Houston Journal of Mathematics
Volume 33, No. 1, 2007

RETRACTIONS FROM $C(X)$ ONTO X AND CONTINUA OF TYPE N

FÉLIX CAPULÍN AND WŁODZIMIERZ J. CHARATONIK

Communicated by Charles Hagopian

Abstract. We show that if a metric continuum X is of type N, then there is no a retraction from the hyperspace of subcontinua $C(X)$ onto X, and X admits no mean. We also give an example which answers a question posed by T. J. Lee related to this topic.

1. Introduction and preliminaries

The symbol $\mathbb{N}$ stands for the set of all positive integers. All considered spaces are assumed to be metric and all mappings are continuous. A ***continuum*** means a nonempty compact and connected space. A continuum is said to be ***hereditarily unicoherent*** provided that the intersection of any two of its subcontinua is connected. A ***dendroid*** is defined as an arcwise connected and hereditarily unicoherent continuum. A point p of a dendroid X is called a ***ramification point*** of X if p is vertex of a simple triod contained in X. A ***fan*** means a dendroid that contains exactly one ramification point, which is called the ***vertex*** of the fan. If X is arcwise connected and $x, y \in X$, we denote by $[x, y]$ any arc in X joining x and y, if the arc is rectilinear we denote it by xy. We said that an arcwise connected continuum X is ***uniformly arcwise connected*** if given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that every arc $[a, b] \subseteq X$ contains a set $\{a = a_1, a_2, ..., a_k = b\}$, $a_1 < a_2 < ... < a_k$ (the natural order in the arc) which satisfies $\operatorname{diam}[a_i, a_i + 1] <\varepsilon$, $i = 1, ..., k-1$. Let X be a continuum, we denote by 2^X and $C(X)$ the **hyperspaces** of all nonempty closed subsets and of all subcontinua of X, respectively, equipped with the Hausdorff metric. It is well known that these hyperspaces are arcwise connected continua, see for example [6, Theorem 14.9,

2000 *Mathematics Subject Classification.* 54B20, 54E40, 54F15.
Key words and phrases. Continuum, hyperspace, retraction, means,type N.

p. 113] or [12, Theorem 1.12, p. 65]. Let $F_1(X) = \{\{x\} : x \in X\} \subset C(X)$ and $F_2(X) = \{\{x, y\} : x, y \in X\} \subset 2^X$. Since $F_1(X)$ is homeomorphic to X, we may assume that $X \subset C(X)$. Recall that for any arc A, the hyperspace $C(A)$ is homeomorphic to a disk, see [6, p. 33]

Let A be a closed subset of X. A ***retraction*** is a mapping $r : X \to A$ such that $r|A = \mathrm{id}\,|A$. A ***mean*** is a mapping $m : X \times X \to X$ such that

(a) $m((x, x)) = x$ for each $x \in X$,
(b) $m((x, y)) = m((y, x))$ for each $x, y \in X$.

The problem of characterizing continua which admit retractions $r : C(X) \to X$ or $r : 2^X \to X$ has been investigated in [11, p. 413] and in [12, Theorem 6.4, p. 270]. A. Illanes in [4] constructs an example of a continuum X which is a retract of $C(X)$ but not of 2^X. It is known that a one-dimensional continuum X which is a retract of either 2^X or $C(X)$ is a dendroid, [3, p. 122]. Moreover, in [2, Theorems 3.1 and 3.3, p. 9 and 10] the following results are proved for one-dimensional continua.

