RETRACEMENTS FROM $C(\mathbb{X})$ ONTO $\mathbb{X}$
AND CONTINUA OF TYPE $N$

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Abstract. We show that if a metric continuum $\mathbb{X}$ is of type $N$, then there is no a retraction from the hyperspace of subcontinua $C(\mathbb{X})$ onto $\mathbb{X}$, and $\mathbb{X}$ admits no mean. We also give an example which answers a question posed by T. J. Lee related to this topic.

1. Introduction and preliminaries

The symbol $\mathbb{N}$ stands for the set of all positive integers. All considered spaces are assumed to be metric and all mappings are continuous. A continuum means a nonempty compact and connected space. A continuum is said to be hereditarily unicoherent provided that the intersection of any two of its subcontinua is connected. A dendroid is defined as an arcwise connected and hereditarily unicoherent continuum. A point $p$ of a dendroid $\mathbb{X}$ is called a ramification point of $\mathbb{X}$ if $p$ is vertex of a simple triod contained in $\mathbb{X}$. A fan means a dendroid that contains exactly one ramification point, which is called the vertex of the fan. If $\mathbb{X}$ is arcwise connected and $x, y \in \mathbb{X}$, we denote by $[x, y]$ any arc in $\mathbb{X}$ joining $x$ and $y$, if the arc is rectilinear we denote it by $xy$. We said that an arcwise connected continuum $\mathbb{X}$ is uniformly arcwise connected if given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that every arc $[a, b] \subseteq \mathbb{X}$ contains a set $\{a = a_1, a_2, ..., a_k = b\}$, $a_1 < a_2 < ... < a_k$ (the natural order in the arc) which satisfies $\text{diam}[a_i, a_{i+1}] < \varepsilon$, $i = 1, ..., k - 1$. Let $\mathbb{X}$ be a continuum, we denote by $2^X$ and $C(\mathbb{X})$ the hyperspaces of all nonempty closed subsets and of all subcontinua of $\mathbb{X}$, respectively, equipped with the Hausdorff metric. It is well known that these hyperspaces are arcwise connected continua, see for example [6, Theorem 14.9,

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Let $F_1(X) = \{ \{x\} : x \in X \} \subset C(X)$ and $F_2(X) = \{ \{x, y\} : x, y \in X \} \subset 2^X$. Since $F_1(X)$ is homeomorphic to $X$, we may assume that $X \subset C(X)$. Recall that for any arc $A$, the hyperspace $C(A)$ is homeomorphic to a disk, see [6, p. 33].

Let $A$ be a closed subset of $X$. A **retraction** is a mapping $r : X \to A$ such that $r|_A = \text{id}|_A$. A **mean** is a mapping $m : X \times X \to X$ such that

(a) $m((x, x)) = x$ for each $x \in X$,
(b) $m((x, y)) = m((y, x))$ for each $x, y \in X$.

The problem of characterizing continua which admit retractions $r : C(X) \to X$ or $r : 2^X \to X$ has been investigated in [11, p. 413] and in [12, Theorem 6.4, p. 270]. A. Illanes in [4] constructs an example of a continuum $X$ which is a retract of $C(X)$ but not of $2^X$. It is known that a one-dimensional continuum $X$ which is a retract of either $2^X$ or $C(X)$ is a dendroid, [3, p. 122]. Moreover, in [2, Theorems 3.1 and 3.3, p. 9 and 10] the following results are proved for one-dimensional continua.

**Theorem 1.1.** Let $X$ be a one-dimensional continuum. If there exists a retraction from either $2^X$ or $C(X)$ onto $X$, then $X$ is a uniformly arcwise connected dendroid.

**Definition 1.** Let $X$ be a continuum and $p, q \in X$. We say that $X$ is a continuum of **type $N$ between $p$ and $q$** if there exist in $X$ an arc $A = [p, q]$ and points $p''_i \in B_i \{ q_i, q_i' \}$ and $q''_i \in A_i \{ p_i, p_i' \}$ (where $i \in \mathbb{N}$) such that

1. $A = \text{Lim} A_i = \text{Lim} B_i$;
2. $p = \text{Lim} p_i = \text{Lim} p''_i$;
3. $q = \text{Lim} q_i = \text{Lim} q''_i$;
4. each arc in $X$ joining $p_i$ and $p''_i$ contains $q''_i$;
5. each arc in $X$ joining $q_i$ and $q''_i$ contains $p''_i$.

