

*ARC PROPERTY OF KELLEY AND ABSOLUTE RETRACTS  
FOR HEREDITARILY UNICOHERENT CONTINUA*

BY

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**Abstract.** We investigate absolute retracts for hereditarily unicoherent continua, and also the continua that have the arc property of Kelley (i.e., the continua that satisfy both the property of Kelley and the arc approximation property). Among other results we prove that each absolute retract for hereditarily unicoherent continua (for tree-like continua, for  $\lambda$ -dendroids, for dendroids) has the arc property of Kelley.

**1. Introduction.** It is well known that absolute retracts (for the class of all compacta) are locally connected. This is also the case of absolute retracts for many smaller but important classes of spaces (see [4] and [9]). Following Maćkowiak's ideas from [25], in the present paper we study the class  $\text{AR}(\mathcal{HU})$  of absolute retracts for the class  $\mathcal{HU}$  of hereditarily unicoherent continua (and also absolute retracts for tree-like continua,  $\lambda$ -dendroids, dendroids). These classes appear in a natural way in various regions of mathematical interest and are among the most extensively studied classes of continua. Therefore their absolute retracts seem to be worth a special attention. According to Maćkowiak's result (see [25, Corollaries 4 and 5, pp. 181 and 183]), which was also independently proved by David P. Bellamy (unpublished; see [25, "added in proof", p. 183]), the buckethandle continuum and the Cantor fan belong to  $\text{AR}(\mathcal{HU})$ . Thus the members of  $\text{AR}(\mathcal{HU})$  need not be locally connected. Recently, the authors proved [5] that every inverse limit of trees with confluent bonding mappings belongs to  $\text{AR}(\mathcal{HU})$ . Consequently, we have a new large class of (not necessarily locally connected) continua that belong to  $\text{AR}(\mathcal{HU})$ . Other authors' results concerning the class  $\text{AR}(\mathcal{HU})$  are presented in [5], [6], [7] and [11].

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The goal of this paper is to show that the members of  $\text{AR}(\mathcal{HU})$  have very regular properties which make them particularly interesting objects to study. First, we present a number of results showing that the members of  $\text{AR}(\mathcal{HU})$  have some basic properties similar to those of regular absolute retracts for all compacta (Theorems 2.5, 2.11 and 2.17). Next we prove that every member of  $\text{AR}(\mathcal{HU})$  has two strong properties that are shared by all Peano continua: the property of Kelley and the arc approximation property. The conjunction of these two properties, called here the *arc property of Kelley*, turned out to be a crucial tool in the authors' study of  $\text{AR}(\mathcal{HU})$  spaces [5]–[8] and lifting properties of confluent mappings [10]. Let us recall that the property of Kelley has been used to study hyperspaces, in particular their contractibility (see e.g. Chapter 16 of [29], where references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting in its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces, as e.g. topological characterizations of solenoids (see [20] and [21]) and of the sinusoidal curve [28, Lemma 2.5, p. 517]. The other property, i.e., the arc approximation property, has been introduced and studied in [14].

In this paper we also present some other results for members of  $\text{AR}(\mathcal{HU})$  and for continua that have the arc property of Kelley. For instance we prove that continua in  $\text{AR}(\mathcal{HU})$  have the *generalized  $\varepsilon$ -push property* (see Definition 2.20 below), which relates such continua to homogeneous ones. Other applications of this property are presented in [6] and [12].

The paper consists of three sections. The notion of a unionable class of spaces as well as two other related concepts are introduced in Section 2 (Definitions 2.1, 2.6 and 2.13). It is shown that the class of hereditarily unicoherent continua and some of its subclasses are unionable (Theorem 2.16) and that members of the class of their absolute retracts have the generalized  $\varepsilon$ -push property. The properties of unionable and related classes of continua are exploited in the next section, in which classes  $\mathcal{K}$  of continua are investigated such that absolute retracts for  $\mathcal{K}$  have the property of Kelley and the arc approximation property.

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By a *space* we mean a metric space. A class of spaces is always meant topologically, i.e., it contains all topological copies of its members. A *mapping* means a continuous function. The reader is referred to [1] and [18] for needed information on retracts.

Following [18, p. 80], if  $\mathcal{K}$  is a class of *compacta* (i.e., of compact metric spaces), then  $\text{AR}(\mathcal{K})$  denotes the *family of all absolute retracts for  $\mathcal{K}$* , i.e.,  $K \in \text{AR}(\mathcal{K})$  provided that if  $Z \in \mathcal{K}$  contains a homeomorphic copy  $K'$  of  $K$ , then  $K'$  is a retract of  $Z$ .

Let  $X$  be a metric space with a metric  $d$ . For a mapping  $f : A \rightarrow B$ , where  $A$  and  $B$  are subspaces of  $X$ , we define  $d(f) = \sup\{d(x, f(x)) : x \in A\}$ . Further, we denote by  $B(p, \varepsilon)$  the (open) ball in  $X$  centered at  $p \in X$  with radius  $\varepsilon$ . For a subset  $A \subset X$  we put  $N(A, \varepsilon) = \bigcup\{B(a, \varepsilon) : a \in A\}$ . The symbol  $\mathbb{N}$  stands for the set of all positive integers, and  $\mathbb{R}$  denotes the space of real numbers.

By a *continuum* we mean a connected compactum. Given a continuum  $X$ , we let  $C(X)$  denote the hyperspace of all nonempty subcontinua of  $X$  equipped with the Hausdorff metric  $H$  (see e.g. [29, (0.1), p. 1 and (0.12), p. 10]). We use the symbols  $\text{Ls } A_n$  and  $\text{Lim } A_n$  to denote the upper limit and the limit of the sequence  $\{A_n\}$  as defined in [22, §29, III and VI, pp. 337 and 339].

A continuum  $X$  is said to be *unicoherent* if the intersection of any two of its subcontinua whose union is  $X$  is connected.  $X$  is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A locally connected dendroid is called a *dendrite*. A *tree* means a graph containing no simple closed curve. A continuum that is the inverse limit of an inverse sequence of trees is called *tree-like*.

A continuum is said to be *decomposable* provided that it can be represented as the union of two of its proper subcontinua. Otherwise it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its nondegenerate subcontinua is decomposable (indecomposable, respectively). A hereditarily unicoherent and hereditarily decomposable continuum is called a  $\lambda$ -*dendroid*.

Let  $\mathcal{D}_0$  denote the class of dendrites,  $\mathcal{D}$  the class of dendroids,  $\lambda\mathcal{D}$  the class of  $\lambda$ -dendroids, and  $\mathcal{TL}$  the class of tree-like continua. Then

$$(1.1) \quad \mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{TL} \subset \mathcal{HU}.$$

According to the result of Borsuk (see [1, (13.5), p. 138]) we have

$$\text{AR}(\mathcal{D}_0) = \mathcal{D}_0 \subset \text{AR}(\mathcal{D}) \cap \text{AR}(\lambda\mathcal{D}) \cap \text{AR}(\mathcal{TL}) \cap \text{AR}(\mathcal{HU}).$$

Note that the class of absolute retracts of all unicoherent continua coincides with the class of retracts of the Hilbert cube, thus it also coincides with the class of absolute retracts of all compacta. This class is relatively well studied, and we do not investigate it here.

