

Dendrites

Janusz J. Charatonik^{1,3} and Włodzimierz J. Charatonik^{2,3}

¹ Instituto de Matemáticas, UNAM
Circuito Exterior, Ciudad Universitaria
04510 México, D. F.
México
e-mail: jjc@hera.math.uni.wroc.pl
jjc@gauss.matem.unam.mx

² Departamento de Matemáticas
Facultad de Ciencias, UNAM
Circuito Exterior, Ciudad Universitaria
04510 México, D. F.
México
e-mail: wjcharat@hera.math.uni.wroc.pl
wjcharat@lya.fcencias.unam.mx

³ Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland

Abstract. A *dendrite* means a locally connected (metric) continuum containing no simple closed curve. Theorems, examples and questions related to this class of continua, dispersed in the literature, are collected and reminded. We start with known characterizations of dendrites: structural as well as mapping ones. Properties concerning various kinds of universal dendrites are discussed in the second section. Facts and problems on homogeneity of dendrites with respect to monotone mappings are next recalled. Theorems concerning chaotic and rigid dendrites are collected in the fourth section. In the next one we discuss some new results on the structure of dendrites with the set of their end points closed. Finally in sixth section some open problems concerning dendrites are recalled.

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All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. A *continuum* means a compact connected space. A property of a continuum X is said to be *hereditary* if every subcontinuum of X has the property. In particular, a continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. A *dendrite* means a locally connected continuum containing no simple closed curve. A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. A dendroid is locally connected if and only if it is a dendrite. The reader is referred to [18] for more information on acyclic curves (see also A.1 in the Appendix at the end of the paper).

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [48, §51, I, p. 274] and A.2 in the Appendix), and we denote by $\text{ord}(p, X)$ the order of the continuum X at a point $p \in X$. Points of order 1 in a continuum X are called *end points* of X ; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X ; the set of all ordinary points of X is denoted by $O(X)$. Points of order at least 3 are called *ramification points* of X ; the set of all ramification points of X is denoted by $R(X)$.

Let X and Y be continua. A mapping $f : X \rightarrow Y$ is said to be:

- a *retraction* provided that $Y \subset X$ and the restriction $f|_Y : Y \rightarrow f(Y) \subset X$ is the identity; then Y is called a *retract* of X ;
- *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$;
- *light* provided that $f^{-1}(y)$ has one-point components for each $y \in Y$ (note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional);
- *open* provided that f maps each open set in X onto an open set in Y ;
- *confluent* provided that for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ the equality $f(C) = Q$ holds;
- *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected.

Thus each monotone mapping is confluent. Also each open mapping is confluent [69, (7.5), p. 148].

Other concepts and notions undefined here (especially those exploited to characterize dendrites in Theorems 1.1 and 1.2) are used according to their definitions given in corresponding references. For the reader's convenience, according to a suggestion of the referee, an appendix is attached at the end of the paper that contains definitions of concepts used, but not defined, in the text. We refer to the n -th definition in the appendix as to A. n .

1. Characterizations.

The following structural characterizations of dendrites are known.

1.1 Theorem. *For a continuum X the following conditions are equivalent:*

- (1) X is a dendrite;
- (2) every two distinct points of X are separated by a third point;
- (3) each point of X is either a cut point (A.3) or an end point of X ;
- (4) each nondegenerate subcontinuum of X contains uncountably many cut points of X (A.3);
- (5) for each point $p \in X$ the number of components of the set $X \setminus \{p\}$ equals $\text{ord}(p, X)$ whenever either of these is finite;
- (6) the intersection of every two connected subsets of X is connected;
- (7) the space $L(X)$ of all locally connected subcontinua of X (A.4) with 0-regular convergence (A.5) is compact;
- (8) for each sequence of subcontinua of X , 0-regular convergence (A.5) is equivalent to the convergence with respect to the Hausdorff metric (A.4);
- (9) X is aposyndetic (A.6) and hereditarily unicoherent;
- (10) X is aposyndetic (A.6) and strongly unicoherent (A.7);
- (11) X is aposyndetic (A.6) and weakly hereditarily unicoherent (A.8);
- (12) there exists an arc-structure A on X (A.9), and (X, A) is arc-smooth at each point of X (A.10);
- (13) there exists an arc-structure A on X (A.9), and (X, A) is locally convex at each point of X (A.11);
- (14) there exists an arc-structure on X (A.9), and for each arc-structure A on X the values of the set function T_A are singletons only, i.e., $T_A(p) = \{p\}$ for each point $p \in X$ (A.12);
- (15) there exists an arc-structure on X (A.9), and for each point $p \in X$ there exists a metric on X which is radially convex at p (A.13);
- (16) for each $\varepsilon > 0$ X can be covered by finitely many subcontinua of diameter less than ε , and X is unicoherent at each of them (A.14);
- (17) for each point $p \in X$ and for each open set U containing p there exists an open connected set V such that $p \in V \subset U$ and X is unicoherent at $\text{cl} V$ (A.14);
- (18) X is locally connected and hereditarily unicoherent;
- (19) X is locally connected, one-dimensional and unicoherent;
- (20) X is locally connected and uniquely arcwise connected (A.15);
- (21) X is locally connected and each cyclic element of X (A.16) is a singleton;
- (22) X is locally connected and every convergent sequence of arcs in X converges to an arc or a singleton;
- (23) X is locally connected and the union of each increasing sequence of arcs in X is contained in an arc;

- (24) X is locally connected and tree-like (A.17);
- (25) X is locally connected and C^* -smooth (A.18);
- (26) X is locally connected and the hyperspace consisting of all singletons and arcs contained in X is compact;
- (27) X is a dendroid which is smooth at each of its points (A.19);
- (28) X is a dendroid which is weakly smooth at each of its points (A.20);
- (29) X is a neat dendroid (A.21 and A.22);
- (30) X is a dendroid, and for each point $p \in X$ the arc components of the set $X \setminus \{p\}$ are open;
- (31) X is a dendroid with the property of Kelley (A.23), and having no improper shore points (A.21);
- (32) X is a dendroid having the property of Kelley (A.23) hereditarily.