This definition was introduced by Oversteegen in [14]. It is proved in that paper that a type $N$-continuum is not contractible.

In this paper we prove that a continuum $X$ of type $N$ admits no retraction from $C(X)$ onto $X$, we give an example which proves that the converse to this result is not true and we prove that a continuum $X$ of type $N$ admits no mean. We also answer a question posed by T. J. Lee in [9].

**Definition 2.** Let $A$ be a subcontinuum of a continuum $X$ and let $B \subset A$. Assume that there are two sequences of subcontinua $\{A_n\}$ and $\{A'_n\}$ with $n \in \mathbb{N}$ satisfying the conditions:
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A continuum $X$ is said to have the bend intersection property provided that for each subcontinuum $A$ of $X$, the intersection of all its bend sets is nonempty.

A space $X$ is said to be connected between its subsets $A$ and $B$ if there is no closed and open subset $F$ of $X$ such that $A \subset F$ and $F \cap B = \emptyset$ (see [8, §46, IV, p. 142]). A subset $C$ of a space $X$ separates $X$ between $A$ and $B$ (or $C$ is a separator between $A$ and $B$) if $X \setminus C$ is not connected between $A$ and $B$; it other words, if there are two sets $M$ and $N$ such that

$$X \setminus C = M \cup N, \quad (\text{cl}(M) \cap N) \cup (M \cap \text{cl}(N)) = \emptyset, \quad A \subset M, \quad B \subset N$$

(see [8, §46, VII, p. 154]).

The following results will be used in this paper.

Theorem 1.2. [8, §47, II, Theorem 3, p. 170]. If a compact space $X$ is connected between two closed sets $A$ and $B$, then there exists a component $C$ of $X$ such that $C \cap A \neq \emptyset \neq C \cap B$.

Theorem 1.3. [8, §47, II, Theorem 6, p. 171]. If $\{C_n\}$ is a sequence of subcontinua of a compact metric space such that $\liminf C_n \neq \emptyset$, then $\limsup C_n$ is a continuum.

Theorem 1.4. [8, §57, III, Theorem 2, p. 438]. Let $X$ be a locally connected, unicoherent continuum. Let $F_0$ and $F_1$ be closed sets in $X$, $p_j \in F_j$ for $j = 0, 1$, and $F_0 \cap F_1 = \emptyset$. Then there exists a separator $C$ between $p_0$ and $p_1$ which is a locally connected continuum disjoint from $F_0 \cup F_1$.

Theorem 1.5. [8, §61, II, Theorem 2, p. 511](θ-curve). If $C$ is a θ-curve (in the plane $\mathbb{R}^2$ consisting of three arcs $L_0$, $L_1$ and $L_2$ having, pairwise, only their end points in common, then $X \setminus C = D_0 \cup D_1 \cup D_2$ and $\text{bd}(D_j) = L_j \cup L_{j+1}$, where the disks $D_0$ and $D_1$ join with the set $D_2$ are the components of $X \setminus C$ (the subscripts being reduced mod 3).

2. Retractions from $C(X)$ onto $X$ and means in $X$

The Theorems 2.1 and 2.2 are the main results of this paper.

Theorem 2.1. Let $X$ be a continuum and assume that there exists a retraction $r : C(X) \to X$. Then $X$ is not a type $N$-continuum.
Proof. Notice that $X$ is arcwise connected, this follows by the fact that $C(X)$ is arcwise connected and $r$ is a continuous mapping from $C(X)$ onto $X$. Let $w, w^* \in X$ and $T = [w, w^*]$ be any arc contained in $X$. We denote $wT = \{[w, t] : t \in T\}$. Therefore $wT$ is an order arc in $C(X)$ whose end points are $\{w\}$ and $T$. Suppose that the continuum $X$ is of type $N$. We will use the notation of Definition 1.