**2. Unionable classes of spaces.** We start by introducing the following concept.

**DEFINITION 2.1.** A class  $\mathcal{S}$  of nonempty compacta is called *unionable* provided that if a space  $Z$  is the union of subspaces  $X$  and  $Y$  with  $X, Y, X \cap Y \in \mathcal{S}$ , then  $X \cup Y \in \mathcal{S}$ .

**OBSERVATION 2.2.** *The following classes of spaces are unionable: compacta of dimension less than or equal to  $n$ , continua, hereditarily unicoherent continua, tree-like continua,  $\lambda$ -dendroids, dendroids, dendrites.*

Indeed, for at most  $n$ -dimensional compacta and for continua see [22, §27, I, Theorem 2, p. 288] and [23, §47, I, Theorem 1, p. 168], respectively. For the remaining classes see [15, Theorems 14 and 15, p. 97 and 98, and Corollary, p. 98].

However, one can see that most classes of compacta are not unionable. Such are, e.g., the classes of hereditarily decomposable continua and of unicoherent ones. The next two examples show this.

**EXAMPLE 2.3.** *The class of hereditarily decomposable continua is not unionable.*

*Proof.* In fact, let  $D$  be the simplest Knaster indecomposable continuum embedded in the plane  $\mathbb{R}^2$  in the standard way, as described in [23, §48, V, Example 1, p. 204 and Fig. 4, p. 205]. Let  $L$  stand for the straight line segment joining the points  $\langle 0, 0 \rangle$  and  $\langle 1, 0 \rangle$  of  $D$ . Define  $X = L \cup \{\langle x, y \rangle \in D : y \geq 0\}$  and  $Y = L \cup \{\langle x, y \rangle \in D : y \leq 0\}$ . Then  $X, Y$ , and  $X \cap Y = L$  are hereditarily decomposable continua, but  $X \cup Y$  is not, since it contains  $D$ . ■

A mapping  $f : X \rightarrow Y$  between continua is said to be:

- *monotone* if it has connected point inverses;
- *atomic* provided that for each subcontinuum  $K$  of  $X$  either  $f(K)$  is a singleton or  $f^{-1}(f(K)) = K$  (it is known [24, (4.14), p. 17] that every atomic mapping of a continuum is monotone).

**EXAMPLE 2.4.** *The class of unicoherent continua is not unionable.*

*Proof.* In the ring  $R = \{z \in \mathbb{R}^2 : 1 \leq |z| \leq 2\}$  consider the closure  $S$  of a spiral approaching the inner circle, i.e.,  $S = \text{cl}\{(1 + 1/t)e^{i\pi t} : t \in [1, \infty)\}$ . Let  $h : R \rightarrow R$  denote the central symmetry defined by  $h(z) = -z$  for  $z \in R$ . Then  $R \setminus (S \cup h(S))$  is the union of two components,  $A$  and  $B$ . Thus  $R = \text{cl} A \cup \text{cl} B$ , and  $\text{cl} A \cap \text{cl} B = S \cup h(S)$ .

We will show that the continua  $\text{cl} A$ ,  $\text{cl} B$  and  $S \cup h(S)$  are unicoherent. Indeed, shrinking the inner circle  $\{z \in \mathbb{R}^2 : |z| = 1\}$  of  $R$  to a point we obtain atomic mappings defined on each of these three continua onto a disk, a disk, and an arc, respectively. Then by [26, Proposition 11(i), p. 537] the continua  $\text{cl} A$ ,  $\text{cl} B$  and  $S \cup h(S)$  are unicoherent, while  $R$  is not. ■

Let  $X$  and  $Y$  be two disjoint spaces,  $U \subset X$  a closed subset of  $X$ , and let  $f : U \rightarrow Y$  be a mapping. In the disjoint union  $X \oplus Y$  define an equivalence

relation  $\sim$  by  $u \sim f(u)$  for each  $u \in U$ . Then the quotient space  $(X \oplus Y)/\sim$  is denoted by  $X \cup_f Y$  (see [17, Definition 6.1, p. 127] and compare [30, 3.18, p. 42]).

**THEOREM 2.5.** *Let  $\mathcal{S}$  be a unionable class of compacta. If  $X \in \text{AR}(\mathcal{S})$  and  $Y \in \mathcal{S}$  is a retract of  $X$ , then  $Y \in \text{AR}(\mathcal{S})$ .*

*Proof.* Let  $Y$  be embedded in a space  $Z \in \mathcal{S}$ . We have to define a retraction from  $Z$  onto  $Y$ . Take the disjoint union of  $Z$  and  $X$ , and let  $e : Y \rightarrow X$  be the embedding. Put  $T = Z \cup_e X$ . Since  $\mathcal{S}$  is unionable,  $T \in \mathcal{S}$ . Thus there exists a retraction  $r_1 : T \rightarrow X$ . If  $r_2 : X \rightarrow Y$  is a retraction, then  $r_2 \circ r_1|_Z : Z \rightarrow Y$  is the required retraction. ■

The assumption that  $\mathcal{S}$  is unionable is necessary in the above theorem, as shown by the following example. Let  $S^1$  be the unit circle,  $I^2$  the unit square, and  $K$  any of the two Kuratowski nonplanable graphs (see [23, §51, VII, Fig. 11, p. 305]). Put  $\mathcal{S} = \{S^1, I^2, K\}$  (understood topologically). Then  $\mathcal{S}$  is not unionable,  $\text{AR}(\mathcal{S}) = \mathcal{S} \setminus \{S^1\}$ , but  $S^1$  is a retract of  $K$ .

**DEFINITION 2.6.** A class  $\mathcal{S}$  of nonempty compacta is called *functionally unionable* provided that for all members  $U, X, Y$  of  $\mathcal{S}$  such that  $U \subset X$  with  $X \cap Y = \emptyset$  and for each mapping  $f : U \rightarrow Y$ , if  $f(U) \in \mathcal{S}$ , then  $X \cup_f Y \in \mathcal{S}$ .

**OBSERVATION 2.7.** *Each functionally unionable class of compacta is unionable.*

**REMARK 2.8.** The opposite implication to that of Observation 2.7 does not hold in general. Indeed, let  $\mathcal{K}$  consist of all one-point sets and of all continua that are unions of finitely many arcs. Then  $\mathcal{K}$  is unionable. To see that it is not functionally unionable consider the following example. Let  $U$  be the straight line segment joining  $\langle 0, 0 \rangle$  to  $\langle 1, 0 \rangle$  in the plane. Let  $B_n$  be the upper semicircle in the plane that has the points  $\langle 1/(n+1), 0 \rangle$  and  $\langle 1/n, 0 \rangle$  as its ends. Thus  $B = \{\langle 0, 0 \rangle\} \cup \bigcup \{B_n : n \in \mathbb{N}\}$  is an arc having  $\langle 0, 0 \rangle$  and  $\langle 1, 0 \rangle$  as its ends. Put  $X = U \cup B$ . Then  $X \in \mathcal{K}$ . If  $Y$  is a singleton and  $f : U \rightarrow Y$  is the constant mapping, then the continuum  $X \cup_f Y$  is not in  $\mathcal{K}$ .