Theorem 1.1. *Let X be a one-dimensional continuum. If there exists a retraction from either 2^X or $C(X)$ onto X, then X is a uniformly arcwise connected dendroid.*

Definition 1. Let X be a continuum and $p, q \in X$. We say that X is a continuum of ***type*** N ***between*** p ***and*** q if there exist in X an arc $A = [p, q]$, two sequences of arcs $\{A_i\} = \{[p_i, p_i']\}$ and $\{B_i\} = \{[q_i, q_i']\}$ and points $p_i'' \in B_i \setminus \{q_i, q_i'\}$ and $q_i'' \in A_i \setminus \{p_i, p_i'\}$ (where $i \in \mathbb{N}$) such that

(1) $A = \mathrm{Lim}\, A_i = \mathrm{Lim}\, B_i$;
(2) $p = \mathrm{Lim}\, p_i = \mathrm{Lim}\, p_i' = \mathrm{Lim}\, p_i''$;
(3) $q = \mathrm{Lim}\, q_i = \mathrm{Lim}\, q_i' = \mathrm{Lim}\, q_i''$;
(4) each arc in X joining p_i and p_i' contains q_i'';
(5) each arc in X joining q_i and q_i' contains p_i''.

This definition was introduced by Oversteegen in [14]. It is proved in that paper that a type N-continuum is not contractible.

In this paper we prove that a continuum X of type N admits no retraction from $C(X)$ onto X, we give an example which proves that the converse to this result is not true and we prove that a continuum X of type N admits no mean. We also answer a question posed by T. J. Lee in [9].

Definition 2. Let A be a subcontinuum of a continuum X and let $B \subset A$. Assume that there are two sequences of subcontinua $\{A_n\}$ and $\{A_n'\}$ with $n \in \mathbb{N}$ satisfying the conditions:

(6) $A_n \cap A'_n \neq \emptyset$ for each $n \in \mathbb{N}$;
(7) $A = \mathrm{Lim}\, A_n = \mathrm{Lim}\, A'_n$;
(8) $B = \mathrm{Lim}(A_n \cap A'_n)$,

then we say that B is a ***bend set*** of A.

A continuum X is said to have the ***bend intersection property*** provided that for each subcontinuum A of X, the intersection of all its bend sets is nonempty.

A space X is said to be ***connected between its subsets*** A ***and*** B if there is no closed and open subset F of X such that $A \subset F$ and $F \cap B = \emptyset$ (see [8, §46, IV, p. 142]). A subset C of a space X ***separates*** X ***between*** A ***and*** B (or C is a ***separator*** **between** A **and** B) if $X \setminus C$ is not connected between A and B; it other words, if there are two sets M and N such that

$$X \setminus C = M \cup N, \quad (\mathrm{cl}(M) \cap N) \cup (M \cap \mathrm{cl}(N)) = \emptyset, \quad A \subset M, \quad B \subset N$$

(see [8, §46, VII, p. 154]).

The following results will be used in this paper.

Theorem 1.2. [8, §47, II, Theorem 3, p. 170]. *If a compact space X is connected between two closed sets A and B, then there exists a component C of X such that $C \cap A \neq \emptyset \neq C \cap B$.*

Theorem 1.3. [8, §47, II, Theorem 6, p. 171]. *If $\{C_n\}$ is a sequence of subcontinua of a compact metric space such that* $\mathrm{Lim\,inf}\, C_n \neq \emptyset$, *then* $\mathrm{Lim\,sup}\, C_n$ *is a continuum.*

Theorem 1.4. [8, §57, III, Theorem 2, p. 438]. *Let X be a locally connected, unicoherent continuum. Let F_0 and F_1 be closed sets in X, $p_j \in F_j$ for $j = 0, 1$, and $F_0 \cap F_1 = \emptyset$. Then there exists a separator C between p_0 and p_1 which is a locally connected continuum disjoint from $F_0 \cup F_1$.*

Theorem 1.5. [8, §61, II, Theorem 2, p. 511]*(θ-curve). If C is a θ-curve (in the plane $\mathbb{R}^2$ consisting of three arcs L_0, L_1 and L_2 having, pairwise, only their end points in common, then $X \setminus C = D_0 \cup D_1 \cup D_2$ and $\mathrm{bd}(D_j) = L_j \cup L_{j+1}$, where the disks D_0 and D_1 join with the set D_2 are the components of $X \setminus C$ (the subscripts being reduced* $\bmod 3$*).*

2. Retractions from $C(X)$ onto X and means in X

The Theorems 2.1 and 2.2 are the main results of this paper.

