Open Mappings of Universal Dendrites

by

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Summary. Let $D_n$ be the standard universal dendrite of order $n$, where $n = 2, 3, 4, \ldots$, (i.e. such a dendrite that all its ramification points are exactly of order $n$ and that they lie densely on each of its arcs). It is proved that all open images of $D_n$ are homeomorphic if and only if $n = 2, 3$ or $\omega$.

All mappings considered in this paper are continuous and all spaces are assumed to be metric. We use the notion of order of a point in a continuum in the sense of Menger–Urysohn (see e.g. [8], §51, I, p. 274). A dendrite means a locally connected continuum containing no simple closed curve. It is known that every image of a dendrite under an open mapping is again a dendrite (see [18], Chapter VIII, (7.7), p. 148 and Chapter X, a statement just before Theorem (1.5) on p. 185). A dendrite is said to be universal if it contains a homeomorphic image of any other dendrite (see [16], Chapter K, p. 137, and [10], Chapter X, §6, p. 318). Observe that if a dendrite $X$ contains a universal dendrite $Y$, then $X$ is universal itself. To avoid any confusion with other universal dendrites, we accept the following definition.

By Ważewski's dendrite we mean the standard universal dendrite, i.e., a dendrite $D_\omega$ such that

1. every ramification point of $D_\omega$ if of order $\omega$,

and

2. for every arc $A$ contained in $D_\omega$ the set of all ramification points of $D_\omega$ which belong to $A$ is a dense subset of $A$.

It is proved in [16], Chapter H, §16, p. 123 and 124, and §17, p. 124 (cf. (1) and (2) above with condition 1°) and 2°) of §17, p. 124) that

3. any two dendrites satisfying (1) and (2) are homeomorphic.

**Lemma.** If $f: X \to f(X)$ is an open mapping of a dendrite $X$ and a point $p \in X$ is of order $\omega$ in $X$, then $f(p)$ is of order $\omega$ in $f(X)$.

**Proof.** We already know that $f(X)$ is a dendrite. Since every dendrite is regular in the sense of the theory of order (see [8], §51, I, p. 275, and VI, theorem 4, p. 389).
p. 301; cf. [18], Chapter V, §5, p. 99) the point \( f(p) \) can be either of a finite order or of order \( \omega \). Suppose on the contrary that \( f(p) \) is of a finite order \( n \). Since \( f \) is continuous, there is \( \delta > 0 \) such that if points \( p \) and \( x \) are of distance less than \( \delta \), then their images under \( f \) are of distance less than \( \varepsilon \). Since \( p \) is of order \( \omega \), there is a component \( C \) of \( X \setminus \{p\} \) whose diameter is less than \( \varepsilon \). The component \( C \) is obviously an open subset of \( X \), thus its image \( f(C) \) is an open subset of \( f(X) \setminus \{f(p)\} \). By the continuity of \( f \) we have \( \text{diam} f(C) < \varepsilon \), thus \( f(C) \) is a proper subset of a component \( K \) of \( f(X) \setminus \{f(p)\} \). So let \( y' \in K \setminus f(C) \). Hence there is a point \( y' \) of the arc \( f(p) \) lying in the boundary \( \text{Bd} \ (f(C)) = (f(C) \setminus f(C)) \) of \( f(C) \) and which is different from \( f(p) \). Take \( C = C \cup \{p\} \) and consider \( f(C) = f(C) = f(C \cup \{p\}) = f(C) \cup \{f(p)\} \). It follows that \( \text{Bd} \ (f(C)) \) consists of exactly one point \( f(p) \), a contradiction, since \( y' \in \text{Bd} \ (f(C)) \setminus \{f(p)\} \).

**Theorem 1.** Every two open images of Ważewski's dendrite are homeomorphic.

**Proof.** By (3) it is sufficient to prove that any open image of \( D_\omega \) satisfies (1) and (2). Let a mapping \( f: D_\omega \to f(D_\omega) \) be open. Thus \( f(D_\omega) \) is a dendrite. Let \( q \) be a ramification point of \( f(D_\omega) \). Since the order of a point in \( D_\omega \) is never increased when \( D_\omega \) undergoes an open mapping (see [18], Chapter VIII, corollary (7.31), p. 147), there is a ramification point \( p \) in \( D_\omega \) with \( f(p) = q \). But all ramification points of \( D_\omega \) are of order \( \omega \) by (1), thus by virtue of the Lemma the point \( q \) is of order \( \omega \), and so (1) holds for \( f(D_\omega) \).

Now let \( B \) be an arc in \( f(D_\omega) \). Let \( C \) be a component of \( f^{-1}(B) \). Then we have \( f(C) = B \) by [18], Chapter VIII, §7, theorem (7.5), p. 148. Since the set \( R \) of ramification points of \( D_\omega \) is dense on every arc \( A \subset D_\omega \), the set \( R \cap C \) is dense in \( C \), and therefore its image \( f(R \cap C) \) is dense in \( B \). Thus (2) follows from the Lemma, and the proof is complete.

