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EVERY MONOTONE OPEN 2-HOMOGENEOUS METRIC CONTINUUM IS LOCALLY CONNECTED

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It is observed that all results of Whittington's note [17] can be extended in two directions. The assumption of homogeneity of the considered spaces can be weakened to monotone open homogeneity, and homeomorphisms used in the results can be replaced by monotone open and closed mappings. In particular, the statement in the title is obtained.

1. Introduction.

In [16, Theorem 3.12, p. 397], Ungar answered a question of Burgess [2] by showing that every 2-homogeneous metric continuum is locally connected. A short, elementary and elegant proof of this result has been given by Whittington [17] who has omitted a powerful result of Effros (viz. Theorem 2.1 of [4, p. 39]) used by Ungar. It is observed in this note that Whittington's proof can be applied to obtain the same conclusion for monotone open 2-homogeneous metric continua.

Let a class $\mathfrak{M}$ of mappings between topological spaces be given which has the composition property, that is, for every two mappings in $\mathfrak{M}$ their composition also is in $\mathfrak{M}$. A topological space X is said to be $\mathfrak{M}$ *homogeneous* provided that for every two points $a, b \in X$ there is a

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surjective mapping $f : X \to X$ such that $f(a) = b$ and $f \in \mathfrak{M}$. If $\mathfrak{M}$ means the class of homeomorphisms, we get the well known concept of a homogeneous space. Let n be a positive integer. A space X is said to be $\mathfrak{M}$ *n-homogeneous* provided that for every two subsets A and B of X of cardinality n each there is a mapping $f : X \to X$ such that $f(A) = B$ and $f \in \mathfrak{M}$. If a and b are distinct points of an $\mathfrak{M}$ 2-homogeneous space X, c is a third point, and $f : X \to X$ is a mapping in $\mathfrak{M}$ carrying $\{a, c\}$ onto $\{b, c\}$, then either f or f^2 maps a to b and is in $\mathfrak{M}$; thus each $\mathfrak{M}$ 2-homogeneous space is $\mathfrak{M}$ homogeneous.

A continuous mapping between topological spaces is said to be *open* provided that it maps open subsets of the domain on open subsets of the range; it is said to be *monotone* if it has compact and connected point inverses (see e.g. [18, p. 127]). Note that if a mapping between continua is considered, then compactness of the point inverses is always satisfied, and thus the definition coincides with the one given in [5, p. 358].

We say that a space X is *locally connected at a point* $p \in X$ provided that X has a local basis at p composed of connected open subsets of X. If each point of X has this property, then X is said to be *locally connected.* The following fact will be used in the sequel.

Fact 1. Local connectedness of a space at a point is an invariant property under open mappings.

Proof. Indeed, if $\mathcal{B}$ is a local basis of connected open sets at a point p in space X, and if a mapping $f : X \to Y = f(X)$ is open, then $\{f(B) : B \in \mathcal{B}\}$ is a local basis of connected open sets at $f(p)$ in Y. □

The lemma below generalizes Lemma 2 of [17, p. 3131]. Its proof is a modification of that from [17].

LEMMA 2. *If X is an open homogeneous complete metric space that is not locally connected, then X has a base of open sets, each component of which is nowhere dense in X.*

Proof. Since X is open homogeneous, it suffices to show that there is a nonempty open subset V of X, each component of which is nowhere dense in X. So, suppose on the contrary that there is no such subset. Let U_1 be some particular nonempty open set of diameter less than 1. Then there is a component C_1 of U_1 such that $\operatorname{int}\operatorname{cl}C_1 \neq \emptyset$. There is a nonempty open set U_2 of diameter less than 1/2 such that $\operatorname{cl}U_2 \subset \operatorname{int}\operatorname{cl}C_1$. Again, U_2 must have a component C_2 such that $\operatorname{int}\operatorname{cl}C_2 \neq \emptyset$. Continuing in this manner, one finds connected sets $\operatorname{cl}C_n$ for $n \in \mathbb{N}$ that form a neighborhood base at some point p in X (the Cantor theorem for complete metric spaces is used here). Since X is open homogeneous, it follows from Fact 1 that X is locally connected, contrary to the assumption. □

The next two theorems are generalizations of [17, Theorems 1 and 2, p. 3132]; namely homogeneity in that theorems is replaced by open homogeneity, and monotone open mappings suffice for homeomorphisms used there. To show the former one we need a lemma from [17], which is formulated below, for sake of completeness only.