To show functional unionability of classes listed in Observation 2.2 we need the following lemma.

**LEMMA 2.9.** *Let  $\mathcal{K} \in \{\mathcal{D}_0, \mathcal{D}, \lambda\mathcal{D}, \mathcal{TL}, \mathcal{HU}\}$ . Let  $X, U, Y \in \mathcal{K}$  with  $U \subset X$  and  $X \cap Y = \emptyset$ , and let a mapping  $f : U \rightarrow Y$  be a surjection. Then  $X \cup_f Y \in \mathcal{K}$ .*

*Proof.* We will prove the lemma for  $\mathcal{K} = \mathcal{HU}$ . For the other classes the proof is the same, or even simpler.

First we show the following claim.

(2.9.1) For each continuum  $K \subset X \cup_f Y$  the set  $K \cap Y$  is connected.

Indeed, suppose on the contrary that there are two nonempty, closed and disjoint sets  $F$  and  $G$  such that  $K \cap Y = F \cup G$ . Applying the boundary bumping theorem (see e.g. [30, 5.6, p. 74]) we infer that there is a component  $C$  of  $K \setminus Y$  such that  $\text{cl}C \cap F \neq \emptyset \neq \text{cl}C \cap G$ . If  $q : X \rightarrow X \cup_f Y$  is the quotient mapping (which, by definition, identifies the pairs  $(u, f(u))$  for  $u \in U$  only), then  $\text{cl}q^{-1}(C) \cap q^{-1}(F) \neq \emptyset \neq \text{cl}q^{-1}(C) \cap q^{-1}(G)$ , and  $\text{cl}q^{-1}(C) \cap U \subset q^{-1}(F) \cup q^{-1}(G)$ . Therefore the continuum  $\text{cl}q^{-1}(C) \cup U$  is a nonunicoherent subcontinuum of  $X$ , a contradiction. Thus (2.9.1) follows.

Let  $K_1$  and  $K_2$  be subcontinua of  $X \cup_f Y$  with  $K_1 \cap K_2 \neq \emptyset$ . For  $i \in \{1, 2\}$  define

$$K'_i = \begin{cases} K_i & \text{if } K_i \cap Y = \emptyset, \\ K_i \cup Y & \text{otherwise.} \end{cases}$$

Then  $L_i = q^{-1}(K'_i)$  are continua with  $L_1 \cap L_2 \neq \emptyset$ . Hence  $L_1 \cap L_2$  is connected. So  $M = q(L_1 \cap L_2)$  is also connected. We have either  $M = K_1 \cap K_2$ , or  $M = (K_1 \cap K_2) \cup Y$ . In the former case  $K_1 \cap K_2$  is connected. In the latter, we see that the sets  $K_1 \cap Y$  and  $K_2 \cap Y$  are connected by (2.9.1). Therefore  $K_1 \cap K_2 \cap Y$  is connected by the hereditary unicoherence of  $Y$ . So  $Y$  intersects at most one component of  $K_1 \cap K_2$ . Since  $M = (K_1 \cap K_2) \cup Y$  is connected, it follows that  $K_1 \cap K_2$  is connected, as required. ■

A simple consequence of the above lemma and of the unionability of the classes listed in Observation 2.2 is the functional unionability of these classes. Thus we have the following result.

**PROPOSITION 2.10.** *All the classes of continua listed in Observation 2.2 are functionally unionable.*

**THEOREM 2.11.** *Let  $\mathcal{S}$  be a functionally unionable class of compacta. Then for each  $Y \in \mathcal{S}$  the following two conditions are equivalent:*

(2.11.1)  $Y \in \text{AR}(\mathcal{S})$ ;

(2.11.2) for each space  $X \in \mathcal{S}$ , for each subspace  $U \subset X$  such that  $U \in \mathcal{S}$ , and for each mapping  $f : U \rightarrow Y$  with  $f(U) \in \mathcal{S}$  there exists a mapping  $f^* : X \rightarrow Y$  such that  $f^*|U = f$ .

*Proof.* To show that (2.11.2) implies (2.11.1), let  $Y$  be embedded in a space  $X \in \mathcal{S}$ . Take  $U = Y$  and let  $f : U \rightarrow X$  be the inclusion map. Then  $f^*$  is a retraction from  $X$  onto  $Y$ .

Assume (2.11.1). Let  $q : X \oplus Y \rightarrow T = X \cup_f Y$  be the quotient mapping. Then  $T \in \mathcal{S}$  by assumption. Let  $r : T \rightarrow Y$  be a retraction. Then the restriction  $f^* = r \circ q|X : X \rightarrow Y$  is an extension of  $f$ , as required. ■

Consider a sequence of compact sets  $X_0, X_1, \dots$  (all situated in a metric space  $M$  with a metric  $d$ ), a sequence of closed subsets  $Y_n$  of  $X_n$  for  $n \in \mathbb{N}$

and of mappings  $f_n : Y_n \rightarrow X_0$  such that

- (2.12.1)  $X_m \cap X_n = \emptyset$  for  $m \neq n$ , and  $m, n \in \{0, 1, \dots\}$ ;  
 (2.12.2)  $\text{Ls } X_n \subset X_0$ ;  
 (2.12.3)  $\lim d(f_n) = 0$ .

Take a decomposition of the union  $X_0 \cup \bigcup \{X_n : n \in \mathbb{N}\}$  induced by identification of each of the pairs  $(y, f_n(y))$  for each  $y \in Y_n$  and each  $n \in \mathbb{N}$ , and observe that this decomposition is upper semicontinuous. Denote the quotient space of this decomposition by  $Q(X_0; \{X_n, Y_n, f_n\})$ . In the particular case when the subsets  $Y_n$  are singletons  $\{p_n\}$  we put  $x_n = f_n(p_n)$  for  $n \in \mathbb{N}$  and we use the symbol  $Q(X_0; \{X_n, p_n, x_n\})$  in place of  $Q(X_0; \{X_n, Y_n, f_n\})$ .

DEFINITION 2.13. A class  $\mathcal{K}$  of nonempty compacta is called *functionally  $\omega$ -unionable* provided that for each sequence  $X_0, X_1, \dots$  of elements of  $\mathcal{K}$  and for any sequences of subsets  $Y_n$  of  $X_n$  with  $Y_n \in \mathcal{K}$  and of mappings  $f_n : Y_n \rightarrow X_0$  for  $n \in \mathbb{N}$  satisfying conditions (2.12.1)–(2.12.3) the quotient space  $Q(X_0; \{X_n, Y_n, f_n\})$  belongs to  $\mathcal{K}$ . In particular, if the sets  $Y_n$  are singletons, then  $\mathcal{K}$  is called *pointwise  $\omega$ -unionable*.