By a universal dendrite of order at most \( n \), where \( n \geq 3 \), we mean a dendrite \( D \) with the property that every ramification point of \( D \) is of order at most \( n \) and that for any dendrite \( D' \) having this property the dendrite \( D \) contains a homeomorphic image of \( D' \). It is known (see [10], Chapter X, §6, p. 322) that a dendrite \( X \) is a universal one of order at most \( n \) if and only if every point of \( X \) is of order less than or equal to \( n \) and \( X \) contains a subdendrite \( D \) such that for every arc \( A \) in \( D \) the set of ramification points of order \( n \) lying in \( A \) is a dense subset of \( A \). Let us accept the following definition. By the standard universal dendrite of order \( n \) we mean a dendrite \( D_n \) such that

(4) every ramification point of \( D_n \) is of order \( n \),
and

(5) for every arc \( A \) contained in \( D_n \) the set of all ramification points of \( D_n \) which belong to \( A \) is a dense subset of \( A \).

It is known (the argumentation is exactly the same as for \( D_\omega \)) that

(6) any two standard universal dendrites \( D_n \) (for the same natural \( n \geq 3 \)) are homeomorphic.
To show some further results concerning open mappings of standard universal dendrites $D_n$ we need a special description of these continua. Namely we describe $D_n$ (for $n \geq 3$) as the inverse limit space of an inverse system $\{X_i, f_{i+1}^i, i=1, 2, \ldots\}$ of finite dendrites $X_i$ (i.e. of dendrites having only a finite number of end points) with monotone onto bonding mappings $f_{i+1}^i: X_{i+1} \to X_i$. The procedure for obtaining $D_n$ is quite similar to that employed by Anderson and Choquet in [1]. To begin with, we recall that an $n$-od means a continuum homeomorphic to the union of $n$ distinct straight line segments of unit length emanating from the origin. The common point of the segments is called the vertex of the $n$-od. Let $T_n^i$ denote an $n$-od composed of $n$ straight line segments of length $4^{-j}$ each. To define $X_i$ and $f_{i+1}^i: X_{i+1} \to X_i$ for $i=1, 2, 3, \ldots$ we shall proceed by induction. Define $X_1$ as the unit straight line segment. Let $x$ be the mid-point of $X_1$ and define $X_2$ as the union of $X_1$ and $T_1^{n-2}$ such that the intersection of $X_1$ and $T_1^{n-2}$ is just $x$, which is the vertex of $T_1^{n-2}$. So $X_2$ is the union of $n$ straight line segments disjoint out of their end-points. Let $f_2^1: X_2 \to X_1$ be the mapping which shrinks $T_1^{n-2}$ to the point $x$, i.e. such that all the point inverses are degenerate except the one of $x$ which is $T_1^{n-2}$. Assume now that a dendrite $X_i$ has been defined as the union of $n_i-1$ straight line segments disjoint out of their end-points, i.e., such that the end points of these straight line segments are either end-points or ramification points of $X_i$, while each interior point of each of them is a point of order 2 in $X_i$. Given such a straight line segment, let $x$ denote its mid-point. To each point $x$ such defined we associate in a one-to-one way a set $T_2^{n-2}$. We take each mid-point $x$ as the vertex of the associated set $T_1^{n-2}$ in such a manner that $X_i$ has only the point $x$ in common with the added copy of $T_1^{n-2}$ and that different copies of $T_1^{n-2}$ are disjoint. All this can clearly be done so carefully that the resulting set $X_{i+1}$ which is by the definition equal to the union of $X_i$ and of $n_i-1$ copies of $T_1^{n-2}$, is a finite dendrite, namely a dendrite which is the union of $n_i$ segments. We define $f_{i+1}^i: X_{i+1} \to X_i$ as the mapping which shrinks each $T_1^{n-2}$ (added to $X_i$ to get $X_{i+1}$) to its vertex, i.e., $f_{i+1}^i$ is the mapping which has degenerate point-inverses for all points of $X_i$ except the mid-points $x$ of the $n_i-1$ straight line segments, the set $X_i$ is composed of. Thus $f_{i+1}^i$ is continuous and monotone (moreover, it is even an $A^*$-mapping, i.e., an atomic mapping with only a finite number of nondegenerate point-inverses, see [1], footnote (3) on p. 347). Let $X$ denote the inverse limit set of the inverse system $\{X_i, f_{i+1}^i, i=1, 2, \ldots\}$. Since every $X_i$ is a dendrite and since the bonding mappings $f_{i+1}^i$ are monotone, the inverse limit set $X$ is a dendrite (see [13], theorem 4 (part 3.), p. 229; cf. [12], theorem 4, p. 413). By theorem I of [1], p. 348, the dendrite $X$ is homeomorphic to $\bigcap_{i=1}^{n} \bigcup_{k=1}^{\infty} X_i$, which is obviously equal to $\bigcup_{i=1}^{\infty} X_i$. Thus it can be verified in a routine way that $X$ satisfies both conditions (5) and (6) and therefore is homeomorphic to the standard universal dendrite $D_n$ of order $n$.