LEMMA 3. [17, Lemma 1, p. 3131]. *If X is a space with a countable base such that for every x, $y \in X$ there is a compact connected set C containing x and y, and an open set U containing C such that the component of U containing C is nowhere dense in X, then X is of the first category.*

THEOREM 4. *If X is an open homogeneous Polish (i.e., separable, complete metric) space such that for each pair of points x, $y \in X$ there is a point $p \in X$ such that for every $\varepsilon > 0$ there is a monotone open and closed mapping $f : X \to X$ and a continuum D contained in the ε-neighborhood of p such that $f(x), f(y) \in D$, then X is locally connected.*

Proof. Since the space X is separable metric, it has a countable base. Suppose, on the contrary, that X is not locally connected. Then, according to Lemma 2, X has a base of open sets, each component of which is nowhere dense in X. We will show that the remaining assumptions of Lemma 3 are fulfilled, contradicting that X is of the second category.

Let x, $y \in X$, and let p be as above. Then there is an $\varepsilon > 0$ such that each component of $B(p, \varepsilon)$, the ε-ball centered at p, is nowhere dense in X. By the assumption, there is a monotone open and closed mapping $f : X \to X$ and a continuum D contained in $B(p, \varepsilon)$ such that $f(x), f(y) \in D$. Since f is closed and all point-inverses are compact, it is a perfect mapping (see [5, 3.7, p.182]), whence it follows by [5, Theorem

3.7.2, p. 182] that $C = f^{-1}(D)$ is compact. Further, since f is monotone and closed, C is a connected by [5, Theorem 6.1.29, p. 358]. Thus C is a continuum. Obviuosly $x, y \in C$. Further, $U = f^{-1}(B(p, \varepsilon))$ is an open set containing C. Since f is open, the component of U which contains C is nowhere dense in X. Thus Lemma 3 can be applied, whence it follows that X is of the first category, contrary to the Baire theorem. □

THEOREM 5. *If X is a compact metric open homogeneous space such that each pair of points can be mapped by monotone open mappings from X to X into connected sets of arbitrarily small diameter, then X is locally connected.*

Proof. Again, as in the above proof, we will apply Lemma 3. Let x, $y \in X$, and suppose on the contrary that X is not locally connected. By Lemma 2, using compactness of X we infer that X can be covered with finitely many open sets, each component of which is nowhere dense in X. Utilizing a Lebesgue number for such a covering, it follows from the assumptions that there is a monotone open mapping $f : X \to X$ such that the points $f(x)$ and $f(y)$ lie in a continuum D contained in an open set V which is an element of the covering. As in the previous proof, the sets $C = f^{-1}(D)$ and $U = f^{-1}(V)$ fulfill the requirements of Lemma 3. Then a contradiction is obtained as previously. □

Since nondegenerate metric continua always contain nondegenerate connected subsets of arbitrarily small diameter, and since a monotone open 2-homogeneous continuum is monotone open homogeneous, Theorem 5 implies the following corollary.

COROLLARY 6. *Every monotone open 2-homogeneous metric continuum is locally connected.*

COROLLARY 7. (Ungar). *Every 2-homogeneous metric continuum is locally connected.*

Remark 8. The concepts of monotone open homogeneous and of homogeneous continua are different, even for plane locally connected curves. This can be seen from the example of the Sierpiński universal plane curve which is known to be monotone open homogeneous (see [12, Corollary 24, p. 38] and [15, Theorem 15, p. 111]) but it is not homogeneous, because the only plane locally connected homogeneous continuum is the simple closed curve.

Remark 9. We cannot replace monotone open mappings in Theorem 5 by open ones because of the following example. Let X be the Cantor ternary set. Then X is homogeneous, is not locally connected, and any two of its points can be mapped under an open mapping f from X onto X to a one-point set. Indeed, take $x_1, x_2 \in X$. Let $h : X \to X \times X$ be a homeomorphism such that the first coordinates of $h(x_1)$ and $h(x_2)$ are equal, and $\pi : X \times X \to X$ be the projection onto the first factor. Define $f : X \to X$ by $f = \pi \circ h$. Then $f(x_1) = f(x_2)$ and f is open.