REMARK 2.14. The class of dendrites is functionally unionable but not pointwise  $\omega$ -unionable. To get a pointwise  $\omega$ -unionable class of compact metric spaces which is not unionable it is enough to consider the smallest pointwise  $\omega$ -unionable class containing a disk. To this end, observe the following sequence of facts.

- (2.14.1) *Let  $\{\mathcal{K}_\alpha\}$  be an arbitrary family of classes of compacta such that each class  $\mathcal{K}_\alpha$  is pointwise  $\omega$ -unionable. Then the intersection of this family is also pointwise  $\omega$ -unionable.*

An easy and standard proof of (2.14.1) is omitted. The next fact is obvious.

- (2.14.2) *The class of all compacta is pointwise  $\omega$ -unionable.*

Facts (2.14.1) and (2.14.2) imply the next one.

- (2.14.3) *For each class  $\mathcal{K}$  of compacta there exists the smallest class  $\widehat{\mathcal{K}}$  of compacta that contains  $\mathcal{K}$  and which is pointwise  $\omega$ -unionable.*

It can easily be shown that

- (2.14.4) *The class of all compacta not containing the 3-book (i.e., the product of an arc and a simple triod) is pointwise  $\omega$ -unionable.*

It follows from (2.14.3) and (2.14.4) that

- (2.14.5) *The smallest pointwise  $\omega$ -unionable class  $\mathcal{K}$  of compacta such that the disk is in  $\mathcal{K}$  does not contain the 3-book.*

Obviously,

(2.14.6) *Every unionable class of compacta that contains a disk contains the 3-book as well.*

Finally, from (2.14.3), (2.14.5) and (2.14.6) we get the needed conclusion:

(2.14.7) *The smallest pointwise  $\omega$ -unionable class of compacta that contains a disk is not unionable.*

Consider the following classes of continua:  $\mathcal{D}$ ,  $\lambda\mathcal{D}$ ,  $\mathcal{TL}$  and  $\mathcal{HU}$ . To show that they are functionally  $\omega$ -unionable we need the following lemma, which is a consequence of [15, Theorem 14, p. 97] for  $\mathcal{HU}$ , of [15, Theorem 15, p. 98] for  $\mathcal{TL}$ , and of [15, Corollary, p. 98] for the classes  $\mathcal{D}$  and  $\lambda\mathcal{D}$ .

LEMMA 2.15. *Let  $\mathcal{K} \in \{\mathcal{D}, \lambda\mathcal{D}, \mathcal{TL}, \mathcal{HU}\}$ . Let  $Z_0, Z_1, \dots$  be continua in  $\mathcal{K}$  such that for each  $m, n \in \mathbb{N}$  with  $m \neq n$ , we have  $Z_0 \subset Z_n$ ,  $(Z_m \setminus Z_0) \cap (Z_n \setminus Z_0) = \emptyset$ , and  $\text{Lim } Z_n = Z_0$ . Then the union  $Y = \bigcup\{Z_n : n \in \mathbb{N}\}$  is a member of  $\mathcal{K}$ .*

THEOREM 2.16. *Each of the following classes of continua:  $\mathcal{D}$ ,  $\lambda\mathcal{D}$ ,  $\mathcal{TL}$  and  $\mathcal{HU}$  is functionally  $\omega$ -unionable. In particular, each of them is pointwise  $\omega$ -unionable.*

*Proof.* Let  $\mathcal{K} \in \{\mathcal{D}, \lambda\mathcal{D}, \mathcal{TL}, \mathcal{HU}\}$ . For each  $n \in \mathbb{N}$  let  $X_0, X_n, Y_n$  and  $f_n$  be as in Definition 2.13, and  $q : X_0 \cup \bigcup\{X_n : n \in \mathbb{N}\} \rightarrow Q(X_0; \{X_n, Y_n, f_n\})$  be the quotient mapping. Put  $Z_0 = q(X_0)$  and  $Z_n = q(X_0 \cup X_n)$  for each  $n \in \mathbb{N}$ . Since the class  $\mathcal{K}$  is functionally unionable according to Observation 2.9 and Proposition 2.10, we get  $Z_0, Z_n \in \mathcal{K}$  for each  $n \in \mathbb{N}$ . Note that all other assumptions of Lemma 2.15 are also satisfied. Therefore  $Y = Q(X_0; \{X_n, Y_n, f_n\})$  is a member of  $\mathcal{K}$ , as required. ■

THEOREM 2.17. *Let  $\mathcal{K}$  be a functionally  $\omega$ -unionable class of continua. Let  $X \in \text{AR}(\mathcal{K})$  be a subcontinuum of a metric space  $M$ . Then for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for each  $Y \subset M$  with  $Y \in \mathcal{K}$ , for each mapping  $f : Y \rightarrow X$  satisfying  $d(f) < \delta$ , and for each continuum  $Z \in \mathcal{K}$  satisfying  $Y \subset Z \subset N(X, \delta)$  there is an extension  $f^* : Z \rightarrow X$  of  $f$  satisfying  $d(f^*) < \varepsilon$ .*

*Proof.* Suppose on the contrary that there is an  $\varepsilon > 0$  having the property that for each  $n \in \mathbb{N}$  there are continua  $Y_n, Z_n \in \mathcal{K}$  and a mapping  $f_n : Y_n \rightarrow X$  such that  $Y_n \subset Z_n \subset N(X, 1/n)$ ,  $d(f_n) < 1/n$ , and  $f_n$  does not admit any extension on  $Z_n$  which moves points by less than  $\varepsilon$ . In the space  $M \times \{0, 1/1, 1/2, 1/3, \dots\}$  put  $X_0 = X \times \{0\}$  and, for each  $n \in \mathbb{N}$ , consider the sets  $X_n = Y_n \times \{1/n\}$ ,  $T_n = Z_n \times \{1/n\}$  and define the mappings  $g_n : X_n \rightarrow X_0$  by  $g_n(y, 1/n) = (f_n(y), 0)$ . Let

$$q : X_0 \cup \bigcup\{T_n : n \in \mathbb{N}\} \rightarrow W = Q(X_0; \{T_n, X_n, g_n\})$$

be the quotient mapping. Since the class  $\mathcal{K}$  is functionally  $\omega$ -unionable,  $W \in \mathcal{K}$ . The restriction  $q|X_0$  is a homeomorphism, hence we can identify  $X_0$  and  $q(X_0)$  under this homeomorphism, and since  $X_0$  is homeomorphic to  $X$ , we have  $X_0 \in \text{AR}(\mathcal{K})$ . Let  $r : W \rightarrow X_0$  be a retraction and, for each  $n \in \mathbb{N}$ , let  $h_n : Z_n \rightarrow T_n$  be the homeomorphism defined by  $h_n(z) = (z, 1/n)$  and  $\pi : X_0 = X \times \{0\} \rightarrow X$  be the projection onto the first factor. Then the mapping  $f_n^* = \pi \circ r \circ q \circ h_n$  is an extension of the corresponding  $f_n$ , and  $\lim d(f_n^*) = 0$ , a contradiction finishing the proof. ■

REMARK 2.18. A theorem similar to Theorem 2.17 holds if we replace the condition that the class  $\mathcal{K}$  is functionally  $\omega$ -unionable by the weaker one that  $\mathcal{K}$  is pointwise  $\omega$ -unionable, provided that  $Y$  is degenerate.