Proof: For conditions (1) through (5) see [69, (1.1), p. 88]. For (6) see [68, Corollary 7.32, p. 92]. Condition (7) is proved in [56, Theorem (15.10), p. 521]. Concerning (8) see [7, Theorem 8, p. 370] and [39, Theorem 1, p. 54]. For (9) see [2, Theorem 7, p. 588]. (10) is shown in [64, Theorem 3.7, p. 155]. Characterizations (11), (12) and (13) are proved in [36, Theorems I-2-E, p. 549, and I-3-C, p. 551]. Condition (14) is shown in [36, Theorem I-4-B, p. 551]. Further, (15) and (16) are shown in [64, Theorem 3.2 and Corollary 3.3, p. 153 and 154]. Condition (17) is shown in [54, Chapter X, §2, Theorems 1 and 2, p. 306] (compare also [48, §51, VI, Theorems 1 and 4, p. 300 and 301]). Concerning (18) see [48, §57, III, Corollary 8, p. 442]. For conditions (19) and (20) see [69, (1.2), p. 89]. For (21), (22) and (23) see [57, 10.46 (1) and (2), p. 188, and 10.50, p. 189], respectively. Conditions (24) and (25) are proved in [56, Theorem (15.11), p. 522, and Theorem (19.21), p. 610], respectively. For (26) see [32, Corollaries 4 and 5, p. 298 and 299]; and (27) is shown in [50, Theorem 8, p. 116]. Finally conditions (28)-(31) are proved to be equivalent to (0) in [63, Theorem 2.1, p. 940]. \square

Our next theorem collects known mapping characterizations of dendrites.

1.2 Theorem. *For a continuum X the following conditions are equivalent:*

- (1) X is a dendrite;
- (2) each subcontinuum of X is a monotone retract of X ;
- (3) each subcontinuum of X is a deformation retract of X (A.24);
- (4) each subcontinuum of X is a strong deformation retract of X (A.24);
- (5) for every compact space Y , for every mapping $F : X \rightarrow 2^Y$ with zero-dimensional values and for every point $(x_0, y_0) \in X \times Y$ with $y_0 \in F(x_0)$ there exists a continuous selection $f : X \rightarrow Y$ of F (A.26) such that $f(x_0) = y_0$;

- (6) for every compact space Y and for every mapping $F : X \rightarrow 2^Y$ with zero-dimensional values there exists a continuous selection $f : X \rightarrow Y$ of F (A.26);
- (7) for every compact space Y , for every light open mapping $f : Y \rightarrow f(Y)$ with $X \subset f(Y)$ and for every point $y_0 \in f^{-1}(X) \subset Y$ there exists a homeomorphic copy X' of X in Y with $y_0 \in X'$ such that the restriction $f|X' : X' \rightarrow f(X') = X$ is a homeomorphism;
- (8) for every compact space Y and for every light open mapping $f : Y \rightarrow f(Y)$ with $X \subset f(Y)$ there exists a homeomorphic copy X' of X in Y such that the restriction $f|X' : X' \rightarrow f(X') = X$ is a homeomorphism;
- (9) there exists an arc-structure on X (A.9), and for each point $p \in X$ there exists a $<_p$ -mapping $f : X \rightarrow [0, 1]$ (A.27) with $f(p) = 0$;
- (10) there exists an arc-structure A on X (A.9), and for each arc $A(x, y)$ in X there exists a \leq_x -retraction $r : X \rightarrow A(x, y)$ (A.27);
- (11) X is a one-dimensional absolute retract (A.28);
- (12) X is one-dimensional and there exists a feebly monotone mapping from a locally connected unicoherent continuum onto X ;
- (13) X is hereditarily unicoherent, and a function $I : 2^X \rightarrow C(X)$ that assigns to a closed subset A of X the continuum $I(A)$ irreducible with respect to containing A is continuous;
- (14) X is a dendroid, and for some $p \in X$ each subcontinuum of X admits a \leq_p -preserving retraction (A.27);
- (15) X is a smooth dendroid, and each subcontinuum of X is a retract of X ;
- (16) X is locally connected and for some (for each) hereditarily indecomposable continuum Y (A.29) there is an embedding of X in the hyperspace $C(Y)$ of subcontinua of Y ;
- (17) X is locally connected and for every one-to-one mapping f from $[0, 1)$ into X the set $\text{clf}([0, 1))$ is an arc;
- (18) X is locally connected and selectable (A.30);
- (19) X is locally connected and one-dimensional, and there exists a retraction from the hyperspace 2^X onto X (A.4);
- (20) X is locally connected and one-dimensional, and there exists a retraction from the hyperspace $C(X)$ onto X (A.4);
- (21) X is locally connected, one-dimensional and contractible (A.31);
- (22) X is locally connected, and each (continuous) mapping $F : X \rightarrow C(X)$ has a fixed point (A.25);
- (23) X is locally connected, and each upper semi-continuous function $F : X \rightarrow C(X)$ has a fixed point (A.25);

- (24) X is locally connected, and each (continuous) mapping $F : X \rightarrow 2^X$ has a fixed point (A.25);
- (25) X is locally connected, and for every two upper semi-continuous functions (A.25) $F_1 : X \rightarrow C(X)$ and $F_2 : X \rightarrow C(X)$ there are two points x_1 and x_2 in X such that $x_1 \in F_2(x_2)$ and $x_2 \in F_1(x_1)$;
- (26) X is locally connected, and for every upper semi-continuous function (A.25) $F : X \rightarrow C(X)$ and for every monotone surjective mapping $g : X \rightarrow X$ there is a point x in X such that $g(x) \in F(x)$;
- (27) X is locally connected, and every monotone mapping defined on X is hereditarily monotone (A.32);
- (28) X is locally connected, and for each nondegenerate subcontinuum P of X and for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each subcontinuum Q of X satisfying $H(P, Q) < \delta$ there exists a surjective ε -translation $g : P \rightarrow Q$ (A.33);
- (29) X is locally connected, and for each continuum Y if $f : X \times Y \rightarrow X$ denotes the natural projection (A.34), the induced mapping $C(f) : C(X \times Y) \rightarrow C(X)$ is open (A.35);
- (30) X is locally connected, and there exists a nondegenerate continuum Y such that if $f : X \times Y \rightarrow X$ denotes the natural projection, then the induced mapping $C(f) : C(X \times Y) \rightarrow C(X)$ is open.

Proof: Condition (1) is proved in [40, Theorem, p. 157]. For (2) and (3) see [42, Section 2, Theorem, p. 139]. Conditions (4)-(7) have been shown to be equivalent to (0) in [30, Corollary 10]. Concerning (8) and (9) see [36, Corollary I-7-D, p. 554]. Condition (10) is proved in [5, Corollary 13.5, p. 138]. For (11) see [24, Theorem 6.9]. For (12) see [38, Theorem 1, p. 3]. Condition (13) is shown in [51, Corollary 2.6, p. 337]. To argue (14) use (1) above and [51, Theorem 2.5, p. 334]. Condition (15) is proved in [56, Theorem (1.74.1), p. 129]. For (16) see [57, 10.46 (3), p. 188]. For (17) see [58, Corollary, p. 371]. To see (18) and (19) use (10) and [56, Theorem 6.4, p. 270]. To see (20) note that each one-dimensional and contractible continuum is a dendroid, [10, Proposition 1, p. 73], and use (17) of Theorem 1.1 above. For the opposite implication note that each dendrite is an absolute retract as stated in (10), and thereby it is contractible, [48, §54, VI, Theorem 3, p. 375]. Condition (21) is shown in [65, Theorem 3, p. 163]. Conditions (22) and (23) are equivalent to (0) as it has been shown in [66, Theorem 3, p. 164]. For a proof concerning (24) and (25) use (22) and the equivalence between (22), (24) and (25) shown in [9, (4.1), p. 337]. Characterization (26) of dendrites is shown in [31, Theorem 4.15, p. 11]. Finally (27), (28) and (29) are shown to be equivalent to (0) in [29, Corollary 35]. The proof is complete. \square

Besides the characterizations of dendrites collected in Theorems 1.1 and 1.2, we recall some other important properties of these continua.

