BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 43, No. 4, 1995

Continuous Images of AR-Spaces

by

Włodzimierz J. CHARATONIK

Presented by C. RYLL-NARDZEWSKI on July 23, 1995

Summary. In a private letter Professor Biagio Ricceri has asked whether every arcwise connected space is a continuous image of an AR($\mathfrak{M}$)-space. This paper contains a positive answer to this question. Some related problems are discussed.

We will consider metric spaces only. An *arc* means a space that is homeomorphic to the closed unit interval $[0,1]$ of reals. A space is said to be *arcwise connected* provided that for every two of its points there is an arc in the space joining these points. A (continuous) mapping $f : X \to Y$ is called a *retraction* if $ff = f$, i.e., if $Y \subset X$ and $f \mid f(X)$ is the identity. A metrizable space Y is called an AR($\mathfrak{M}$)-*space* (writing $Y \in$ AR($\mathfrak{M}$)) (see [1, Chapt. IV, 1, p. 85]) provided that if it is embedded into a metrizable space Y as a closed subspace under an embedding $e : Y \to e(Y) = \mathrm{cl}_X e(Y) \subset X$, there is a retraction from X onto $e(Y)$.

The symbol card A stands for the cardinality of the set A. Given a cardinal number α we define a space $J(\alpha)$ called the *hedgehog space of spininess* α as follows (see [2, Example 4.1.5, p. 251]). Let A be a set of cardinality α and let $I_a = [0,1] \times \{a\}$ for each $a \in A$. By letting

$$\langle x, a_1\rangle E \langle y, a_2\rangle \text{ if and only if } x = y = 0 \text{ or } x = y \text{ and } a_1 = a_2$$

we define an equivalence relation E on the union $\bigcup\{I_a : a \in A\}$. Roughly speaking, the relation E shrinks all end points $\langle 0, a\rangle$ in I_a to one point v. The formula

$$d(\langle x, a_1\rangle, \langle y, a_2\rangle) = \begin{cases} x - y & \text{if } a_1 = a_2, \\ x + y & \text{if } a_1 \neq a_2 \end{cases}$$

AMS Classification: 54C05 54C55.

Key words: AR-space, arcwise connected, retract.

defines a metric on the set of equivalence classes of E. The obtained metric space is just $J(\alpha)$.

LEMMA. *For every cardinal number $\alpha > 0$ the space $J(\alpha)$ is an $AR(\mathfrak{M})$-space.*

P r o o f. By [1, p. 85] it is enough to show that $J(\alpha)$ is homeomorphic to a retract of a convex subset of a normed linear space. Consider all real-valued functions $f : A \to \mathbb{R}$. Denote by $\mathcal{F}$ the set composed of all such functions $f : A \to \mathbb{R}$ which take non-zero values for finitely many elements of A only, i.e. $\mathcal{F} = \{f : A \to \mathbb{R} : f(x) = 0$ for all $x \in A$ except finitely many of them$\}$. Observe that if $f, g \in \mathcal{F}$ and $\lambda \in \mathbb{R}$, then $f + g \in \mathcal{F}$ and $\lambda f \in \mathcal{F}$. Putting $\|f\| = \sum_{x \in A} |f(x)|$ for each $f \in \mathcal{F}$ we see that $\mathcal{F}$ is a normed linear space. The set $\mathcal{C} = \{f \in \mathcal{F} : 0 \leqslant f(x) \leqslant 1$ for each $x \in A\}$ is a convex subset of $\mathcal{F}$. Put $\mathcal{H} = \{f \in \mathcal{C} : f(x) \neq 0$ for at most one $x \in A\}$. It is easy to observe that the two spaces $\mathcal{H}$ and $J(\alpha)$ are homeomorphic.

We will construct a retraction r of $\mathcal{C}$ onto $\mathcal{H}$. Let $f \in \mathcal{C}$ be fixed. Since f has at most finitely many non-zero values, the number $m = \max\{f(x) : x \in A\}$ is well-defined. Take $x_m \in A$ such that $f(x_m) = m$, and put $n = \max\{f(x) : x \in A \setminus \{x_m\}\}$. Define $r : \mathcal{C} \to \mathcal{H} \subset \mathcal{C}$ by

$$r(f)(x) = \begin{cases} 0 & \text{if } x \neq x_m, \\ m - n & \text{if } x = x_m. \end{cases}$$

Observe that if $f \in \mathcal{H}$, then there is at most one element $x_m \in A$ for which $0 < f(x_m) = m \leqslant 1$, whence $n = 0$, and therefore $r(f) = f$, so r is a retraction. This finishes the proof of the lemma.