Claim. If $C$ is the component of $r^{-1}(p) \cap C(A)$ which contains the point $p$, then $C \cap qA \neq \emptyset$.

To prove this claim suppose that

\begin{equation}
qA \cap C = \emptyset
\end{equation}

and denote by $C_i$ the component of the compact set $r^{-1}(p_i^\alpha) \cap C(B_i)$ which contains $p_i^\alpha$. Since each $C_i$ is an element of the compact, metric space $C(C(A)) \cup \left\{ \bigcup_{i \in \mathbb{N}} \{C(B_i)\} \right\}$, then the sequence $\{C_i\}$ contains a subsequence which converges to a continuum $T$ contained in $r^{-1}(p) \cap C(A)$. Without loss of generality we assume that $\text{Lim}C_i = T$. Moreover, since $p \in T$, $T$ is a subcontinuum of $C$. Assume that for infinitely many $i \in \mathbb{N}$,

\[ C_i \cap (q_iB_i \cup q_i'B_i) \neq \emptyset. \]

Then

\[ T \cap qA \neq \emptyset. \]

But $T \subset C$, so $C \cap qA \neq \emptyset$ contrary to our assumption (10).

Hence except for a finite number of indices $k \in \mathbb{N}$, we have

\begin{equation}
C_k \cap (q_kB_k \cup q_k'B_k) = \emptyset.
\end{equation}

Let $k \in \mathbb{N}$ be an index for which (11) is true. Let

\[ L = q_kB_k \cup q_k'B_k, \]

\[ F = r^{-1}(p_k^\alpha) \cap C(B_k) \]

and $Z = F \cup L$.

We will prove that $Z$ is not connected between $L$ and $p_k^\alpha$. Assume that $Z$ is connected between $L$ and $p_k^\alpha$. Then by Theorem 1.2 there exists a component $K$ of $Z$ such that $K \cap L \neq \emptyset$ and $p_k^\alpha \in K$. Therefore we have two cases:

First case. $K$ is not contained in $F$.

Then $K \cap F$ is a closed proper subset of the continuum $K$ and $p_k^\alpha \in K \cap F$. Therefore the component $K'$ of $K \cap F$ which contains $p_k^\alpha$ meets the closure of the set $K \setminus F$ ([13, Theorem 5.6, p. 74]) and since

\[ K \setminus F \subset Z \setminus F \subset L, \]
hence $K' \cap L \neq \emptyset$.

Second case. $K$ is contained in $F$.

We take $K' = K$; in both cases, we obtain a continuum $K'$ contained in $F$ and containing the point $p_k''$ such that $K' \cap L \neq \emptyset$. Then $K'$ would be contained in $C_k$ and this would imply that $C_k \cap L \neq \emptyset$, contrary to (11).

This proves that $C(B_k) \setminus Z$ separates $C(B_k)$ between $L$ and $p_k''$. Since $C(B_k)$ is unicoherent and locally connected, then by Theorem 1.4, there exists a locally connected continuum $H \subset C(B_k) \setminus Z$ which also separates $C(B_k)$ between $L$ and $p_k''$.

The point $p_k''$ cuts the arc $[q_k, q_k']$ into two subarcs $M$ and $M'$ and each one of these arcs connects $p_k'$ with $L$. Then there exist points $u, u'$ such that $u \in H \cap M$ and $u' \in H \cap M'$. Let $N$ and $N'$ be subarcs of $M$ and $M'$, whose end points are $u$ and $q_k, u'$ and $q_k'$ respectively. The union $R = H \cup N \cup N'$ is a locally connected continuum. On the other hand

$$H \subset C(B_k) \setminus Z \subset C(B_k) \setminus F = C(B_k) \setminus \{r^{-1}(p_k'' \cap C(B_k)) \subset C(X) \setminus r^{-1}(p_k'').$$