1.3 Theorem.

- (a) *Every subcontinuum of a dendrite is a dendrite.*
- (b) *Every connected subset of a dendrite is arcwise connected.*

Proof: See e.g. [69, (1.3) (i) and (ii), p. 89]. □

1.4 Remark. a) It is a consequence of (a) of Theorem 1.3 that any property possessed by a dendrite is hereditary. In particular, each dendrite is hereditarily locally connected.

b) Note that (b) in Theorem 1.3 does not characterize dendrites: for example, any linear graph (as defined in [69, p. 182]), in particular the circle, has the same property. On the other hand, the property mentioned in assertion (b) cannot be extended from dendrites to dendroids, because if X stands for the harmonic fan, i.e., the cone over the compact countable set with exactly one limit point, and c denotes the mid point of the limit segment, then $X \setminus \{c\}$ is a connected, but not arcwise connected subset of X . For more information on connected subsets of dendrites the reader is referred to the authors' paper [25].

1.5 Theorem. *For every dendrite X the following assertions hold:*

- (a) $\dim E(X) = 0$ (hence $E(X) \neq \emptyset$);
- (b) $E(X) \in G_\delta$;
- (c) $O(X)$ is dense in X ;
- (d) $\text{card } R(X) \leq \aleph_0$ (i.e., every dendrite has at most countably many ramification points);
- (e) if $p \in R(X)$, then $\text{ord}(p, X) = m$ for some $m \in \{3, 4, \dots, \omega\}$ (A.2); i.e., no dendrite contains points of order \aleph_0 or \mathfrak{c} ;
- (f) $E(X)$ is dense if and only if $R(X)$ is dense.

Proof: Assertions (a), (b), (c), (d) and (e) are shown in [48, §51, V, Theorem 2, p. 292; II, Theorem 2, p. 278; VI, Theorems 8 and 7, p. 302; and Theorem 4, p. 301], respectively. Assertion (f) is proved in [17, Theorem 2.4, p. 167]. □

1.6 Remark. None of the properties listed in Theorem 1.5 can be extended from dendrites to dendroids, even if the notion of the order of a point is understood in the classical sense (see [8, p. 229] and A.36). In fact, a dendroid with one-dimensional set of end points has been constructed in [49, §9, p. 314] (see also [33], where some characterizations of the continuum are obtained, and [6]). A construction of a dendroid X for which $E(X)$ is not a G_δ -set is presented in [49, §8, p. 311]. Dendroids X without ordinary points, i.e., with $O(X) = \emptyset$ are constructed in [14, Theorems 1 and 2, p. 62

and 69]. See also [16]. Dendroids X for which the set $R(X)$ of their ramification points is uncountable (in fact, with $R(X)$ being a nondegenerate continuum) are known from [8, Example E 4, p. 240 and Corollary 6, p. 241]. It is known that for each dendroid X the condition $O(X) = \emptyset$ implies $\text{cl}R(X) = X$ [16, Statement 2, p. 322]; a dendroid with the above properties and else with the set $E(X)$ of end points closed has been constructed in [61, Chapter 9, Theorem 9.2, p. 97]. Dendroids containing points of order \aleph_0 or of order \mathfrak{c} (in any sense) are well-known: such are the harmonic fan described above, and the Cantor fan (i.e., the cone over the Cantor ternary set), respectively.

The property of being a dendrite is invariant under various classes of mappings. The next theorem summarizes the known results about this. The reader is referred to Table IV of Maćkowiak dissertation [52, Chapter 7, p. 69-70] for detailed argument. Note that the table contains also an information on some classes of mappings that do not preserve the property of being a dendrite. See A.37, A.32 and A.38 for definitions of classes of mappings mentioned below.

1.7 Theorem. *The image $f(X)$ of a dendrite X is again a dendrite provided that the mapping f is one of the following: 1) local homeomorphism; 2) hereditarily monotone; 3) atomic; 4) monotone; 5) open; 6) MO-mapping; 7) locally monotone; 8) locally MO-mapping; 9) OM-mapping; 10) hereditarily confluent; 11) quasi-monotone; 12) weakly monotone; 13) confluent; 14) locally confluent; 15) semi-confluent; 16) hereditarily weakly confluent.*

2. Universal dendrites.

Let a class \mathcal{S} of spaces be given. A member U of \mathcal{S} is said to be *universal for \mathcal{S}* if every member of \mathcal{S} can be embedded in U , i.e., if for every $X \in \mathcal{S}$ there exists a homeomorphism $h : X \rightarrow h(X) \subset U$. Accordingly, a dendrite is said to be *universal* if it contains a homeomorphic image of any other dendrite. Similarly, if the order of each point of a dendrite U is bounded by a number $m \in \{3, 4, \dots, \omega\}$, and U contains homeomorphic copies of all other dendrites whose points have orders not greater than m , then U is called a *universal dendrite of order m* . Thus, since no dendrite contains points of order exceeding ω (according to (e) of Theorem 1.5), a universal dendrite of order ω is a universal one in the sense of the former definition.

Observe that if a space V contains a universal space U , then V is universal itself. Applying this observation to universal dendrites of order $m \in \{3, 4, \dots, \omega\}$, to avoid any confusion with other universal dendrites, we will consider some special universal dendrites whose definition is taken from [34, Section 6, p. 228].

For a given nonempty set $S \subset \{3, 4, \dots, \omega\}$ we denote by D_S any dendrite X satisfying the following two conditions:

$$(2.1) \text{ if } p \in R(X), \text{ then } \text{ord}(p, X) \in S;$$

- (2.2) for each arc A contained in X and for every $m \in S$ there is in A a point p with $\text{ord}(p, X) = m$.

It is shown in [34, Theorem 6.2, p. 229] that the dendrite D_S is topologically unique:

- (2.3) If two dendrites satisfy conditions (2.1) and (2.2) with the same set $S \subset \{3, 4, \dots, \omega\}$, then they are homeomorphic.