Theorem 2.1. *Let X be a continuum and assume that there exists a retraction $r : C(X) \to X$. Then X is not a type N-continuum.*

PROOF. Notice that X is arcwise connected, this follows by the fact that $C(X)$ is arcwise connected and r is a continuous mapping from $C(X)$ onto X. Let $w, w^* \in X$ and $T = [w, w^*]$ be any arc contained in X. We denote $wT = \{[w,t] : t \in T\}$. Therefore wT is an order arc in $C(X)$ whose end points are $\{w\}$ and T. Suppose that the continuum X is of type N. We will use the notation of Definition 1.

CLAIM. If C is the component of $r^{-1}(p) \cap C(A)$ which contains the point p, then $C \cap qA \neq \emptyset$.

To prove this claim suppose that

$$qA \cap C = \emptyset \tag{10}$$

and denote by C_i the component of the compact set $r^{-1}(p_i'') \cap C(B_i)$ which contains p_i''. Since each C_i is an element of the compact, metric space $C\big(C(A) \cup \{\bigcup_{i\in\mathbb{N}} \{C(B_i)\}\big)$, then the sequence $\{C_i\}$ contains a subsequence which converges to a continuum T contained in $r^{-1}(p) \cap C(A)$. Without loss of generality we assume that $\mathrm{Lim} C_i = T$. Moreover, since $p \in T$, T is a subcontinuum of C. Assume that for infinitely many $i \in \mathbb{N}$,

$$C_i \cap (q_i B_i \cup q_i' B_i) \neq \emptyset,$$

Then

$$T \cap qA \neq \emptyset.$$

But $T \subset C$, so $C \cap qA \neq \emptyset$ contrary to our assumption (10).

Hence except for a finite number of indices $k \in \mathbb{N}$, we have

$$C_k \cap (q_k B_k \cup q_k' B_k) = \emptyset. \tag{11}$$

Let $k \in \mathbb{N}$ be an index for which (11) is true. Let

$$L = q_k B_k \cup q_k' B_k,$$
$$F = r^{-1}(p_k'') \cap C(B_k)$$

and $Z = F \cup L$.

We will prove that Z is not connected between L and p_k''. Assume that Z is connected between L and p_k''. Then by Theorem 1.2 there exists a component K of Z such that $K \cap L \neq \emptyset$ and $p_k'' \in K$. Therefore we have two cases:

First case. K is not contained in F.

Then $K \cap F$ is a closed proper subset of the continuum K and $p_k'' \in K \cap F$. Therefore the component K' of $K \cap F$ which contains p_k'' meets the closure of the set $K \setminus F$ ([13, Theorem 5.6, p. 74])

and since

$$K \setminus F \subset Z \setminus F \subset L,$$

hence $K' \cap L \neq \emptyset$.

Second case. K is contained in F.

We take $K' = K$; in both cases, we obtain a continuum K' contained in F and containing the point p''_k such that $K' \cap L \neq \emptyset$. Then K' would be contained in C_k and this would imply that $C_k \cap L \neq \emptyset$, contrary to (11).

This proves that $C(B_k) \setminus Z$ separates $C(B_k)$ between L and p''_k. Since $C(B_k)$ is unicoherent and locally connected, then by Theorem 1.4, there exists a locally connected continuum $H \subset C(B_k) \setminus Z$ which also separates $C(B_k)$ between L and p''_k.