This implies that $H \cap r^{-1}(p_k'') = \emptyset$. Then $p_k'' \notin r(H)$. Moreover $N \cup N' \subset \{q_k, q_k'\} \setminus \{p_k''\}$, therefore $r(N \cup N') \subset \{q_k, q_k'\} \setminus \{p_k''\}$ since $r$ is a retraction. This implies that $r(R)$ does not contain $p_k''$. Now since $q_k, q_k' \in N \cup N' \subset R$, it follows that $q_k, q_k' \in r(R) \subset X$. But $r(R)$ is a locally connected continuum, so it is arcwise connected; in particular any arc in $r(R)$ containing $q_k$ and $q_k'$ must contain the point $p_k''$, by (5) in Definition 1. But this is impossible because $p_k'' \notin r(R)$. This contradiction proves that $C \cap qA \neq \emptyset$.

Let $D$ be the component of $r^{-1}(q) \cap C(A)$ containing $q$. Then

$$pA \cap D \neq \emptyset.$$  

(12)

The proof of 12 is similar to the proof above interchanging $q$ by $p$ and replacing $B_i, C, p, p_i'', q_i$ and $q_i'$ by $A_i, D, q, q_i''$ and $p_i$ respectively.

Since $p \neq q$, then

$$r^{-1}(p) \cap r^{-1}(q) = \emptyset.$$  

(13)

Since $C(A)$ is a normal space there is an open set $U$ such that $C \subset U \subset C(A)$ and $cl(U) \cap D = \emptyset$. Let $V$ be the component of $U$ containing $C$. Since $C(A)$ is locally connected, it follows from ([13, Exercise 5.22, p. 74]) that $V$ is open set and by [13, Theorem 8.26], $V$ is arcwise connected. Notice that $D \cap pA \setminus V \neq \emptyset$.

Let $\alpha$ be an arc contained in $V$ from $p$ to $z$, $z \in qA \cap C$. For $a, b \in \alpha$ such that $a$ is between $p$ and $b$ in $\alpha$, we denote by $\alpha_{ab}$ the subarc of $\alpha$ whose extreme points are $a$ and $b$. 
Since \( pA \cap D \neq \emptyset \), there exist points \( p_0, q_0 \) and \( x \) such that \( p_0 \in \alpha \cap pA \) (it may happen that \( p_0 = p \)), and either \( q_0 \in \alpha \cap pA \) or \( q_0 \in \alpha \cap qA \) (it may happen that \( q_0 = z \)) such that \( \alpha_{p_0q_0} \cap (pA \cup qA) = \{p_0, q_0\} \) and either \( x \in (pA)_{p_0q_0} \cap D \) or \( x \in (pA)_{p_0,q_0} \cap D \), where \( (pA)_{p_0q_0} \) and \( (pA)_{p_0,q_0} \) denotes the subarcs of \( pA \) whose extreme points in \( C(X) \) are \( p_0, q_0 \) and \( p_0, A \) respectively.

Let
\[
\triangleright = pA \cup qA \cup A'.
\]
where \( A' = \{x | x \in A\} \). So we have the following cases:

(i) \( q_0 \in \alpha \cap pA \) and \( \alpha_{p_0q_0} \cap \triangleright = \{p_0, q_0\} \),
(ii) \( q_0 \in \alpha \cap qA \) and \( \alpha_{p_0q_0} \cap \triangleright = \{p_0, q_0\} \),
(iii) \( q_0 \in (\alpha \cap qA) \cup (\alpha \cap pA) \) and \( \alpha_{p_0q_0} \cap A' \neq \emptyset \). In this case we consider a point \( r_0 \in \alpha_{p_0q_0} \) such that \( \alpha_{r_0q_0} \cap A' = \{r_0\} \).