If $S = \{m\}$ for some $m \in \{3, 4, \dots, \omega\}$, then we will write simply D_m in place of $D_{\{m\}}$. The dendrite D_m is called the *standard universal dendrite of order m* . In particular, D_ω is called the *standard universal dendrite*. Its construction, known from Ważewski's doctoral dissertation [67, Chapter K, p. 137], and simplified by Menger in [54, Chapter X, §6, p. 318], has been presented in full in a more contemporary way in [17, p. 168]. Another description of these continua, for finite m , which uses inverse limits (A.40) of inverse sequences of trees (i.e., dendrites having a finite number of end points only) with monotone onto bonding mappings is given in [13, p. 491]. Observe that since Ważewski's dendrite D_ω has been constructed in the plane, every dendrite is planable, i.e., it can be homeomorphically embedded in the plane.

According to (2.3) for each $m \in \{3, 4, \dots, \omega\}$ the standard universal dendrite D_m is characterized by the following two conditions:

- (2.4) each ramification point of D_m is of order m , and
 (2.5) for each arc A contained in X the set of all ramification points of D_m which belong to A is a dense subset of A .

The following universality properties of the dendrites D_S for a nonempty set $S \subset \{3, 4, \dots, \omega\}$ are known (see [34, Theorems 6.6-6.8, p. 230]).

- (2.6) If $\omega \in S$, then the dendrite D_S is universal.
 (2.7) If the set S is finite with $\max S = m$, then D_S is universal for the family of all dendrites having orders of ramification points at most m .
 (2.8) If the set S is infinite and $\omega \notin S$, then D_S is universal in the family of all dendrites having finite orders of ramification points.

The above universality properties of dendrites D_S together with the uniqueness property (2.3) justify their name: given a nonempty set $S \subset \{3, 4, \dots, \omega\}$, the dendrite D_S is called the *standard universal dendrite of orders in S* .

Recall that an *open connected domain* in a space X means a connected subspace of X that equals the interior of its closure in X . The following result is known (see [19, Proposition 3.1, p. 461]).

2.9 Proposition. *For each set $S \subset \{3, 4, \dots, \omega\}$ the closure of any open connected subdomain of the dendrite D_S is homeomorphic to D_S .*

Homeomorphisms of universal dendrites were studied in [17, Sections 4 and 5, p. 171-178], and in [19]. We recall the main results proved there.

2.10 Theorem. [19, Theorem 3.13, p. 464]. *Let a nonempty set $S \subset \{3, 4, \dots, \omega\}$ be given, and let x and y be two points of the standard universal dendrite D_S of orders in S . Then there is a homeomorphism $h : D_S \rightarrow D_S$ such that $h(x) = y$ if and only if $\text{ord}(x, D_S) = \text{ord}(y, D_S)$.*

2.11 Theorem. [19, Theorem 3.16, p. 466]. *Let $X = D_S$ be the standard universal dendrite D_S of orders in a nonempty set $S \subset \{3, 4, \dots, \omega\}$. Then*

- (a) *the group $\mathcal{H}(X)$ of autohomeomorphisms of X has exactly n orbits, where $n = 2 + \text{card } S$ (A.39);*
- (b) *for each orbit B of X and for each arc $A \subset X$ the intersection $A \cap B$ is a dense subset of A ;*
- (c) *each orbit is a dense subset of X .*

Conditions (a) and (b) of Theorem 2.11 characterize standard universal dendrites D_S in the sense of the following theorem which is another formulation of (2.3).

2.12 Theorem. [19, Theorem 3.22, p. 467]. *Let a dendrite X satisfy the following conditions:*

- (a) *the group $\mathcal{H}(X)$ of autohomeomorphisms of X has exactly n orbits, where $n \geq 3$ (A.39);*
- (b) *for each orbit B of X and for each arc $A \subset X$ the intersection $A \cap B$ is a dense subset of A .*

Then X is homeomorphic to D_S , for some $S \subset \{3, 4, \dots, \omega\}$ and $\text{card } S = n - 2$. In particular, if $n = 3$, then X is homeomorphic to the standard universal dendrite D_m of order m for some $m \in \{3, 4, \dots, \omega\}$.

Two important classes of mappings relatively close to that of homeomorphisms are open mappings and monotone mappings. For the class of open mappings, when considered on universal dendrites, recall the following result (see [34, Corollary 6.10, p. 232] and compare [13, Corollary, p. 491]) which resembles the well-known property of an arc to be homeomorphic to any nonconstant open image of the space, [69, (1.3), p. 184].

2.13 Theorem. *Nonconstant open images of standard universal dendrites D_S are homeomorphic to D_S if and only if S is a nonempty subset of $\{3, \omega\}$.*

In connection with this result remind that the following problem, formulated in [13, p. 491] is still open.

2.14 Problem. Characterize all dendrites X having the property that every open image of X is homeomorphic to X .

Three uncountable families of dendrites having the above property are constructed in [31, Theorem 6.45 and Remark 6.47, p. 30]. The universal dendrites either D_3 or D_ω or $D_{\{3,\omega\}}$ are used as building blocks in the construction of each member of any of these families.

Monotone mappings of universal dendrites were studied in [45], [17] and also in [31]. Kato has shown in [45, Proposition 2.3, p. 222] that a surjection $f : D_3 \rightarrow D_3$ is monotone if and only if it is a near homeomorphism. Recall that a mapping $f : X \rightarrow Y$ is called a *near homeomorphism* provided that for each $\varepsilon > 0$ there is a homeomorphism $h : X \rightarrow Y$ such that $\sup\{d(f(x), h(x)) : x \in X\} < \varepsilon$. For a more general formulation of his result see [17, Proposition 5.2, p. 176]. The result has been strengthened as follows in [17, Corollary 5.5, p. 178].

2.15 Theorem. *Let D_m be the standard universal dendrite of some order $m \in \{3, 4, \dots, \omega\}$. Then each monotone surjection of D_m onto itself is a near homeomorphism if and only if $m = 3$.*

In connection with this result recall that the following Problem 5.1 of [17, p. 176] is still open.

2.16 Problem. Characterize all dendrites X having the property that each monotone mapping of X onto itself is a near homeomorphism.

The next result on monotone mappings of universal dendrites has been shown in [17, Corollary 6.5, p. 180]. For a more general statement see Theorem 2.22 below.

2.17 Theorem. *Every dendrite is the image of any standard universal dendrite D_m for $m \in \{3, 4, \dots, \omega\}$ under a monotone mapping.*

Let \mathfrak{M} be a class of mappings. Two spaces X and Y are said to be *equivalent with respect to \mathfrak{M}* (shortly *\mathfrak{M} -equivalent*) if there are two mappings, both in \mathfrak{M} , one from X onto Y and the other from Y onto X . If \mathfrak{M} means the class of monotone mappings, we say that X and Y are *monotonely equivalent*. Monotone equivalence of universal dendrites is discussed in Section 6 of [17, p. 178-185]. We recall the main results proved there. As a consequence of Theorem 2.17 we see the following assertion [17, Corollary 6.6, p. 180].