Remark 10. A. Illanes has constructed in [6] an example of a metric monotone homogeneous non-locally connected continuum X every two points of which can be mapped to a one-point set under a monotone surjection from X onto itself. The example shows that in Theorem 5 we cannot replace open homogeneity by monotone homogeneity and (simultaneously) monotone open mappings by monotone ones.

The following question is of some interest.

Question 11. Is every monotone 2-homogeneous metric continuum locally connected?

Recall that a continuum X is said to be *irreducible* (between a and b) provided that there are two points a and b in X such that each subcontinuum of X that contains these points equals X. Note that each indecomposable continuum is irreducible (between any two points that belong to distinct composants of the continuum, [7, § 48, VI, Theorem 7', p. 213]). In connection with Question 11 we have the following assertion and a corollary.

Assertion 12. If a continuum is monotone 2-homogeneous, then it is not irreducible.

Proof. Let a continuum X be monotone 2-homogeneous, and let a and b be two points of X. Choose c and d in X such that there is a proper subcontinuum C of X that contains both c and d. If $f : X \to X$ is a surjective monotone mapping such that $f(\{a, b\}) = \{c, d\}$, then $f^{-1}(C)$ is a proper subcontinuum of X that contains a and b. Thus X is not ireducible. □

COROLLARY 13. *If a continuum is monotone 2-homogeneous, then it is decomposable.*

A mapping $f : X \to Y$ between continua is said to be *confluent* provided that for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ the equality $f(C) = Q$ holds. It is known that confluent mappings are common generalization of monotone and of open ones, [18, Theorem 7.5, p. 148].

Corollaires 6 and 7, Assertion 12 and Corollary 13 cannot be extended to confluent 2-homogeneity because of the following known example.

EXAMPLE 14. *For each $n \in \mathbb{N}$ the pseudo-arc is confluent n-homogeneous.*

Proof. Combine Corollary 10 of [11, p. 263] and Theorem 4 of [3, p. 243]. □

To show the next result we recall two results that concern some special homeomorphisms of the pseudo-arc. The former is due to Lehner, [8, Theorems 6 and 7, p. 369], and it is formulated here as in [10, Theorem 13, p. 92]. The latter is due to Lewis, [9, Corollary 2, p. 83].

THEOREM 15. (Lehner). *Let n be a positive integer. If $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ are two sets of subcontinua of the pseudo-arc P such that, for each distinct i and j, A_i and A_j are in distinct composants of P, and similarly for B_i and B_j, and if, for each i, there is a homeomorphism $h_i : A_i \to B_i$, then there exists a homeomorphism $h : P \to P$ such that $h|A_i = h_i$ for each i.*

THEOREM 16. (Lewis). *For each positive integer n there exists a homeomorphism of the pseudo-arc of period n having exactly one fixed point.*

Recall that a mapping between spaces is said to be *simple* if its point inverses consist of at most two points.

PROPOSITION 17. *Let $h : P \to P$ be a period 2 homeomorphism of the pseudo-arc P having exactly one fixed point p_0. Then points x and $h(x)$ are in the same composant of P if and only if p_0 and x are in the same composant of P. Moreover, if an equivalence relation $\sim$ on P is defined by*

$$x \sim y \Longleftrightarrow y = x \text{ or } y = h(x),$$

then the quotient space $P/\sim$ is homeomorphic to the pseudo-arc, and the quotient mappings $s : P \to P/\sim$ is simple and open.

Proof. Let C be the composant of P containing p_0. If $x \in C$, then the continuum Q irreducible between p_0 and x is a proper subcontinuum of P, whence $Q \cup h(Q)$ is a proper subcontinuum of P containing x and $h(x)$; thereby they are in the same composant of P, namely in C.

To show the converse implication, let Q be the subcontinuum of P which is irreducible between x and $h(x)$. Then $h(Q)$ is irreducible between $h(x)$ and $h(h(x)) = x$, so $Q = h(Q)$, and $h|Q : Q \to Q$ is an authohomeomorphism on Q. By the fixed point property of the pseudo-arc, Q contains the (only) fixed point of h, i.e., $p_0 \in Q$.