We notice that in the cases (i) and (ii) the set \( \alpha_{p_0q_0} \cup \triangleright \) is a \( \theta \)-curve in \( C(A) \) and in the case (iii), \( \alpha_{r_0q_0} \cup \triangleright \) is a \( \theta \)-curve in \( C(A) \). Therefore the set \( D \) satisfies either \( D \subset C(A) \setminus \alpha_{p_0q_0} \) or \( D \subset C(A) \setminus \alpha_{r_0q_0} \) and \( D \) intersects either both components of \( C(A) \setminus \alpha_{p_0q_0} \) or \( D \) intersects both components of \( C(A) \setminus \alpha_{r_0q_0} \) respectively. But this is a contradiction because \( D \) is a connected in \( C(A) \). This contradiction proves the theorem. \( \square \)

If in the last theorem we replace \( C(X) \) by \( F_2(X) \), all the steps of the proof remain true, because \( F_2(A) \) is homeomorphic to \( C(A) \), if \( A \) is an arc. Then we have the following theorem:

**Theorem 2.2.** If a continuum \( X \) is of type \( N \), then there is no retraction from \( F_2(X) \) onto \( X \).

Since a retraction \( r: F_2(X) \to X \) defines a mean \( m(x, y) = r(\{x, y\}) \), we obtain the following corollary.

**Corollary 2.3.** A continuum of type \( N \) admits no mean.

We know that for each continuum \( X \), the existence of a retraction from \( 2^X \) onto \( X \), implies the existence of a mean on \( X \), see [2, Proposition 5.11 and 5.16, pp. 19, 20]. Therefore we obtain the following corollary

**Corollary 2.4.** A continuum \( X \) of type \( N \) admits no retraction from \( 2^X \) onto \( X \).

**Definition 3.** [1, Definition 3.3, p. 41]. Let \( X \) be a compact, metric space with metric \( d \). A continuous mapping \( f \) between two subspaces of \( X \) is said to be
an \( \varepsilon \)-idy map if for all \( x \), \( d(x, f(x)) < \varepsilon \). A sequence of arcs \( \{[a_n, b_n]\} \) is said to strongly converge to an arc \( A = [a, b] \) if for each \( \varepsilon > 0 \), there exists \( n' \) such that for every \( n \geq n' \) there is a \( \varepsilon \)-idy map \( h : [a, b] \to [a_n, b_n] \) such that \( h(a) = a_n \) and \( h(b) = b_n \). For later clarity we emphasize that order is relevant to this definition; i.e., \([a_n, b_n]\) strongly converges to \([a, b]\) is not the same as \([a_n, b_n]\) strongly converges to \([b, a]\).

Corollary 2.3 implies the following theorem which has been proved by Bell and Watson [1, Theorem 3.5, p. 42].

**Theorem 2.5.** Let \( X \) be a compact, metric space with metric \( d \). If \( X \) contains an arc \( A = [a, b] \) and four sequences of arcs \( \{[a_n, c_n]\}, \{[a_n, d_n]\}, \{[e_n, b_n]\} \) and \( \{[f_n, b_n]\} \) with each of these sequences strongly converging to \( A \) and \( X \) is such that for every \( n \), every subcontinuum containing \( c_n \) and \( d_n \) contains \( a_n \) and every subcontinuum containing \( e_n \) and \( f_n \) contains \( b_n \). Then \( X \) does not admit a mean.

**Proof.** The hypothesis implies that \( X \) is a type \( N \) continuum. By corollary 2.3, \( X \) admits no mean. \( \square \)

Now we consider the following definition.

**Definition 4.** [7, Definition 2.1, p. 99]. Let \( X \) be a continuum and \( A \) be an arc-like subcontinuum of \( X \) which has one end point \( a \). A sequence \( \{A_n\} \) of subcontinua of \( X \) is called a folding sequence with respect to the point \( a \) if it satisfies the following conditions: for each \( n \in \mathbb{N} \) there are two subcontinua \( E_n \) and \( F_n \) of \( A_n \) such that

(i) \( A_n = E_n \cup F_n \), and \( \lim E_n \cap F_n = \{a\} \),

(ii) \( \lim E_n = \lim F_n = A \).

The following theorem has been proved by Kawanura and Tymchatyn in [7, Theorem 2.2, p. 99].

**Theorem 2.6.** Let \( X \) be a hereditarily unicoherent continuum which has an arc-like subcontinuum \( X \) with the following properties:

(iii) \( A \) has \( a \) and \( b \) as its opposite end points, and

(iv) there exist folding sequences \( \{A_n\} \) and \( \{B_n\} \) with respect to \( a \) and \( b \) respectively.