2.18 Corollary. *Any two standard universal dendrites D_m and D_n of some orders $m, n \in \{3, 4, \dots, \omega\}$ are monotonely equivalent.*

However, the family of all dendrites which are monotonely equivalent to a dendrite D_m contains also other members. To characterize the family we have to construct a special dendrite, denoted L_0 (see [17, Example 6.9, p. 182]). It is defined as the closure of the union of an increasing sequence of dendrites in the plane. We start with the unit straight line segment denoted by L_1 . Divide L_1 into three equal subsegments and in the middle of them, M , locate a thrice diminished copy of the Cantor ternary set C . At the mid point of each contiguous interval K to C (i.e., a component K of $M \setminus C$) we erect perpendicularly to L_1 a straight line segment whose length equals length of K . Denote by L_2 the union of L_1 and of all erected segments (there are countably many of them, and their lengths tend to zero). We perform the same construction on each of the added segments: divide such a segment into three equal parts, locate in the middle part M a copy of the Cantor set C properly diminished, at the mid point of any component K of $M \setminus C$ construct a perpendicular to K segment as long as K is, and denote by L_3 the union of L_2 and of all attached segments. Continuing in this manner we get a sequence of dendrites $L_1 \subset L_2 \subset L_3 \subset \dots$. Finally we put

$$L_0 = \text{cl} \left(\bigcup \{L_i : i \in \mathbb{N}\} \right). \quad (2.19)$$

Note that all the ramification points of L_0 are of order 3, and the set $R(L_0)$ is discrete. The above mentioned characterization runs as follows (see [17, Theorem 6.14, p. 185] and [31, Theorem 5.35, p. 17]).

2.20 Theorem. *The following conditions are equivalent for a dendrite X :*

- (a) *X is monotonely equivalent to D_3 ;*
- (b) *X is monotonely equivalent to D_ω ;*
- (c) *X is monotonely equivalent to D_m for each $m \in \{3, 4, \dots, \omega\}$;*
- (d) *X is monotonely equivalent to D_S for each nonempty set $S \subset \{3, 4, \dots, \omega\}$;*
- (e) *X is monotonely equivalent to every dendrite Y having dense set $R(Y)$ of its ramification points;*
- (f) *X is monotonely equivalent to some dendrite Y having dense set $R(Y)$ of its ramification points;*
- (g) *X contains a homeomorphic copy of every dendrite L such that its set $R(L)$ of ramification points is discrete and consists of points of order 3 exclusively;*
- (h) *X contains a homeomorphic copy of the dendrite L_0 defined by (2.19).*

2.21 Remark. a) If the concept of the equivalence with respect to the class of open mappings is applied to universal dendrites, then it follows from Theorem 2.13 that a dendrite is openly equivalent to D_3 (to D_ω) if and only if it is homeomorphic to D_3 (to D_ω , respectively).

b) A class of mappings that is larger than class of monotone mappings is one of confluent mappings. So, one can ask on a characterization of the family of dendrites that

are confluently equivalent to D_3 , for example. However, it is shown in [31, Proposition 5.6, p. 14] that there exists a monotone mapping between dendrites X and Y if and only if there exists a confluent mapping between X and Y . Thus “monotonely equivalent” can be replaced by “confluently equivalent” in Theorem 2.20.

c) Since a dendrite has a dense set of its ramification points if and only if the set of its end points is dense (see (f) of Theorem 1.5), density of $R(Y)$ can be replaced by density of $E(Y)$ in (e) and (f) of Theorem 2.20.

As a consequence of Theorem 2.20 we get the following generalizations of Theorem 2.17 and Corollary 2.18.

2.22 Theorem. *Every dendrite is the image of any standard universal dendrite D_S of orders in a nonempty set $S \subset \{3, 4, \dots, \omega\}$ under a monotone mapping.*

2.23 Corollary. *Any two standard universal dendrites D_{S_1} and D_{S_2} of orders in nonempty sets $S_1, S_2 \subset \{3, 4, \dots, \omega\}$ are monotone equivalent.*

Some results concerning universal elements for the class of dendrites with the set of their end points closed are mentioned in Section 5 (Theorems 5.5 and 5.6).

3. Monotone homogeneity of dendrites.

Let \mathfrak{M} be a class of mappings. A space X is said to be *homogeneous with respect to \mathfrak{M}* (or shortly *\mathfrak{M} -homogeneous*) provided that for every two points p and q of X there is a surjective mapping $f : X \rightarrow X$ such that $f(p) = q$ and $f \in \mathfrak{M}$. If \mathfrak{M} is the class of homeomorphisms, we get the concept of a *homogeneous* space. A class \mathfrak{M} of mappings is said to be *neat* provided that if all homeomorphisms are in \mathfrak{M} and the composition of any two mappings in \mathfrak{M} is also in \mathfrak{M} . A connection between \mathfrak{M} -equivalence and \mathfrak{M} -homogeneity is given in the following statement (see [27, Statement 12, p. 364]).

3.1 Statement. Let \mathfrak{M} be a neat class of mappings between continua. If a continuum X is \mathfrak{M} -equivalent to an \mathfrak{M} -homogeneous continuum Y , then X is \mathfrak{M} -homogeneous.

If the spaces under consideration are dendrites, and we study their homogeneity with respect to a class \mathfrak{M} of mappings, then \mathfrak{M} cannot be the class of homeomorphisms, because the only planar locally connected homogeneous continuum is the simple closed curve, [53, p. 137]. Also the class of open mappings is out of discussion, since no dendroid, in particular no dendrite, is openly homogeneous, [15, Theorem, p. 409]. A number of interesting results are obtained if \mathfrak{M} stands for the class of monotone mappings.

Kato has proved (see [44, Example 2.4, p. 59] and [45, Proposition 2.4, p. 223]) that the standard universal dendrite D_3 of order 3 is monotone homogeneous. This result has

first been extended in [17, Theorem 7.1, p. 186] to all standard universal dendrites D_m of order m for each $m \in \{3, 4, \dots, \omega\}$, and next it has been generalized to all dendrites D_S in Theorem 3.3 of [20, p. 292] as follows (see also [27, Proposition 18, p. 365]).

3.2 Theorem. *For each nonempty set $S \subset \{3, 4, \dots, \omega\}$ the standard universal dendrite D_S of orders in S is monotonely homogeneous.*

Consequently, there are uncountably many topologically different monotonely homogeneous dendrites, [20, Corollary 3.8, p. 293]. Further, it follows from Statement 3.1 and Theorems 3.2 and 2.22 that if a dendrite X is monotonely equivalent to some dendrite D_S , then X is monotonely homogeneous. However, in the monotone equivalence between X and D_S only one mapping is essential, because by Theorem 2.22 for every dendrite X there is a monotone mapping of D_S onto X . Thus we have the following proposition that generalizes Corollary 14 of [27, p. 364].