For  $Z = X \subset M$  (in Theorem 2.17) we even have a simpler formulation.

COROLLARY 2.19. *Let  $\mathcal{K}$  be a functionally  $\omega$ -unionable class of continua. Let  $X \in \text{AR}(\mathcal{K})$  be a subcontinuum of a metric space  $M$ . Then for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for each  $Y \in \mathcal{K}$  with  $Y \subset X$  and each mapping  $f : Y \rightarrow X$  satisfying  $d(f) < \delta$  there is an extension  $f^* : X \rightarrow X$  of  $f$  satisfying  $d(f^*) < \varepsilon$ .*

In particular, by Remark 2.18, if  $Y$  is a singleton we obtain the next corollary. Though the guaranteed mapping need not be a homeomorphism, the formulation of this corollary resembles the Effros theorem for homogeneous continua (see e.g. [3, p. 735]). Thus it indicates some kind of a “weak homogeneity” of absolute retracts for the classes considered. Because of further applications (see Corollary 2.22, Proposition 3.1, as well as [6, Theorem 4.6, p. 140] and [12, Corollary 4.1]) we define the property separately.

DEFINITION 2.20. A continuum  $X$  is said to have the *generalized  $\varepsilon$ -push property* provided that

(2.20.1) for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for any two points  $x, y \in X$  with  $d(x, y) < \delta$  there exists a mapping  $f : X \rightarrow X$  satisfying  $f(x) = y$  and  $d(f) < \varepsilon$ .

Now the above-mentioned corollary is as follows.

COROLLARY 2.21. *Let  $\mathcal{K}$  be a pointwise  $\omega$ -unionable class of continua. Then each member of  $\text{AR}(\mathcal{K})$  has the generalized  $\varepsilon$ -push property.*

Corollary 2.21 and Theorem 2.16 imply the following.

COROLLARY 2.22. *Let  $\mathcal{K} \in \{\mathcal{D}, \lambda\mathcal{D}, \mathcal{TL}, \mathcal{HU}\}$ . Then each member of  $\text{AR}(\mathcal{K})$  has the generalized  $\varepsilon$ -push property.*

Note that the properties of (classic) AR's and even ANR's for the class  $\mathcal{K}$  of all compacta indicated in Corollaries 2.19 and 2.21 are well known [1, Theorem 3.1, p. 103].

**3. The arc property of Kelley.** In this section we introduce a new notion of the arc property of Kelley, which turns out to be the conjunction of the property of Kelley and the arc approximation property. We start by recalling the needed definitions.

A continuum  $X$  is said to have the *property of Kelley* provided that for each point  $p \in X$  and for each subcontinuum  $K$  of  $X$  containing  $p$  and for each sequence of points  $p_n$  converging to  $p$  there exists a sequence of subcontinua  $K_n$  of  $X$  containing  $p_n$  and converging (with respect to the Hausdorff metric) to the continuum  $K$  (see e.g. [19, p. 167] or [29, Definition 16.10, p. 538]).

The generalized  $\varepsilon$ -push property (see Definition 2.20) appears to be stronger than the property of Kelley.

**PROPOSITION 3.1.** *Each continuum having the generalized  $\varepsilon$ -push property has the property of Kelley.*

*Proof.* Fix a point  $x_0 \in X$ , and let  $K \in C(X)$  with  $x_0 \in K$ . Take a sequence of points  $x_n$  in  $X$  converging to  $x_0$ . By (2.20.1) for each  $n \in \mathbb{N}$  there is a mapping  $f_n : X \rightarrow X$  such that  $f_n(x_0) = x_n$ , and we have  $\lim d(f_n) = 0$ . Defining  $K_n = f_n(K)$  we get the required sequence of continua. ■

A continuum  $X$  is said to have the *arc approximation property* provided that for each point  $x \in X$  and for each subcontinuum  $K$  of  $X$  containing  $x$  there exists a sequence of arcwise connected subcontinua  $K_n$  of  $X$  containing  $x$  and converging to the continuum  $K$  (see [14, Section 3, p. 113]). The following proposition is known (see [14, Proposition 3.10, p. 116]).

**PROPOSITION 3.2.** *If a continuum has the arc approximation property, then each arc component of the continuum is dense.*

Investigating absolute retracts for some classes of continua we have found that the following concept of the *arc property of Kelley* that joins the arc approximation property and the property of Kelley turns out to be both natural and useful.

**DEFINITION 3.3.** A metric space  $X$  is said to have the *arc property of Kelley* provided that for each point  $p \in X$ , for each subcontinuum  $K$  of  $X$  containing  $p$  and for each sequence of points  $p_n \in X$  converging to  $p$  there exists a sequence of arcwise connected subcontinua  $K_n$  of  $X$  containing  $p_n$  and converging to the continuum  $K$ .

The following proposition is a consequence of the definitions. Since its proof is straightforward, it is left to the reader.

**PROPOSITION 3.4.** *A continuum has the arc property of Kelley if and only if it has the arc approximation property and the property of Kelley.*

Below we show that continua which are absolute retracts for some classes of continua enjoy the arc property of Kelley.

**THEOREM 3.5.** *Let  $\mathcal{K}$  be a pointwise  $\omega$ -unionable class of continua. Then each member of  $\text{AR}(\mathcal{K})$  has the property of Kelley. If, moreover, a continuum containing an arc belongs to  $\mathcal{K}$ , then each member of  $\text{AR}(\mathcal{K})$  has the arc property of Kelley.*

*Proof.* Let  $X \in \text{AR}(\mathcal{K})$ . The first part of the conclusion follows from Corollary 2.21 and Proposition 3.1. To show the second part we will use the concept of and well known facts about  $Z$ -sets in the Hilbert cube  $Q$ . The reader can find the needed information about  $Z$ -sets for example in [27, Section 6.2, pp. 262ff] (see also [19, Chapter III, Section 9, pp. 76–79]; for (b) below, we refer to [2, Theorem 7.6, p. 322]).

- (a)  $Z$ -sets that are arcs form a dense subset of the hyperspace  $C(Q)$ .
- (b) Each arc  $L \subset Q$  that is a  $Z$ -set has compact neighborhoods  $Q'$  arbitrarily close to  $L$  with respect to the Hausdorff metric and such that every  $Q'$  is homeomorphic to  $Q$  and  $L$  is a  $Z$ -set in  $Q'$ .