The point p''_k cuts the arc $[q_k, q'_k]$ into two subarcs M and M' and each one of these arcs connects p''_k with L. Then there exist points u, u' such that $u \in H \cap M$ and $u' \in H \cap M'$. Let N and N' be subarcs of M and M', whose end points are u and q_k, u' and q'_k respectively. The union $R = H \cup N \cup N'$ is a locally connected continuum. On the other hand

$$H \subset C(B_k) \setminus Z \subset C(B_k) \setminus F = C(B_k) \setminus [r^{-1}(p''_k) \cap C(B_k)] \subset C(X) \setminus r^{-1}(p''_k).$$

This implies that $H \cap r^{-1}(p''_k) = \emptyset$. Then $p''_k \notin r(H)$. Moreover $N \cup N' \subset [q_k, q'_k] \setminus \{p''_k\}$, therefore $r(N \cup N') \subset [q_k, q'_k] \setminus \{p''_k\}$ since r is a retraction. This implies that $r(R)$ does not contain p''_k. Now since $q_k, q'_k \in N \cup N' \subset R$, it follows that $q_k, q'_k \in r(R) \subset X$. But $r(R)$ is a locally connected continuum, so it is arcwise connected; in particular any arc in $r(R)$ containing q_k and q'_k must contain the point p''_k, by (5) in Definition 1. But this is impossible because $p''_k \notin r(R)$. This contradiction proves that $C \cap qA \neq \emptyset$.

Let D be the component of $r^{-1}(q) \cap C(A)$ containing q. Then

$$pA \cap D \neq \emptyset. \tag{12}$$

The proof of 12 is similar to the proof above interchanging q by p and replacing B_i, C, p, p''_i, q, q_i and q'_i by A_i, D, q, q''_i, p, p_i and p'_i respectively.

Since $p \neq q$, then

$$r^{-1}(p) \cap r^{-1}(q) = \emptyset. \tag{13}$$

Since $C(A)$ is a normal space there is an open set U such that $C \subset U \subset C(A)$ and $cl(U) \cap D = \emptyset$. Let V be the component of U containing C. Since $C(A)$ is locally connected, it follows from ([13, Exercise 5.22, p. 74]) that V is open set and by [13, Theorem 8.26], V is arcwise connected. Notice that $D \cap pA \setminus V \neq \emptyset$.

Let α be an arc contained in V from p to z, $z \in qA \cap C$. For $a, b \in \alpha$ such that a is between p and b in α, we denote by α_{ab} the subarc of α whose extreme points are a and b.

Since $pA \cap D \neq \emptyset$, there exist points p_0, q_0 and x such that $p_0 \in \alpha \cap pA$ (it may happen that $p_0 = p$), and either $q_0 \in \alpha \cap pA$ or $q_0 \in \alpha \cap qA$ (it may happen that $q_0 = z$) such that $\alpha_{p_0q_0} \cap (pA \cup qA) = \{p_0, q_0\}$ and either $x \in (pA)_{p_0q_0} \cap D$ or $x \in (pA)_{p_0A} \cap D$, where $(pA)_{p_0q_0}$ and $(pA)_{p_0A}$ denotes the subarcs of pA whose extreme points in $C(X)$ are p_0, q_0 and p_0, A respectively.

Let

$$\triangleright = pA \cup qA \cup A'. \tag{14}$$

where $A' = \{\{x\} | x \in A\}$. So we have the following cases:

(i) $q_0 \in \alpha \cap pA$ and $\alpha_{p_0q_0} \cap \triangleright = \{p_0, q_0\}$,
(ii) $q_0 \in \alpha \cap qA$ and $\alpha_{p_0q_0} \cap \triangleright = \{p_0, q_0\}$,
(iii) $q_0 \in (\alpha \cap qA) \cup (\alpha \cap pA)$ and $\alpha_{p_0q_0} \cap A' \neq \emptyset$. In this case we consider a point $r_0 \in \alpha_{p_0q_0}$ such that $\alpha_{r_0q_0} \cap A' = \{r_0\}$.

