). But H. Cook has recently proved, [5], that every \( \lambda \)-dendroid, not necessarily embeddable in the plane, is tree-like. It can be seen from Corollary 1 in [11], p. 379, that \( X \) has the fixed point property. It is not known if all \( \lambda \)-dendroids have this property. For a partial solution see, e.g., [4], where a list of references is given.

**References**