So, assume that  $\mathcal{K}$  contains a continuum  $P$  with an arc  $A \subset P$ . We may assume that  $P$  and  $X$  are subsets of the Hilbert cube  $Q$ , and that  $A$  is a  $Z$ -set in  $Q$ . Let  $K \in C(X)$ ,  $x_0 \in K$ , and take a sequence  $\{x_n\}$  of points in  $X$  converging to  $x_0$ . By (a) we can choose a sequence of arcs  $A_n$  that are  $Z$ -sets and converge to  $K$  with respect to the Hausdorff metric. For each  $A_n$  take, according to (b), a closed neighborhood  $Q_n$  in  $Q$  in such a way that  $\text{Lim } Q_n = \text{Lim } A_n = K$  and  $A_n$  is a  $Z$ -set in  $Q_n$ . For each  $n \in \mathbb{N}$  let  $h_n : A \rightarrow A_n$  be a homeomorphism. Since  $A$  and  $A_n$  are  $Z$ -sets in  $Q$  and  $Q_n$ , respectively, the homeomorphisms  $h_n$  can be extended to homeomorphisms  $h_n^* : Q \rightarrow Q_n$  by the Anderson theorem (see e.g. [27, Theorem 6.4.6, p. 278] or [19, Theorem 11.9.1, p. 93]). Put  $P_n = h_n^*(P)$ . Thus, we have in  $Q$  a sequence  $\{P_n\}$  of copies of  $P$  with arcs  $A_n \subset P_n$  such that  $\text{Lim } P_n = \text{Lim } A_n = K$  and a sequence of points  $a_n \in A_n$  with  $\lim a_n = x_0$ . Applying Theorem 2.17 in the version of Remark 2.18 we see that the mappings  $a_n \mapsto x_n$  have extensions  $f_n^* : P_n \rightarrow X$  such that  $\lim d(f_n^*) = 0$ . Then the sets  $f_n^*(A_n)$  are those required by the definition of the arc property of Kelley. The proof is complete. ■

To formulate the next result some definitions are in order. A dendroid  $X$  is said to be *smooth* provided that there is a point  $v \in X$  such that for each point  $x \in X$  and for each sequence  $\{x_n\}$  of points of  $X$  which tends to  $x$  the sequence of arcs  $vx_n$  is convergent to the arc  $vx$ . Since each dendroid having the property of Kelley is smooth (see [16, Corollary 5, p. 730]), Theorem 3.5 implies the following result.

**COROLLARY 3.6.** *Each member of  $\text{AR}(\mathcal{D})$  is a smooth dendroid (with the property of Kelley).*

Theorems 2.16 and 3.5 imply the following two results, new and not obvious for the classes  $\lambda\mathcal{D}$ ,  $\mathcal{TL}$  and  $\mathcal{HU}$ .

**COROLLARY 3.7.** *Let  $\mathcal{K}$  be any class of continua listed in (1.1). Then any member of  $\text{AR}(\mathcal{K})$  has the arc property of Kelley.*

By Theorem 3.5 and [14, Proposition 3.10, p. 116] we get the next corollary.

**COROLLARY 3.8.** *Let a pointwise  $\omega$ -unionable class  $\mathcal{K}$  of continua contain a continuum that contains an arc. Then each arc component of any member  $X$  of  $\text{AR}(\mathcal{K})$  is dense in  $X$ . In particular, the conclusion holds for any class  $\mathcal{K}$  of continua listed in (1.1).*

**PROBLEM 3.9.** Does there exist a pointwise  $\omega$ -unionable class  $\mathcal{K}$  of continua, each containing no arc, such that there is a nondegenerate member of  $\text{AR}(\mathcal{K})$ ?

In the remaining part of this section we study metric spaces (in particular continua) which have some (or all) of the following properties: having all arc components dense; the arc approximation property; the property of Kelley; hereditary unicoherence. All results of this study can be applied to the members of  $\text{AR}(\mathcal{D})$ ,  $\text{AR}(\lambda\mathcal{D})$ ,  $\text{AR}(\mathcal{TL})$ ,  $\text{AR}(\mathcal{HU})$ , or more generally, to the members of  $\text{AR}(\mathcal{K})$ , where  $\mathcal{K}$  is any  $\omega$ -pointwise unionable class of hereditarily unicoherent continua with some  $K \in \mathcal{K}$  containing an arc (see Theorem 3.5). Some of these results may have more general applications.

We start with the following lemma, whose proof is left to the reader.

**LEMMA 3.10.** *Let  $X$  be a space having the arc property of Kelley, let  $A = pq$  be an arc in  $X$ , and let  $p_n$  be a sequence of points converging to  $p$ . Then there is a sequence of arcs  $A_n = p_nq_n$  converging to the arc  $A$  such that the sequence  $q_n$  converges to  $q$ .*

Previously we have shown that continua in  $\text{AR}(\mathcal{HU})$  have the generalized  $\varepsilon$ -push property (see Corollary 2.22). In the next lemma we observe a similar phenomenon for trees contained in spaces satisfying the more general arc property of Kelley.

**LEMMA 3.11.** *Let  $X$  be a space having the arc property of Kelley. Then for each tree  $T \subset X$ , each point  $p \in T$ , and each sequence  $p_n$  in  $X$  converging to  $p$  there are mappings  $f_n : T \rightarrow X$  such that  $f_n(p) = p_n$  and  $\lim d(f_n) = 0$ .*

*Proof.* Given  $\varepsilon > 0$ , it suffices to prove that there are mappings  $f_n : T \rightarrow X$  with  $f_n(p) = p_n$  and  $d(f_n) < \varepsilon$  for almost all  $n \in \mathbb{N}$ . Let  $\mathcal{M} = \{v_1, \dots, v_k, A_1, \dots, A_m\}$  be a simplicial complex structure on  $T$  such that  $p = v_1, \dots, v_k$  are vertices of  $\mathcal{M}$ , and  $A_1, \dots, A_m$  are 1-dimensional edges, each being an arc with some end points  $v_i, v_j$  such that  $\text{diam } A_l < \varepsilon$  for each  $l \in \{1, \dots, m\}$ . Let  $q \in \{v_1, \dots, v_k\}$  be a vertex connected with  $p$  by

an edge  $A_l \in \{A_1, \dots, A_m\}$ . By Lemma 3.10 there are arcs  $A_{l,n} \subset X$  with end points  $p_n$  and  $q_n$ , respectively, such that  $\text{Lim}_n A_{l,n} = A_l$  and  $\lim q_n = q$ . We can inductively continue this construction so that at the next step the role of  $p$  is played by  $q$  and that of  $p_n$ 's by  $q_n$ 's. In this way we eventually obtain a collection of sequences  $\{v_{i,n}\}$  with  $i \in \{1, \dots, k\}$ , and arcs  $\{A_{l,n}\}$  with  $l \in \{1, \dots, m\}$ , such that

- (i)  $v_{1,n} = p_n$  for each  $n \in \mathbb{N}$ ;
- (ii) if  $v_i$  is an end point of  $A_l$ , then  $v_{i,n}$  is an end point of  $A_{l,n}$  for each  $n \in \mathbb{N}$ ;
- (iii)  $\lim_n v_{i,n} = v_i$  and  $\text{Lim}_n A_{l,n} = A_l$ .