To see openness of s observe that the decomposition $\{\{x, h(x)\} : x \in P\}$ of P is continuous. Since open mappings preserve arc-likeness of continua, [14, Theorem 1.0, p. 259], and since confluent (in particular open) mappings preserve their hereditary indecomposability, the conclusion follows from the Bing characterization of the pseudo-arc, [1, Theorem 1, p. 653]. The proof is complete. □

Recently J. R. Prajs has obtained in [13] the following result.

THEOREM 18. (Prajs). *For each subcontinuum Q of a pseudo-arc P there is an open retraction from P onto Q.*

COROLLARY 19. (Prajs). *There exists an open mapping between pseudo-arcs such that some proper subcontinuum of the domain is mapped onto the whole range.*

Theorems 15 and 16, Proposition 17 and Corollary 19 will be used to show the following theorem which is the main result of this part of the paper.

THEOREM 20. *Let a, b, c, d be points of the pseudo-arc P such that $a \neq b$ and $c \neq d$. Then the following implications hold.*

(20.1) *If P is irreducible between a and b and between c and d, then there is a homeomorphism $h : P \to P$ with $h(a) = c$ and $h(b) = d$.*

(20.2) *If P is irreducible neither between a and b nor between c and d, then there is a homeomorphism $h : P \to P$ with $h(a) = c$ and $h(b) = d$.*

(20.3) *If P is irreducible between a and b and it is not irreducible between c and d, then there is a simple open mapping $g : P \to P$ with $g(a) = c$ and $g(b) = d$.*

(20.4) *If P is not irreducible between a and b and is irreducible between c and d, then there is an open mapping $g : P \to P$ with $g(a) = c$ and $g(b) = d$.*

Proof. (20.1) is an immediate consequence of Theorem 15.

To prove (20.2) denote by P_1 and P_2 the subcontinua of P which are irreducible between a and b and between c and d, respectively. Then P_1 and P_2 are proper subcontinua of P. Substituting in Theorem 15 the singletons $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$ for the sets A_1, A_2, B_1 and B_2 respectively, we conclude that there is a homeomorphism $h^* : P_1 \to P_2$ such that $h^*(a) = c$ and $h^*(b) = d$. Applying Theorem 15 once more with P_1 for A_1 and P_2 for B_1 and with h^* as h_1 we infer that there exists a homeomorphism $h : P \to P$ such that $h|P_1 = h^*$, whence $h(a) = c$ and $h(b) = d$, so (20.2) follows.

To show implication (20.3) take a period 2 homeomorphism $h : P \to P$ with exactly one fixed point p_0 (see Theorem 16), and let $s : P \to P$ be the simple open mapping of Proposition 17. Let $h_1 : P \to P$ be a homeomorphism such that $h_1(c)$ and $s(p_0)$ are in different composants of P. Then $h_1(c)$ and $h_1(d)$ are in the same composant of P, which is different from the composant containing $s(p_0)$. By Proposition 17 there are points c' and d' in different composants of P such that $s(c') = h_1(c)$ and $s(d') = h_1(d)$. By implication (20.1) there is a homeomorphism $h_2 : P \to P$ such that $h_2(a) = c'$ and $h_2(b) = d'$. Then for the composition

$g = h_1^{-1} \circ s \circ h_2$ we have $g(a) = c$ and $g(b) = d$, so g is the needed simple open mapping.

To prove (20.4) we apply Corollary 19. So, let $f : P \to P$ be an open mapping such that some proper subcontinuum of P is mapped by f onto P. Let Q be a minimal of the subcontinua of P having the property $f(Q) = P$, i.e., such that no proper subcontinuum of Q is mapped onto P under f. Let $c, d \in P$ be points of irreducibility of P. Choose $c' \in f^{-1}(c) \cap Q$ and $d' \in f^{-1}(d) \cap Q$. Then Q is irreducible between c' and d' by its mininimality. By (20.2) there exists a homeomorphism $h : P \to P$ such that $h(a) = c'$ and $h(b) = d'$. Then $g = f \circ h$ is the needed open mapping. □

Theorem 20 implies the following important corollary.

COROLLARY 21. *The pseudo-arc is open 2-homogeneous.*

Corollary 21 shows that assumption of monotone open 2-homogeneity in Corollary 6 cannot be weakened to open 2-homogeneity, and that the corresponding assumptions made in Theorems 4 and 5 are essential ones.

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