Now we are going to show that Theorem 1 on open mappings of $D_n$ cannot be extended to a similar result on open mappings of $D_n$ for $n \geq 4$. Namely we have the following
THEOREM 2. Given two natural numbers \( n \) and \( m \) with \( n > m \geq 3 \), there exists an open mapping of \( D_n \) onto \( D_m \).

Proof. Let \( D_n \) and \( D_m \) be the inverse limit spaces \( X \) and \( Y \) of the inverse systems \( \{ X_i, f_i^{i+1}, i = 1, 2, \ldots \} \) and \( \{ Y_i, g_i^{i+1}, i = 1, 2, \ldots \} \) respectively, where—as above—all \( X_i \) and \( Y_i \) are unit segments, \( X_i \) and \( Y_i \) are finite dendrites for \( i = 2, 3, \ldots \) obtained in the procedure described above from their predecessors \( X_{i-1} \) and \( Y_{i-1} \) respectively by adding to them in the proper way \( n^{i-2} \) and \( m^{i-2} \) sets \( T_i^{n-2} \) and \( T_i^{m-2} \) correspondingly. For every \( i = 1, 2, \ldots \) we define an open surjection \( \varphi_i: X_i \rightarrow Y_i \).

To this end let \( A_1, A_2, \ldots, A_m, A_{m+1}, \ldots, A_{n-2} \) be the straight line segments which form the \((n-2)\)-od \( T_i^{n-2} \), i.e. \( T_i^{n-2} = \bigcup_{j=1}^{n-2} A_j \), and let \( x \) denote the vertex of \( T_i^{n-2} \).

Let \( \gamma_i: T_i^{n-2} \rightarrow T_i^{m-2} \) be a mapping which is the identity on every of \( A_1, A_2, \ldots, A_m \) and which maps every of \( A_{m+1}, \ldots, A_{n-2} \) isometrically onto a fixed segment of \( T_i^{m-1} \), say \( A_1 \). Thus in particular \( x \) is a fixed point under \( \gamma_i \). It is evident that \( \gamma_i \) is an open surjection. Now we are ready to proceed by induction. Let \( \varphi_1: X_1 \rightarrow Y_1 \) be the identity mapping. We define \( \varphi_2: X_2 \rightarrow Y_2 \) putting \( \varphi_2 = \varphi_1 \) and \( \varphi_2|T_2^{n-2} = \gamma_1 \) (recall that \( X_2 = X_1 \cup T_1^{n-2} \)). Assume that an open mapping \( \varphi_i: X_i \rightarrow Y_i \) is defined in such a way that it is an isometry on each of straight line segments which form the dendrite \( X_i \).

We define \( \varphi_{i+1} \) as the composite of two open mappings \( \alpha \) and \( \beta \) such that \( \alpha: X_{i+1} \rightarrow \alpha(X_{i+1}) \subseteq X_{i+1} \) and \( \beta: \alpha(X_{i+1}) \rightarrow Y_{i+1} \). Recall that every \( X_{i+1} \) is the union of \( X_i \) and of \( n^{i-1} \) copies of \((n-2)\)-od \( T_i^{n-2} \). We admit \( \alpha| X_1 \) to be the identity, and \( \alpha| T_i^{n-2} = \gamma_i \) for every copy of \( T_i^{n-2} \). Thus \( \alpha(X_{i+1}) \) is the union of \( X_i \) and of \( n^{i-1} \) copies of \((m-2)\)-od \( T_i^{m-2} \). Further, we let \( \beta| X_i = \varphi_i \) and for every \( T_i^{m-2} \) which is associated with a straight line segment \( A \) contained in \( X_i \) (i.e. such that the vertex \( x \) of \( T_i^{m-2} \) is the mid-point of \( A \)) the partial mapping \( \beta|T_i^{m-2} \) is defined as a homeomorphism taking \( T_i^{m-2} \) onto the \((m-2)\)-od which is associated with the straight line segment \( \beta(A) \). It is evident from the above definition that \( \varphi_{i+1} \) is an open surjection. Furthermore, it is a routine procedure to verify that the mapping \( \varphi_{i+1} \) is defined in such a way that the diagram