We notice that in the cases (i) and (ii) the set $\alpha_{p_0q_0} \cup \triangleright$ is a θ -curve in $C(A)$ and in the case (iii), $\alpha_{r_0q_0} \cup \triangleright$ is a θ -curve in $C(A)$. Therefore the set D satisfies either $D \subset C(A) \setminus \alpha_{p_0q_0}$ or $D \subset C(A) \setminus \alpha_{r_0q_0}$ and D intersects either both components of $C(A) \setminus \alpha_{p_0q_0}$ or D intersects both components of $C(A) \setminus \alpha_{r_0q_0}$ respectively. But this is a contradiction because D is a connected in $C(A)$. This contradiction proves the theorem. □

If in the last theorem we replace $C(X)$ by $F_2(X)$, all the steps of the proof remain true, because $F_2(A)$ is homeomorphic to $C(A)$, if A is an arc. Then we have the following theorem:

Theorem 2.2. *If a continuum X is of type N, then there is no retraction from $F_2(X)$ onto X.*

Since a retraction $r : F_2(X) \to X$ defines a mean $m(x, y) = r(\{x, y\})$, we obtain the following corollary.

Corollary 2.3. *A continuum of type N admits no mean.*

We know that for each continuum X, the existence of a retraction from 2^X onto X, implies the existence of a mean on X, see [2, Proposition 5.11 and 5.16, pp. 19, 20]. Therefore we obtain the following corollary

Corollary 2.4. *A continuum X of type N admits no retraction from 2^X onto X.*

Definition 3. [1, Definition 3.3, p. 41]. Let X be a compact, metric space with metric d. A continuous mapping f between two subspaces of X is said to be an ***ϵ-idy map*** if for all x, $d(x, f(x)) < \varepsilon$. A sequence of arcs $\{[a_n, b_n]\}$ is said to ***strongly converges*** to an arc $A = [a, b]$ if for each $\varepsilon > 0$, there exists n' such that for every $n \geq n'$ there is a ε-idy map $h : [a, b] \longrightarrow [a_n, b_n]$ such that $h(a) = a_n$ and $h(b) = b_n$. For later clarity we emphasize that order is relevant to this definition; i.e., $[a_n, b_n]$ strongly converges to $[a, b]$ is not the same as $[a_n, b_n]$ strongly converges to $[b, a]$.

Corollary 2.3 implies the following theorem which has been proved by Bell and Watson [1, Theorem 3.5, p. 42].

Theorem 2.5. *Let X be a compact, metric space with metric d. If X contains an arc $A = [a, b]$ and four sequences of arcs $\{[a_n, c_n]\}$, $\{[a_n, d_n]\}$, $\{[e_n, b_n]\}$ and $\{[f_n, b_n]\}$ with each of these sequences strongly converging to A and X is such that for every n, every subcontinuum containing c_n and d_n contains a_n and every subcontinuum containing e_n and f_n contains b_n. Then X does not admit a mean.*

PROOF. The hypothesis implies that X is a type N continuum. By corollary 2.3, X admits no mean. □

Now we consider the following definition.

Definition 4. [7, Definition 2.1, p. 99]. Let X be a continuum and A be an arc-like subcontinuum of X which has one end point a. A sequence $\{A_n\}$ of subcontinua of X is called a ***folding sequence*** with respect to the point a if it satisfies the following conditions: for each $n \in \mathbb{N}$ there are two subcontinua E_n and F_n of A_n such that

(i) $A_n = E_n \cup F_n$, and $\operatorname{Lim} E_n \cap F_n = \{a\}$,
(ii) $\operatorname{Lim} E_n = \operatorname{Lim} F_n = A$.

The following theorem has been proved by Kawamura and Tymchatyn in [7, Theorem 2.2, p. 99].

Theorem 2.6. *Let X be a hereditarily unicoherent continuum which has an arc-like subcontinuum X with the following properties:*

(iii) *A has a and b as its opposite end points, and*
(iv) *there exist folding sequences $\{A_n\}$ and $\{B_n\}$ with respect to a and b respectively.*

Then X admits no mean.

Corollary 2.3 is similar to this theorem, does not require the hypothesis of X to be hereditarily unicoherent. Instead it requires that the continuum A in the theorem 2.6 is actually an arc.