Let  $A_l$  be an edge in  $\mathcal{M}$  with end points  $v_i$  and  $v_j$ . For any  $n$  we fix a homeomorphism  $f_{l,n} : A_l \rightarrow A_{l,n}$  such that  $f_{l,n}(v_i) = v_{i,n}$  and  $f_{l,n}(v_j) = v_{j,n}$ . Combining the mappings  $f_{l,n}$  together, we obtain a mapping  $f_n : T \rightarrow X$ . The reader can observe that for sufficiently large  $n$  we have  $d(f_n) < \varepsilon$ , as required. ■

For any tree  $T$  let  $E(T)$  be the set of end points of  $T$  and, for a vertex  $p \in T$ , denote by  $\text{ord}(p, T)$  the *order of  $p$  in  $T$* , i.e., the number of edges of  $T$  to which  $p$  belongs. Finally,  $\text{ord} T$  is the maximum of the orders of the vertices of  $T$ .

LEMMA 3.12. *Let  $T$  be a tree in a metric space  $X$ , and for each  $n \in \mathbb{N}$  let  $f_n : T \rightarrow X$  be mappings such that  $\lim d(f_n) = 0$ . Then, for sufficiently large  $n$ , there are trees  $T_n \subset f_n(T)$  and monotone surjective mappings  $g_n : T_n \rightarrow T$  such that*

- (1)  $f_n(E(T)) = E(T_n)$  and  $g_n(E(T_n)) = E(T)$ ;
- (2)  $(f_n \circ g_n)|_{E(T_n)} = \text{id}|_{E(T_n)}$ ;
- (3)  $\lim d(g_n) = 0$ ;
- (4) if  $\text{card } g_n^{-1}(x) > 1$ , then  $\text{ord}(x, T) > 3$ .

*In particular, if  $\text{ord} T \leq 3$ , then the mappings  $g_n$  are homeomorphisms.*

*Proof.* We apply induction with respect to the number of end points in  $T$ . First, we assume that  $T$  is an arc with end points  $a$  and  $b$ . Let  $\prec$  be the natural order in  $T = ab$  from  $a$  to  $b$ . If  $x, y \in ab$ , the symbol  $xy$  denotes the arc in  $ab$  from  $x$  to  $y$ . Let  $x_0, x_1, \dots, x_k$  be points in  $ab$  such that  $a = x_0 \prec x_1 \prec \dots \prec x_k = b$ . Then for sufficiently large  $n$  and  $j > i + 1$  we have  $f_n(x_i x_{i+1}) \cap f_n(x_j x_{j+1}) = \emptyset$ , and also  $f_n(a) \notin f_n(x_1 b)$  and  $f_n(b) \notin f_n(ax_{k-1})$ .

Let  $p_0 = f_n(a)$ , and  $p_0 p_1$  be an irreducible arc in  $f_n(x_0 x_1)$  connecting  $p_0$  and  $f_n(x_1 b)$ . Then, by the assumption,  $p_1 \in f_n(x_1 x_2) \setminus f_n(x_2 b)$ . Similarly,  $p_1 p_2$  is defined as an irreducible arc in  $f_n(x_1 x_2)$  connecting  $p_1$  and  $f_n(x_2 b)$ . As previously, we observe that  $p_2 \in f_n(x_2 x_3) \setminus f_n(x_3 b)$ . We inductively con-

tinue this procedure until we obtain  $p_{k-1} \in f_n(x_{k-1}x_k) \setminus \{f_n(b)\}$ . Finally, we choose  $p_{k-1}p_k$  to be an irreducible arc in  $f_n(x_{k-1}b)$  from  $p_{k-1}$  to  $p_k = f_n(b)$ .

Note that the union  $p_0p_1 \cup p_1p_2 \cup \dots \cup p_{k-1}p_k$  is an arc with end points  $p_0 = f_n(a)$  and  $p_k = f_n(b)$ . We denote this arc by  $T_n = p_0p_k$ . For each  $i \in \{0, \dots, k-1\}$  we choose a homeomorphism from  $p_i p_{i+1}$  to  $x_i x_{i+1}$  that sends  $p_i$  to  $x_i$  and  $p_{i+1}$  to  $x_{i+1}$ . The combination of these homeomorphisms is the desired mapping  $g_n : T_n \rightarrow T$ .

Given  $\varepsilon > 0$ , observe that if our initial choice of points  $x_i$  is such that

$$\max\{\text{diam } x_i x_{i+1} : i \in \{0, \dots, k-1\}\} < \varepsilon/2,$$

and  $n$  is sufficiently large, then  $d(g_n) < \varepsilon$ . Note that the mappings  $g_n$  are homeomorphisms as required in the case  $\text{ord } T \leq 3$ . The case when  $T$  is an arc is therefore proved.

Suppose the lemma is proved for all trees having fewer than  $k$  end points, where  $k > 2$ , and assume that  $T$  has exactly  $k$  end points. Let  $a$  be a ramification point of  $T$ , and  $D_1, \dots, D_m$  be the closures of the components of  $T \setminus \{a\}$ . Then  $D_1, \dots, D_m$  are subtrees of  $T$  such that  $a$  is an end point of each  $D_i$ , and  $D_i$  has fewer than  $k$  end points for  $i \in \{1, \dots, m\}$ . According to the inductive assumption, there are trees  $D_{i,n} \subset f_n(D_i)$  and mappings  $g_{i,n} : D_{i,n} \rightarrow D_i$  satisfying the conclusion of the lemma for the mappings  $f_n|_{D_i}$ . In particular, we have  $\lim_n d(g_{i,n}) = 0$  for each  $i \in \{1, \dots, m\}$ . Let  $a_n = f_n(a)$ , and note that  $a_n$  is an end point of each  $D_{i,n}$ . Since the trees  $D_{i,n}$  approximate  $D_i$ , and the trees  $D_1, \dots, D_m$  are mutually disjoint except at  $a$ , there are compact connected neighborhoods  $V_n$  of  $a_n$  in  $D_{1,n} \cup \dots \cup D_{m,n}$  with  $\lim \text{diam } V_n = 0$  such that for each  $n$  the trees  $D_{1,n}, D_{2,n}, \dots, D_{m,n}$  are mutually disjoint except in  $V_n$ . For sufficiently large  $n$  we may additionally assume that  $V_n$  is the union of arcs  $A_{i,n} = a_n b_{i,n} \subset D_{i,n}$ , and  $b_{i,n} \notin A_{j,n}$  for  $i, j \in \{1, \dots, m\}$  with  $i \neq j$ . In the remaining part of the proof we only consider such indices  $n$ .