3.3 Proposition. *If there exists a monotone mapping of a dendrite X onto D_S for some nonempty set $S \subset \{3, 4, \dots, \omega\}$, then X is monotonely homogeneous.*

Another sufficient condition for monotone homogeneity of a dendrite is the following (see [27, Proposition 15, p. 364]).

3.4 Proposition. *If, for a dendrite X , the set $R(X)$ of its ramification points is dense in X , then X is monotonely homogeneous.*

The condition $\text{cl } R(X) = X$, being sufficient, is far from being necessary. Namely the dendrite L_0 defined by (2.19) has the set $R(L_0)$ discrete, and it is monotonely homogeneous. Moreover, by the equivalence of conditions (a) and (h) of Theorem 2.20 the following statement holds, [27, Proposition 20, p. 366].

3.5 Proposition. *If a dendrite X contains a homeomorphic copy of the dendrite L_0 defined by (2.19), then X is monotonely homogeneous.*

It would be interesting to know if the converse to Proposition 3.5 holds true, i.e., if containing the dendrite L_0 characterizes monotonely homogeneous dendrites. In other words, we have the following question.

3.6 Question. Does every monotonely homogeneous dendrite contains a homeomorphic copy of the dendrite L_0 defined by (2.19)?

The above question is closely related to a more general problem.

3.7 Problem. Give any structural characterization of monotonely homogeneous dendrites.

- (B) Do chaotic spaces of the cardinality of the continuum exist?
 (C) Do completely normal, connected and locally connected chaotic spaces exist?

Questions (A) and (B) have been answered in the affirmative by Berney [3], who has constructed — assuming the continuum hypothesis — an example of a separable metric space X of the continuum cardinality, linearly ordered (the linear order inducing the topology on X ; in fact, X is a subspace of the unit interval of reals), and chaotic. The reader is referred to [12, Chapter III, p. 223] for more information about constructive examples of chaotic spaces described as early as in the twenties.

In 1925 Zarankiewicz [71] asked the following question.

- (a) Is every dendrite homeomorphic to some proper subset of itself?

The question has been answered in the negative in 1932 by Miller [55] who has constructed a dendrite that was not homeomorphic to any proper subset of itself. The Miller's dendrite was not, however, a chaotic space since it contained some open arcs as its open subsets. An answer to all four questions (A)-(D) in one example is given in [12, Statement 13, p. 231]. An outline of the construction is given in [11]. Below we summarize properties of this example.

4.2 Theorem. *There exists a dendrite D such that*

- (a) $\text{ord}(p, D) \leq 4$ for each point $p \in D$;
- (b) for each $n \in \{1, 2, 3, 4\}$ the set of all points of D which are of order n is dense in D ;
- (c) D is strongly rigid;
- (d) D is chaotic.

A further progress concerning the structural properties of chaotic dendrites has been made in [22], where the following result is shown [22, Theorem 13].

4.3 Theorem. *If a dendrite has a dense set of its ramification points, and if it contains at most one ramification point of order n for each $n \in \{3, 4, \dots, \omega\}$, then it is chaotic but not strongly rigid.*

The simplest example of a chaotic dendrite that satisfies assumptions of Theorem 4.3 is due to Johann J. de Iongh. Outline of its construction is in [41, p. 443], and is recalled in [12, Section 5, pp. 227-228]. The dendrite has exactly one ramification point of each finite order, and it does not contain points of order ω .

The concept of a strongly chaotic space has been introduced in [26], where various conditions are shown to be either necessary or sufficient for a dendrite to be strongly chaotic. The following result is shown as Theorem 5.5 of [26, p. 185].

4.4 Theorem. *For any two integers m and n with $3 \leq n < m$ there exists a strongly chaotic dendrite X containing ramification points of orders m and n only, the sets of which, as well as the set of end points of X , are dense in X , and such that each arc in X contains points of order n .*

Solutions of the following problems seem to be important for a future research in the area.

4.5 Problem. Give any structural characterization of (strongly) chaotic and of (strongly) rigid dendrites.

The four above discussed concepts of (strongly) chaotic and (strongly) rigid spaces have been generalized in [23] in such a way that homeomorphisms that appear in their definitions are replaced by some particular mappings as monotone or open, for example. It is shown that no dendrite is rigid (and hence either strongly rigid, or chaotic, or strongly chaotic) with respect to monotone mappings, [23, Corollary 4.17]. A chaotic but not openly chaotic dendrite is constructed in [23, Theorem 5.11]. A modification of a dendrite mentioned in Theorem 4.4 above leads to a construction of a dendrite which is chaotic, not strongly chaotic, strongly rigid and openly rigid, [23, Theorem 5.26]. A countable family of strongly chaotic and openly rigid dendrites is constructed in [23, Theorem 5.36].

5. Dendrites with the set of end points closed.

All results contained in this section have recently been obtained in [1]. We start with some notation.

Let F_ω denote the union of countably many straight line segments in the plane any two of which intersect at their common point v only and such that for each $\varepsilon > 0$ at most finitely many segments have lengths greater than ε . In other words, F_ω is a dendrite such that $R(F_\omega) = \{v\}$ and $\text{ord}(v, F_\omega) = \omega$.

Given two points p and q in the plane, we denote by pq the straight line segment with end points p and q . For each $n \in \mathbb{N}$ put $a_n = (1/n, 1/n)$, $b_n = (1/n, 0)$, $c = (-1, 0)$, and define

$$W = cb_1 \cup \bigcup \{a_n b_n : n \in \mathbb{N}\}.$$

Then W is a dendrite.

Note that neither $E(W)$ nor $E(F_\omega)$ are closed. Moreover, the following theorem characterizes dendrites with the set of their end points closed.

5.1 Theorem. *A dendrite has the set of its end points closed if and only if it contains no homeomorphic copy of either F_ω or W .*

(The property of being a dendrite with its set of end points closed is hereditary, i.e., each subcontinuum of a dendrite with its set of end points closed is a dendrite with its set of end points closed.)

5.2 Corollary. *End points closed is hereditary, i.e., each subcontinuum of a dendrite with its set of end points closed is a dendrite with its set of end points closed.*

One of the classical examples of dendrites with its set of end points closed is the *Gehman dendrite*. It can be described as a dendrite G having the set $E(G)$ of its end points homeomorphic to the Cantor ternary set C in $[0, 1]$, such that all ramification points of G are of order 3 (see [37, the example on p. 42]; see also [59, pp. 422-423] for a detailed description, and [60, Fig. 1, p. 203] and [18, Fig. 1, p. 5] for a picture). This concept has been generalized as follows.