The following is an example of a dendroid X (actually a fan) which is not of type N and it is not uniformly arcwise connected. Therefore, by [2, Theorem 3.1, p. 9] there are no retractions from $C(X)$ onto X. This proves that the converse to Theorem 2.1 is not true.

Example 1.

Description of the fan X which will be contained in $\mathbb{R}^3$.
For $p, q \in \mathbb{R}^3$, we denote by pq the rectilinear segment joining p with q.
We consider
$\mathbb{L}_1 = \{(0, y, 0) : 0 \leq y \leq 1\} \subset \mathbb{R}^3$ and $w_n = (0, 1/2^n, 0) \in \mathbb{L}_1$, $n = 1, 2, 3, ...$
For each $z \in w_n w_{n+1}$ define $z^* \in w_n w_{n+1}$, such that $d(z, w_n) = d(z^*, w_{n+1})$.
$\mathbb{L}_2 = \{(0, 0, z) : 0 \leq z \leq 1\} \subset \mathbb{R}^3$, for each $n \in \mathbb{N}$, let
$p_j^n = (1/j, 0, 1/n)$ and $q_j^n = (1/j, 1, 1/n)$, $j \in \{n, n+1, n+2, n+3, ..., 2n-1\}$,
$p_n = (0, 0, 1/n)$. Notice that $\{p_j^n\}_{j=n}^{2n-1}$ is contained in the plane
$\mathbb{P}_n = \{(x, y, 1/n) : x, y \in \mathbb{R}^3\}$.

Define $L_n = p_n^n q_n^n \cup q_n^n p_{n+1}^n \cup p_{n+1}^n q_{n+1}^n \cup q_{n+1}^n p_{n+2}^n \cup ... \cup q_{2n-1}^n p_n$, so $L_n \subset \mathbb{P}_n$, $n = 1, 2, 3, ...$ Let $Y = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \{\bigcup_{n=1}^{\infty} L_n\}$.

The continuum X is a quotient space $Y/\sim$ obtained by defining the following equivalence relation for pairs of elements in Y. Let $z_1, z_2 \in Y$, $z_1 \sim z_2$ if and only if $z_1, z_2 \in \mathbb{L}_2$ or $z_1, z_2 \in \mathbb{L}_1$ and $z_1 = z_2^*$. The quotient $X = Y/\sim$ is homeomorphic to the space Z in figure 1.

Let F_ω be the dendrite which is a fan whose vertex a has infinite order. It is not difficult to see that $Z = F_\omega \cup \{\bigcup_{n=1}^{\infty} W_n\}$ where

(a) For each $n \in \mathbb{N}$, W_n is an arc and one of its extreme points is the vertex a of F_ω,
(b) $\operatorname{Lim} W_n = F_\omega$,
(c) $W_n \cap W_m = \{a\}$ if $n \neq m$,
(d) W_n "turns around F_ω", $2n$ times.

In order to prove that Z is not a type N-continuum, we use the notation in definition 1. We only consider arcs $A = [p, q]$ such that $A \subset F_\omega$ and we notice that if there exists a sequence $\{A_i = [p_i, p_i']\}$ of arcs which satisfy the conditions of definition 1, then there exists no sequence $\{B_i = [q_i, q_i']\}$ which satisfies the corresponding conditions of definition 1.

On the other hand, we will prove that Z is not uniformly arcwise connected. In the figure 1, arcs W_1, W_2 and W_3 are shown and they are distinguished by