CASE 1. Assume  $m = 3$ . Let  $W_n$  be an arc in  $V_n$  from  $b_{1,n}$  to  $b_{2,n}$ , and  $Z_n$  be an arc in  $V_n$  irreducibly connecting  $b_{3,n}$  and  $W_n$ . The condition  $b_{i,n} \notin A_{j,n}$  for  $i \neq j$  implies that the union  $K_n = W_n \cup Z_n$  cannot be an arc, and thus  $K_n$  is a trioid in  $V_n$  irreducibly connecting  $b_{1,n}, b_{2,n}$ , and  $b_{3,n}$ . Let  $h_n : K_n \rightarrow T$  be an embedding such that  $h_n(b_{i,n}) = g_{i,n}(b_{i,n})$  for  $i \in \{1, 2, 3\}$ . We let  $T_n = ((D_{1,n} \cup D_{2,n} \cup D_{3,n}) \setminus V_n) \cup K_n$ , and we define a homeomorphism  $g_n : T_n \rightarrow T$  as follows:

$$g_n(x) = \begin{cases} g_{i,n}(x) & \text{for } x \in D_{i,n} \setminus V_n \text{ and } i \in \{1, 2, 3\}, \\ h_n(x) & \text{for } x \in K_n. \end{cases}$$

CASE 2. Assume  $m > 3$ . We have  $b_{i,n} \notin D_{j,n}$  for  $i \neq j$ . Therefore we can choose points  $c_{i,n} \in A_{i,n} \setminus \{a_n, b_{i,n}\}$  so near to  $b_{i,n}$  that letting  $C_{i,n}$  be the arc in  $A_{i,n}$  from  $a_n$  to  $c_{i,n}$ , and  $B_{i,n}$  the arc from  $c_{i,n}$  to  $b_{i,n}$ , we have

$D_{i,n} \cap D_{j,n} \subset C_{1,n} \cup \dots \cup C_{m,n}$  for  $i \neq j$ . The union  $U_n = C_{1,n} \cup \dots \cup C_{m,n}$  is a locally connected continuum, so it contains a tree  $L_n$  irreducibly connecting the points  $c_{1,n}, \dots, c_{m,n}$ . Define  $T_n = ((D_{1,n} \cup \dots \cup D_{m,n}) \setminus U_n) \cup L_n$ . Let  $h_{i,n} : B_{i,n} \rightarrow T$  be an embedding such that  $h_{i,n}(c_{i,n}) = a$  and  $h_{i,n}(b_{i,n}) = g_{i,n}(b_{i,n})$ . We define  $g_n : T_n \rightarrow T$  as follows:

$$g_n(x) = \begin{cases} g_{i,n}(x) & \text{for } x \in D_{i,n} \setminus V_n \text{ and } i \in \{1, \dots, m\}, \\ h_{i,n}(x) & \text{for } x \in B_{i,n} \text{ and } i \in \{1, \dots, m\}, \\ a & \text{for } x \in L_n. \blacksquare \end{cases}$$

Among consequences of Lemmas 3.11 and 3.12 we have Theorem 3.13 and Corollary 3.15 below, which are the most important results of this part of the paper. They show, for trees in spaces with the arc property of Kelley, stronger properties than the one proved in Lemma 3.11. The properties proved are applied in [6], [7] and [12].

**THEOREM 3.13.** *Let  $X$  be a space having the arc property of Kelley,  $T$  be a tree in  $X$ , and  $p$  be an end point of  $T$ . Then for each sequence  $\{p_n\}$  of points in  $X$  converging to  $p$  and for sufficiently large  $n$  there are trees  $T_n \subset X$  with  $p_n \in T_n$  and monotone mappings  $g_n : T_n \rightarrow T$  such that*

- (a)  $\lim d(g_n) = 0$ ;
- (b) for each  $x \in X$ , if  $g_n^{-1}(x)$  is nondegenerate, then  $\text{ord}(x, T) > 3$ .

**DEFINITION 3.14.** Let  $C, C_1, C_2, \dots$  be compacta in a metric space  $X$ , with corresponding points  $p \in C, p_1 \in C_1, p_2 \in C_2, \dots$ . We say that the pairs  $(C_n, p_n)$  converge homeomorphically to the pair  $(C, p)$  provided that there exists a sequence of homeomorphisms  $h_n : C \rightarrow C_n$  such that  $h_n(p) = p_n$  and  $\lim d(h_n) = 0$ .

Note that if the tree  $T$  in Theorem 3.13 is such that  $\text{ord } T \leq 3$ , then the mappings  $g_n$  are homeomorphisms by (b). Using the inverses  $g_n^{-1}$  we get the following.

**COROLLARY 3.15.** *Let  $X$  be a metric space having the arc property of Kelley,  $p$  be an end point of a tree  $T$  in  $X$  with  $\text{ord } T \leq 3$ , and  $\{p_n\}$  be a sequence of points in  $X$  converging to  $p$ . Then for almost all  $n \in \mathbb{N}$  there are trees  $T_n \subset X$  with  $p_n \in T_n$  such that the pairs  $(T_n, p_n)$  converge homeomorphically to the pair  $(T, p)$ .*

We end the paper with the following example which shows that the assumption  $\text{ord } T \leq 3$  in Corollary 3.15 is essential. This example also shows that the mappings  $g_n$  in Theorem 3.13 not always can be homeomorphisms. Recall that the union of four arcs each two of which are disjoint except at their common end point  $p$  is called a 4-od.

EXAMPLE 3.16. *There exist a continuum  $X$  which is in  $\text{AR}(\mathcal{HU})$  (so it has the arc property of Kelley by Corollary 3.7), a tree  $T \subset X$  with  $\text{ord } T > 3$ , an end point  $p \in T$ , and a sequence of points  $p_n \in X$  converging to  $p$  with no trees  $T_n$  in  $X$  such that the pairs  $(T_n, p_n)$  converge homeomorphically to the pair  $(T, p)$ .*

*Proof.* In the Euclidean 3-space define

$$A_0 = ([-1, 1] \times \{(0, 0)\}) \cup (\{0\} \times [-1, 1] \times \{0\}),$$

and for each  $n \in \mathbb{N}$  put

$$A_n = ([-1, 1] \times \{(0, 1/n)\}) \\ \cup (\{-1/n\} \times [-1, 0] \times \{1/n\}) \cup (\{1/n\} \times [0, 1] \times \{1/n\}).$$

Then  $A_0$  is a 4-od, for  $n > 1$  each  $A_n$  is homeomorphic to the letter H, and the sequence of continua  $A_n$  converges to  $A_0$ . Define

$$X = (\{(1, 0)\} \times [0, 1]) \cup A_0 \cup \bigcup \{A_n : n \in \mathbb{N}\}.$$

Then putting  $p = \langle -1, 0, 0 \rangle$ ,  $T = A_0$ , and  $p_n = \langle -1, 0, 1/n \rangle$  for each  $n \in \mathbb{N}$ , we see that in  $X$  there are no 4-ods  $T_n$  containing  $p_n$  and such that the pairs  $(T_n, p_n)$  converge homeomorphically to  $(T, p)$ . On the other hand, the continuum  $X$  has the arc property of Kelley, and by [5, Corollary 4.6] it is even in  $\text{AR}(\mathcal{HU})$ . ■

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