Then \( X \) admits no mean.

Corollary 2.3 is similar to this theorem, does not require the hypothesis of \( X \) to be hereditarily unicoherent. Instead it requires that the continuum \( A \) in the theorem 2.6 is actually an arc.
The following is an example of a dendroid $X$ (actually a fan) which is not of type $N$ and it is not uniformly arcwise connected. Therefore, by [2, Theorem 3.1, p. 9] there are no retractions from $C(X)$ onto $X$. This proves that the converse to Theorem 2.1 is not true.

**Example 1.**

Description of the fan $X$ which will be contained in $\mathbb{R}^3$.

For $p, q \in \mathbb{R}^3$, we denote by $pq$ the rectilinear segment joining $p$ with $q$. We consider

$\mathbb{L}_1 = \{(0, y, 0) : 0 \leq y \leq 1\} \subset \mathbb{R}^3$ and $w_n = (0, 1/2^n, 0) \in \mathbb{L}_1$, $n = 1, 2, 3, ...$

For each $z \in w_n w_{n+1}$ define $z^* \in w_n w_{n+1}$, such that $d(z, w_n) = d(z^*, w_{n+1})$.

$\mathbb{L}_2 = \{(0, z, 0) : 0 \leq z \leq 1\} \subset \mathbb{R}^3$, for each $n \in \mathbb{N}$, let

$p_j^p = (1/j, 0, 1/n)$ and $q_j^q = (1/j, 1, 1/n)$, $j \in \{n, n + 1, n + 2, n + 3, ..., 2n - 1\}$,

$p_n = (0, 0, 1/n)$. Notice that $\{p_j^p\}_{j=n}^{2n-1}$ is contained in the plane $\mathbb{P}_n = \{(x, y, 1/n) : x, y \in \mathbb{R}^3\}$.

Define $L_n = p_n p_n^p \cup q_n q_n^q \cup p_{n+1} p_{n+1}^p \cup q_{n+1} q_{n+1}^q \cup \bigcup_{n=1}^{\infty} p_{2n-1} p_n$, so $L_n \subset \mathbb{P}_n$, $n = 1, 2, 3, ...$ Let $Y = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \bigcup_{n=1}^{\infty} L_n$.

The continuum $X$ is a quotient space $Y/\sim$ obtained by defining the following equivalence relation for pairs of elements in $Y$. Let $z_1, z_2 \in Y$, $z_1 \sim z_2$ if and only if $z_1, z_2 \in \mathbb{L}_2$ or $z_1, z_2 \in \mathbb{L}_1$ and $z_1 = z_2^*$. The quotient $X = Y/\sim$ is homeomorphic to the space $Z$ in figure 1.

Let $F_\omega$ be the dendrite which is a fan whose vertex $a$ has infinite order. It is not difficult to see that $Z = F_\omega \cup \bigcup_{n=1}^{\infty} W_n$ where

(a) For each $n \in \mathbb{N}$, $W_n$ is an arc and one of its extreme points is the vertex $a$ of $F_\omega$,

(b) $\text{Lim } W_n = F_\omega$,

(c) $W_n \cap W_m = \{a\}$ if $n \neq m$,

(d) $W_n$ ”turns around $F_\omega$,” $2n$ times.

In order to prove that $Z$ is not a type $N$-continuum, we use the notation in definition 1. We only consider arcs $A = [p, q]$ such that $A \subset F_\omega$ and we notice that if there exists a sequence $\{A_i = [p_i, q_i]\}$ of arcs which satisfy the conditions of definition 1, then there exists no sequence $\{B_i = [q_i, q_i']\}$ which satisfies the corresponding conditions of definition $1$.