THEOREM 1. *Any arcwise connected space Y is a continuous image of $J(\operatorname{card} Y)$.*

P r o o f. Let $\alpha = \operatorname{card} Y$. We denote points of $J(\alpha)$ by $\langle t, y \rangle$, where $t \in [0,1]$ and $y \in Y$ with $\langle 0, y_1 \rangle = \langle 0, y_2 \rangle$ for $y_1, y_2 \in Y$. Let B be the set of all end points of $J(\alpha)$, i.e. the set of points of the form $\langle 1, y \rangle$ for $y \in Y$, we have $\operatorname{card} B = \alpha$. Hence there is a surjection $\varphi : B \to Y$. We extend this surjection continuously to obtain a mapping $\varphi^* : J(\alpha) \to Y$ such that $\varphi^* \mid B = \varphi$. If v is the vertex of $J(\alpha)$, we choose $\varphi^*(v) \in Y$ arbitrarily. Since Y is arcwise connected, for each $b \in B$ there is an arc L in Y from $\varphi^*(v)$ to $\varphi^*(b)$. Let I be the arc in $J(\alpha)$ from its vertex v to an end point $b \in B$. We define the restriction $\varphi^* \mid I : I \to L$ separately for each arc I of $J(\alpha)$ as a surjection of I onto L in the following way.

We order the arc L from $\varphi^*(v)$ to $\varphi^*(b)$. For every $n \in \mathbb{N}$ let x_n be the first point of L whose distance to $\varphi^*(v)$ is $1/2^{n-1}$ (if there is no such a point for some several values of n, then we put $x_n = \varphi(b)$). Thus we have a sequence of points x_n in L such that $\varphi^*(v) < \ldots < x_{n+1} < x_n < \ldots < x_2 < x_1 = \varphi^*(b)$ and that $\lim x_n = \varphi^*(v)$. For every $n \in \mathbb{N}$ let $x_{n+1}x_n$ denote the subarc of the arc L with end points x_{n+1} and x_n. Recall that, by the definition of $J(\alpha)$, the arc I is isometric to $[0,1]$. It will be convenient to use this isometry and to think on the closed unit interval $[0,1]$ rather then on I, with 0 in place of v and with 1 in place of b. For every $n \in \mathbb{N}$ put $\varphi^*(1/2^{n-1}) = x_n$, and define $\varphi^*|[1/2^n, 1/2^{n-1}] : [1/2^n, 1/2^{n-1}] \to x_{n+1}x_n$ as a surjection which is either a homeomorphism (if $x_{n+1} \neq x_n$) or a constant mapping (if $x_{n+1} = x_n$). In this way the partial mapping $\varphi^* \mid I : I \to L$ is well defined, and so is the whole mapping $\varphi^* : J(\alpha) \to Y$. Observe that the definition assures continuity of the mapping at the vertex v of $J(\alpha)$. In fact, it follows from the definition of φ^* that for the open ball Q in Y centered at the point $\varphi^*(v)$ and of radius $1/2^n$ the point v is an interior point of its inverse image $(\varphi^*)^{-1}(Q)$. Continuity of φ^* at other points of the domain is obvious. Thus the proof is complete.

It would be interesting to know which spaces are continuous images of the hedgehog of a given spininess. In the next theorem we characterize such spaces if the spininess is countable.

THEOREM 2. *The following conditions for a space X are equivalent:*

(a) *X is a continuous image of the real line;*

(b) *X is a continuous image of $J(\aleph_0)$;*

(c) *X is arcwise connected and it is the union of countably many locally connected continua.*

P r o o f. To prove (a) $\Rightarrow$ (b) we need to define a mapping f of $J(\aleph_0)$ onto $\mathbb{R}$. Denote points of $J(\aleph_0)$ by $\langle t, n\rangle$ with $\langle 0, m\rangle = \langle 0, n\rangle$ for any $m, n \in \mathbb{N}$ and put

$$f(\langle t,n\rangle) = \begin{cases} 0 & \text{if } t < 1/2, \\ (t-1/2)n & \text{if } t \geqslant 1/2 \text{ and } n \text{ is even}, \\ -(t-1/2)n & \text{if } t \geqslant 1/2 \text{ and } n \text{ is odd}. \end{cases}$$

Then f is continuous and surjective.

To see (b) $\Rightarrow$ (c) note that both arcwise connectedness and being a locally connected continuum are invariant under continuous mappings.

To show (c) $\Rightarrow$ (a) assume that X is arcwise connected and there are locally connected continua $X_1, X_2, \ldots$ such that $X = \bigcup_{n=1}^{\infty} X_n$. We define a mapping $f : \mathbb{R} \to X$ as follows. Let $f \mid [2n, 2n+1]$ be any map onto X_n. Define $f(x) = f(2)$ for $x < 2$ and let $f \mid [2n+1, 2n+2]$ be any homeomorphism of $[2n+1, 2n+2]$ onto an arc with end points $f(2n+1)$ and $f(2n+2)$. Such a defined function f maps $\mathbb{R}$ onto X, so the implication (c) $\Rightarrow$ (a) is shown. This finishes the proof of Theorem 2.

Remark. Let X be the Cantor fan, i.e. the cone over the Cantor set. According to Theorem 2 the continuum X is not a continuous image of $J(\aleph_0)$, so we cannot substitute weight of Y in place of cardinality of Y in Theorem 1.

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, 50–384 WROCŁAW
(INSTYTUT MATEMATYKI, UNIWERSYTET WROCŁAWSKI)
E-mail: wjcharat@math.uni.wroc.pl

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