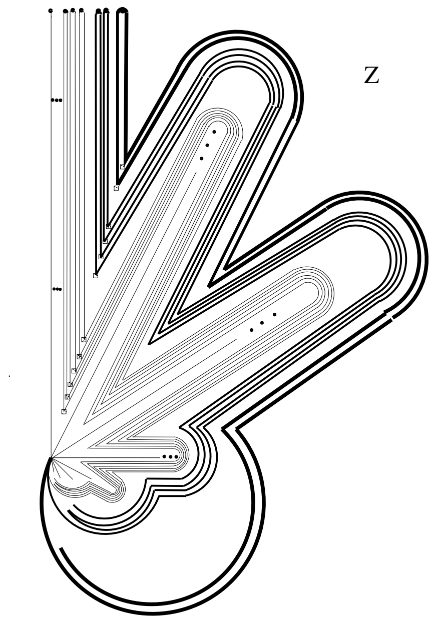

Figure 1

the thickness of the line. Denote by w_n^m the points in W_n, $m = 1, 2, 3, ..., n$ which are drawn with a point $\bullet$ in figure 1 and $\operatorname{Lim} w_n^m = b$ $(n \to \infty)$. Each w_n^m is the extreme point of two arcs $A_n^m = [w_n^m, u_n^m]$ and $B_n^m = [w_n^m, v_n^m]$, $(m = 1, 2, ..., n)$ where u_n^m and $v_n^m \in W_n$ are drawn with small square $\square$ in figure 1 and $\operatorname{Lim} u_n^m = a = \operatorname{Lim} v_n^m$ $(n \to \infty)$. The arcs A_n^m and B_n^m are both contained in W_n, $\operatorname{Lim} A_n^m = ab = \operatorname{Lim} B_n^m$ $(n \to \infty)$ and $\operatorname{Lim} diam A_n^m = 1 = \operatorname{Lim} diam B_n^m$ $(n \to \infty)$. If $\varepsilon > 0$ is small enough, we need at least four points in each arc A_n^m and B_n^m, $m = 1, ..., n$, so that the diameter of the subarc $[a_k, a_k + 1]$ is less than ε. Therefore we need at least $2n(4) = 8n$ points $\{a_1, a_2, ..., a_{8n}\}$ in the arc W_n, in order that each subarc $[a_k, a_k + 1] \subset W_n$ has diameter less than ε. This proves that Z is not uniformly arcwise connected.

A question by T. J. Lee. In [9] T. J. Lee gives an example of a dendroid which is not of type N and which does not have the bend intersection property. He also asks if there exists a fan Z which does not have the bend intersection property and which is not of type N. The following example proves that there exists such a fan.

Example 2.

Description of the fan W.

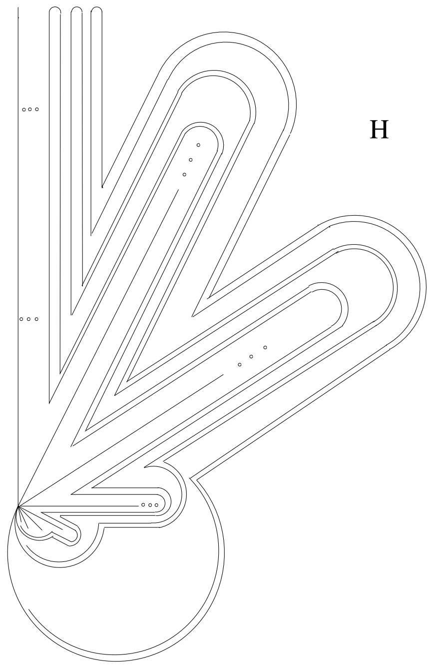

Figure 2

Let $a = (0,0,0)$, $b = (0,1,0)$, $a_i = (1/i, 1, 0)$, $b_i = (-1/i, 1, 0)$, $c_i = (-1/i, 0, 0)$ for $i = 1, 2, 3, ...$ Denote by S_i the semicircle
$S_i = \{(x, y, 0) : x^2 + (y-1)^2 = 1/i\}$ joining the points a_i and b_i for $i = 1, 2, 3, ...$
Then $K = ab \cup \{\bigcup_{i=1}^{\infty} (pa_i \cup S_i \cup b_i c_i)\}$ is a planar fan.

The continuum W is a quotient space $K/\sim$ obtained by defining the following equivalence relation for pairs of elements in K. Let $z_1, z_2 \in K$, $z_1 \sim z_2$ if and only if $z_1, z_2 \in \mathbb{L}_1$ and $z_1 = z_2^*$ or $z_1 = p$ and $z_2 = w_n$, $n = 1, 2, 3, ...$ where w_n was defined in example 1. The quotient $W = K/\sim$ is homeomorphic to the space H in figure 2.