5.3 Theorem. *For each natural number $n \geq 3$ there exists one and only one dendrite G_n such that*

- (a) $\text{ord}(p, G_n) = n$ for each point $p \in R(G_n)$;
- (b) $E(G_n)$ is homeomorphic to the Cantor ternary set.

Note that G_3 is just the Gehman dendrite G .

5.4 Theorem. *There exists a dendrite G_ω such that*

- (a) $\text{ord}(p, G_\omega)$ is finite for each point $p \in G_\omega$;
- (b) $E(G_\omega)$ is homeomorphic to the Cantor ternary set;
- (c) for each natural number n and for each maximal arc A contained in G_ω there is a point $q \in A$ such that $\text{ord}(q, G_\omega) \geq n$.

The dendrites G_n and G_ω have the following universality properties.

5.5 Theorem. *For each natural number $n \geq 3$ the dendrite G_n is universal for the class of all dendrites X such that $\text{cl} E(X) = E(X)$ and that $\text{ord}(p, X) \leq n$ for each point $p \in X$.*

5.6 Theorem. *Each dendrite G_ω is universal for the class of all dendrites X such that $\text{cl} E(X) = E(X)$.*

Finally let us cite two results that concern mapping properties of the considered class of dendrites. The former theorem implies that the discussed property of being a dendrite with its set of end points closed is invariant with respect to open mappings.

5.7 Theorem. *A dendrite Y has its set of end points closed if and only if there are a dendrite X with the set $E(X)$ of its end points closed and an open surjective mapping $f : X \rightarrow Y$.*

5.8 Theorem. *A dendrite Y does not contain any homeomorphic copy of the dendrite W if and only if there are a dendrite X with the set $E(X)$ of its end points closed and a monotone surjective mapping $f : X \rightarrow Y$.*

We close this section with recalling an application of the concept of the Gehman dendrite to the dynamical systems theory. To formulate the result some definitions are in order first.

Let a continuum X and a mapping $f : X \rightarrow X$ be given. For each natural number n denote by f^n the n -th iteration of f . A point $p \in X$ is called:

- a *periodic point of f* provided that there is $n \in \mathbb{N}$ such that $f^n(p) = p$;
- a *recurrent point of f* provided that for every neighborhood U of p there is $n \in \mathbb{N}$ such that $f^n(p) \in U$.

The set of periodic points and of recurrent points of a mapping $f : X \rightarrow X$ are denoted by $P(f)$ and $R(f)$ respectively. Clearly $P(f) \subset R(f)$. A continuum X is said to have the *periodic-recurrent property* (shortly *PR-property*) provided that for every mapping $f : X \rightarrow X$ the equality $\text{cl}P(f) = \text{cl}R(f)$ holds. The reader is referred to [21] and [28] and references therein for more information about this property. It has been shown in [70, Theorem 2.6, p. 349] that every tree has the PR-property. Kato has shown in [46] that this result cannot be extended to all dendrites because the Gehman dendrite does not have the PR-property. And recently Illanes has proved in [43, Theorem 2] the following characterization of dendrites with the PR-property.

5.9 Theorem. *A dendrite has the periodic-recurrent property if and only if it does not contain any homeomorphic copy of the Gehman dendrite.*

6. Problems.

In this section we recall some open problems concerning dendrites and mappings between them.

As it was already pointed out in Question 3.6 and Problem 3.7, the problem of finding any structural characterization of monotonely homogeneous dendrites can be solved if we will show that every monotonely homogeneous dendrite contains a homeomorphic copy of the dendrite L_0 defined by (2.19), i.e., if Question 3.6 has a positive answer.

Similarly, we do not have any internal characterization of (strongly) chaotic and of (strongly) rigid dendrites (Problems 4.5).

Besides, there are some open problems that concern mapping hierarchies of dendrites, see [31, Chapter 7, p. 51]. We recall some of them without using the notion of a mapping hierarchy. 1) As it was stated in Problem 2.14, we do not know any characterization of dendrites whose open images are homeomorphic to them.

2) We say that a dendrite X is *openly minimal* provided that every open image of X can be openly mapped onto X . Thus, in particular, all dendrites which are

homeomorphic to their open images are openly minimal. We do not know if the converse is true, [31, Q2(\mathbb{O}), p. 51].

3) No structural characterization of openly minimal dendrites is known, [31, Q1(\mathbb{O}), p. 51].

We finish the paper with the following question (see [31, Q3(a) (\mathbb{M}), p. 51]).

6.1 Question. Is there an infinite set \mathcal{A} of dendrites such that for any two distinct elements of \mathcal{A} there is no monotone surjection between them?

Appendix.

This appendix contains definitions of concepts used but not included in the text, in particular those ones that are exploited to characterize dendrites in Theorems 1.1 and 1.2. They are ordered as they appear in the text for the first time.

A.1. A continuum X is said to be *acyclic* provided that each mapping from X into the unit circle \mathbb{U}^1 is homotopic to a constant mapping, i.e., for all mappings $f : X \rightarrow \mathbb{U}^1$ and $c : X \rightarrow \{p\} \subset \mathbb{U}^1$ there exists a mapping $h : X \times [0, 1] \rightarrow \mathbb{U}^1$ such that for each point $x \in X$ we have $h(x, 0) = f(x)$ and $h(x, 1) = c(x)$. By a *curve* we mean a one-dimensional continuum. It is known that every acyclic curve is hereditarily unicoherent, see property (b) in [69, p. 226].

A.2. A concept of an *order* of a point p in a continuum X (in the sense of Menger-Urysohn), written $\text{ord}(p, X)$, is defined as follows. Let \mathfrak{n} stand for a cardinal number. We write:

$\text{ord}(p, X) \leq \mathfrak{n}$ provided that for every $\varepsilon > 0$ there is an open neighborhood U of p such that $\text{diam } U \leq \varepsilon$ and $\text{card } \text{bd } U \leq \mathfrak{n}$;

$\text{ord}(p, X) = \mathfrak{n}$ provided that $\text{ord}(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\text{ord}(p, X) \leq \mathfrak{m}$ does not hold;

$\text{ord}(p, X) = \omega$ provided that the point p has arbitrarily small open neighborhoods U with finite boundaries $\text{bd } U$ and $\text{card } \text{bd } U$ is not bounded by any $n \in \mathbb{N}$.

Thus, for any continuum X we have

$$\text{ord}(p, X) \in \{1, 2, \dots, \mathfrak{n}, \dots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [48, §51, I, p. 274].

A.3. A point p of a continuum X is called a *cut point* of X provided that $X \setminus \{p\}$ is not connected (see [69, p. 41]).