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i^{i+1}} & X_{i+1} \\
\downarrow \varphi_i & & \downarrow \varphi_{i+1} \\
Y_i & \xleftarrow{g_i^{i+1}} & Y_{i+1}
\end{array}
\]

is exact for every \( i = 1, 2, \ldots \) (it means that the diagram is commutative and \( \varphi_i(x_i) = \varphi_{i+1}(y_{i+1}) \) implies \( (f_i^{i+1})^{-1}(x_i) \cap \varphi_{i+1}^{-1}(y_{i+1}) \neq \emptyset \), see [7], §3, IV, p. 19). Therefore it follows that the inverse limit mapping \( \varphi: X \rightarrow Y \) is continuous ([5], Chapter VIII, theorem 3.13, p. 218), surjective (since for every \( i = 1, 2, \ldots \) all four mappings in diagram (7) are surjective) and open ([6], theorem 3, p. 58; see also [14], Theorem 4, p. 61). The conclusion of the Theorem follows from (6).

Although the result discussed in Theorem 1 for \( D_o \) cannot be extended by Theorem 2 to other universal dendrites \( D_n \) where \( n \geq 4 \), it can be proved for the dendrite \( D_3 \).
THEOREM 3. Every two open images of the standard universal dendrite $D_3$ of order 3 are homeomorphic.

Indeed, let a mapping $f: D_3 \rightarrow f(D_3)$ be open. Thus $f(D_3)$ is a dendrite. Let $q$ be a ramification point of $f(D_3)$. By the same argument as in the corresponding part of the proof of Theorem 1 we see that there is a ramification point $p$ in $D_3$ with $f(p) = q$. Since all ramification points of $D_3$ are of order 3 by (4), and since the order of a point does not increase under an open mapping ([18], (7.31), p. 147), we see that the point $q$ is of order 3, and thus (4) holds for $f(D_3)$. Further, the same argument implies that the image of an end point in $D_3$ is an end point in $f(D_3)$. But in a dendrite the set of all ramification points is dense if and only if the set of all end points is dense. Hence the set of end points of $D_3$ is dense, and thereby $f(D_3)$ also has a dense set of end points. Thus the set $R$ of ramification points of $f(D_3)$ is dense, and since all of them are of order 3, the intersection $R \cap B$ must be a dense subset of $B$ for every arc $B$ in $f(D_3)$. Thus (5) holds for $f(D_3)$ and the Theorem follows by (6).

It is natural to consider the unit straight line segment $[0, 1]$ as the standard universal dendrite $D_2$ of order 2. Since every open image of an arc is an arc (see [17], theorem (3.3), p. 375 and [18], theorem (1.3), p. 184; cf. [9], §4, (a), (iii), p. 190, and [11], theorem 2, p. 818) we see that $D_2=[0, 1]$ has the same property as $D_3$ and $D_3$ that any two of its open images are homeomorphic. Thus the following corollary is a consequence of Theorems 1, 2 and 3.

COROLLARY. Among all standard universal dendrites $D_2, D_3, ..., D_n, ..., D_\omega$ only $D_2, D_3$ and $D_\omega$ are homeomorphic with all their open images.

We describe now another dendrite having this property. Let $F$ be the union of countably many straight line segments $A_1, A_2, ...$ of lengths $2^{-1}, 2^{-2}, ...$ respectively, emanating from a fixed point $p$. In other words $F$ is a fan of order $\omega$ with the vertex $p$. It is easy to note that $F$ is the universal dendrite in the class of all dendrites having exactly one ramification point (i.e. in the class of locally connected fans). Similarly to $D_2, D_3$ and $D_\omega$, also $F$ has the discussed property. In fact, if a mapping $f: F \rightarrow f(F)$ is open, then $f(F)$ is a dendrite, and since the order of a point does not increase under $f$, the dendrite $f(F)$ has at most one ramification point and the end-points of $F$ are mapped to end-points of $f(F)$. Thus $f(F)$ is either an arc or a (locally connected) fan (compare [3], Theorem 12, p. 32, and [4], Corollary I.2, p. 410, where this statement is established for a wider class of mappings, namely for confluent ones, [2], p. 213 and 214). Since the vertex $p$ of $F$ is the only accumulation point of the set of end-points of $F$ and since every end-point of $F$ is mapped to an end-point of $f(F)$, hence $f(F)$ cannot be an arc. Further, the same argument shows that $f(F)$ cannot be a finite fan (i.e., an $n$-od for some natural $n$). Thus it must be an infinite locally connected fan, but every such fan is obviously homeomorphic to $F$.

In the light of the above results the following problem seems to be interesting.

PROBLEM. Characterize all dendrites $X$ which have the property that

(*) every open image of $X$ is homeomorphic to $X$. 
This is a particular case of a more general problem concerning a characterization of all continua $X$ having property ($\ast$). Recall that the pseudo-arc in such a continuum ([15], theorem 1.3, p. 260).

REFERENCES