In order to prove that W is not of type N continuum, we use the notation in definition 1. We only consider arcs $A = [p, q]$ such that $A \subset F_\omega$ and we notice that if there exists a sequence $\{A_i = [p_i, p'_i]\}$ of arcs which satisfy the conditions of definition 1, then there exists no sequence $\{B_i = [q_i, q'_i]\}$ which satisfies the corresponding conditions of definition 1.

On the other hand, the sets $\{a\}$, $\{b\}$ are bend sets of F_ω. Indeed, if $A_i = [a, b_i]$, $A'_i = [b_i, c_i]$, $A = F_\omega$ and $B = \{a\}$, then $\{A_i\}$, $\{A'_i\}$ and B satisfy (6), (7) and (8) in definition 2. On the other hand if $A_i = [c_i, a]$, $A'_i = [a, c_{i+1}]$, $A = F_\omega$ and $B = \{b\}$, then $\{A_i\}$, $\{A'_i\}$ and B satisfy (6), (7) and (8) in definition 2. Therefore the intersection of the bend sets of F_ω is empty. Therefore the fan W does not have the bend intersection property.

ACKNOWLEDGMENT

We wish to thank Isabel Puga for her valuable remarks during the preparation of this paper.

References

[1] M. Bell and S. Watson, *Not all dendroids have a mean*, Houston J. Math. **22** (1996), 39–50.

[2] J. J. Charatonik, W. J. Charatonik, K. Omiljanowski and J. R. Prajs, *Hyperspace retractions for curves*, Dissertationes Math. (Rozprawy Mat.) **370** (1997), 1–34.

[3] J. T. Goodykoontz, Jr., *Some retractions and deformation retractions on* 2^X *and* $C(X)$, Topology Appl. **21** (1985), 121–133.

[4] A. Illanes, *A continuum* X *which is a retract of* $C(X)$ *but not of* 2^X, Proc. Amer. Math. Soc. **100** (1987), 199–200.

[5] A. Illanes, *Two examples concerning hyperspace retraction*, Topology Appl. **29** (1988), 67–72.

[6] A. Illanes and S. B. Nadler, Jr., *Hyperspaces*, M. Dekker, New York and Basel, 1999.

[7] K. Kawamura and E. D. Tymchatyn, *Continua which admit no mean*, Colloq. Math. **71** (1996), 97–105.

[8] K. Kuratowski, *Topology*, vol. 2, Academic Press and PWN, New York, London and Warszawa, 1968.

[9] T. J. Lee, *Bend intersection property and dendroids of type* N, Period. Math. Hungar. **23** (1991), 121–127.

[10] T. Maćkowiak, *Continuous selections for* $C(X)$, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **26** (1978), 547–551.

[11] S. B. Nadler, Jr., *Inverse Limits and multicoherence*, Bull. Amer. Math. Soc. **76** (1970), 411–414.

[12] S. B. Nadler, Jr., *Hyperspaces of sets*, M. Dekker, New York and Basel, 1978.

[13] S. B. Nadler, Jr., *Continuum theory*, M. Dekker, New York, Basel and Hong Kong, 1992.

[14] L. G. Oversteegen, *Non-contractibility of continua*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **26** (1978), 837–840.

[15] L. E. Ward, Jr., *Rigid selections and smooth dendroids*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **19** (1971), 1041–1044.

[16] M. Wojdysławski, *Rétractes absolus et hyperespaces des continus*, Fund. Math. **32** (1939), 184–192.

Received April 23, 2004
Revised version received September 29, 2005

(F. Capulín) Departamento de Matemáticas, Facultad de Ciencias, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D. F., México

(W. J. Charatonik) Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409-0020, U. S. A.

E-mail address, FCP: fcapulin@correo.unam.mx

E-mail address, WJC: wjcharat@umr.edu