On the other hand, we will prove that $Z$ is not uniformly arcwise connected. In the figure $1$, arcs $W_1, W_2$ and $W_3$ are shown and they are distinguished by
the thickness of the line. Denote by \( w^m_n \) the points in \( W_n \), \( m = 1, 2, 3, ..., n \) which are drawn with a point • in figure 1 and \( \text{Lim} w^m_n = b (n \to \infty) \). Each \( w^m_n \) is the extreme point of two arcs \( A^m_n = [w^m_n, u^m_n] \) and \( B^m_n = [w^m_n, v^m_n] \), \( (m = 1, 2, ..., n) \) where \( u^m_n \) and \( v^m_n \) \( \in W_n \) are drawn with small square □ in figure 1 and \( \text{Lim} u^m_n = a = \text{Lim} v^m_n (n \to \infty) \). The arcs \( A^m_n \) and \( B^m_n \) are both contained in \( W_n \), \( \text{Lim} A^m_n = ab = \text{Lim} B^m_n (n \to \infty) \) and \( \text{Lim} \text{diam} A^m_n = 1 = \text{Lim} \text{diam} B^m_n (n \to \infty) \). If \( \varepsilon > 0 \) is small enough, we need at least four points in each arc \( A^m_n \) and \( B^m_n \), \( m = 1, ..., n \), so that the diameter of the subarc \([a_k, a_k + 1]\) is less than \( \varepsilon \). Therefore we need at least \( 2n(4) = 8n \) points \( \{a_1, a_2, ..., a_{8n}\} \) in the arc \( W_n \), in order that each subarc \([a_k, a_k + 1] \subset W_n \) has diameter less than \( \varepsilon \). This proves that \( Z \) is not uniformly arcwise connected.

**A question by T. J. Lee.** In [9] T. J. Lee gives an example of a dendroid which is not of type \( N \) and which does not have the bend intersection property. He also asks if there exists a fan \( Z \) which does not have the bend intersection property and which is not of type \( N \). The following example proves that there exists such a fan.
Example 2.

Description of the fan $W$.

Let $a = (0,0,0), b = (0,1,0), a_i = (1/i,1,0), b_i = (-1/i,1,0), c_i = (-1/i,0,0)$ for $i = 1, 2, 3, ...$. Denote by $S_i$ the semicircle

$S_i = \{(x,y,0) : x^2 + (y-1)^2 = 1/i\}$ joining the points $a_i$ and $b_i$ for $i = 1, 2, 3, ...$.

Then $K = ab \cup \left( \bigcup_{i=1}^{\infty} (pa_i \cup S_i \cup b_i c_i) \right)$ is a planar fan.

The continuum $W$ is a quotient space $K/\sim$ obtained by defining the following equivalence relation for pairs of elements in $K$. Let $z_1, z_2 \in K, z_1 \sim z_2$ if and only if $z_1, z_2 \in L_1$ and $z_1 = z_2^2$ or $z_1 = p$ and $z_2 = w_n, n = 1, 2, 3, ...$ where $w_n$ was defined in example 1. The quotient $W = K/\sim$ is homeomorphic to the space $H$ in figure 2.

In order to prove that $W$ is not of type $N$ continuum, we use the notation in definition 1. We only consider arcs $A = [p,q]$ such that $A \subset F_\omega$ and we notice
that if there exists a sequence \( \{ A_i = [p_i, p'_i] \} \) of arcs which satisfy the conditions of definition 1, then there exists no sequence \( \{ B_i = [q_i, q'_i] \} \) which satisfies the corresponding conditions of definition 1.

On the other hand, the sets \( \{ a \}, \{ b \} \) are bend sets of \( F_\omega \). Indeed, if \( A_i = [a, b], A'_i = [b, c_i], A = F_\omega \) and \( B = \{ a \} \), then \( \{ A_i \}, \{ A'_i \} \) and \( B \) satisfy (6), (7) and (8) in definition 2. On the other hand if \( A_i = [c_i, a], A'_i = [a, c_i+1], A = F_\omega \) and \( B = \{ b \} \), then \( \{ A_i \}, \{ A'_i \} \) and \( B \) satisfy (6), (7) and (8) in definition 2. Therefore the intersection of the bend sets of \( F_\omega \) is empty. Therefore the fan \( W \) does not have the bend intersection property.

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