A.4. Given a continuum X with a metric d , we let 2^X to denote the *hyperspace* of all nonempty closed subsets of X equipped with the *Hausdorff metric* H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\} \quad \text{for } A, B \in 2^X$$

(see e.g. [56, (0.1), p. 1 and (0.12), p. 10] for an equivalent formulation). If $H(A, A_n)$ tends to zero as n tends to infinity, we put $A = \text{Lim } A_n$. Further, we denote by $F_1(X)$ the hyperspace of singletons of X , and by $C(X)$ the hyperspace of all subcontinua of X , i.e., of all connected elements of 2^X . Since X is homeomorphic to $F_1(X)$, there is a natural embedding of X into $C(X)$, and so we can write $X \subset C(X) \subset 2^X$. Thus one can consider a retraction from either $C(X)$ or 2^X onto X .

A.5. A sequence of sets $\{A_n\}$ contained in a metric space X with a metric d is said to *converge 0-regularly* to its limit $A = \text{Lim } A_n$ (see A.4) provided that for each $\varepsilon > 0$ there is a $\delta > 0$ and there is an index $n_0 \in \mathbb{N}$ such that if $n > n_0$ then for every two points $p, q \in A_n$ with $d(p, q) < \delta$ there is a connected set $C_n \subset A_n$ satisfying conditions $p, q \in C_n$ and $\text{diam } C_n < \varepsilon$ (see [69, Chapter 9, §3, p. 174]).

A.6. A continuum X is said to be *aposyndetic* provided that for each point $p \in X$ and for each $q \in X \setminus \{p\}$ there exists a subcontinuum K of X and an open set U of X such that $p \in U \subset K \subset X \setminus \{q\}$ (see e.g. [57, Exercise 1.22, p. 12]).

A.7. A continuum X is said to be *strongly unicoherent* provided that it is unicoherent and for each pair of its proper subcontinua A and B such that $X = A \cup B$, each of A and B is unicoherent (see [2, p. 587]).

A.8. A continuum is said to be *weakly hereditarily unicoherent* if the intersection of any two of its subcontinua with nonempty interiors is connected (see [64, p. 152] and references therein).

A.9. By an *arc-structure* on an arbitrary space X we understand a function $A : X \times X \rightarrow C(X)$ such that for every two distinct points x and y in X the set $A(x, y)$ is an arc from x to y and that the following metric-like axioms are satisfied for every points x, y and z in X :

- (1) $A(x, x) = \{x\}$;
 - (2) $A(x, y) = A(y, x)$;
 - (3) $A(x, z) \subset A(x, y) \cup A(y, z)$,
- with equality prevailing whenever $y \in A(x, z)$.

We put (X, A) to denote that the space X is equipped with an arc-structure A (see [36, p. 546]). Note that if there exists an arc-structure on a continuum, then the continuum is arcwise connected.

A.10. Given a continuum X with an arc-structure A (see A.9), the pair (X, A) (see A.9) is said to be *arc-smooth at a point* $v \in X$ provided that the induced function $A_v : X \rightarrow C(X)$ defined by $A_v(x) = A(v, x)$ is continuous. Then the point v is called an *initial point* of (X, A) . The pair (X, A) is said to be *arc-smooth* provided that there exists a point in X at which (X, A) is arc-smooth. An arbitrary space X is said to be *arc-smooth at a point* $v \in X$ provided that there exists an arc-structure A on X for which (X, A) is arc-smooth at v . The space X is said to be *arc-smooth* if it is arc-smooth at some point (see [36, p. 546]). Note that a dendroid is smooth (A.19) if and only if it is arc-smooth.

A.11. Given a continuum X with an arc-structure A (see A.9), a subset Z of X is said to be *convex* provided that for each pair of points x and y of Z the arc $A(x, y)$ is a subset of Z . If Z is a convex subcontinuum of X , then $A|Z \times Z$ is an arc-structure on Z . We define X to be *locally convex at a point* $p \in X$ provided that for each open set U containing p there is a convex set Z such that $p \in \text{int } Z \subset \text{cl } Z \subset U$ (see [36, I.2, p. 548-549]).

A.12. For each point x of a continuum X with an arc-structure A (see A.9) we define (see [36, I.3, p. 550])

$$T_A(x) = \{y \in X : \text{each convex subcontinuum of } X \\ \text{with } y \text{ in its interior contains } x\}.$$

Since $T_A(x)$ is always closed, we have $T_A : X \rightarrow 2^X$.

A.13. Given a continuum X with an arc-structure A (see A.9) and a point $p \in X$, a metric d on X is said to be *radially convex at* p provided that $d(p, z) = d(p, y) + d(y, z)$ for every points $y, z \in X$ with $y \in A(p, z)$ (see [36, I.4, p. 551]).

A.14. Let a continuum X and its subcontinuum Y be given. Then X is said to be *unicoherent at* Y provided that for each pair of proper subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B \cap Y$ is connected (see [64, p. 146]).

A.15. A continuum is said to be *uniquely arcwise connected* provided that for every two of its points there is exactly one arc in the continuum joining these points.

A.16. Let a continuum X be locally connected and let $p \in X$. By a *cyclic element* of X containing p we mean either the singleton $\{p\}$ if p is a cut point or an end point of X , or the set consisting of p and of all points $x \in X$ such that no point of X cuts X between p and x , otherwise (see e.g. [69, Chapter 4, §2, p. 66]).

A.17. A concept of a tree-like continuum can be defined in several (equivalent) ways. One of them is the following. A *tree* means a one-dimensional acyclic (A.1) connected polyhedron, i.e., a dendrite with finitely many end points. A continuum X is said to be *tree-like* provided that for each $\varepsilon > 0$ there is a tree T and a surjective mapping $f : X \rightarrow T$ such that f is an ε -mapping (i.e., $\text{diam } f^{-1}(y) < \varepsilon$ for each $y \in T$). Let us mention that a continuum X is tree-like if and only if it is the inverse limit (A.40) of an inverse sequence of trees with surjective bonding mappings. Compare e.g. [57, p. 24].

Using a concept of a nerve of a covering, one can reformulate the above definition saying that a continuum X is tree-like provided that for each $\varepsilon > 0$ there is an ε -covering of X whose nerve is a tree.

Finally, the original definition using tree-chains can be found e.g. in Bing's paper [4, p. 653].

A.18. Let X be a continuum. Define $C^* : C(X) \rightarrow C(C(X))$ by $C^*(A) = C(A)$. It is known that for any continuum X the function C^* is upper semi-continuous, [56, Theorem 15.2, p. 514], and it is continuous on a dense G_δ subset of $C(X)$, [56, Corollary 15.3, p. 515]. A continuum X is said to be *C^* -smooth at* $A \in C(X)$ provided

that the function C^* is continuous at A . A continuum X is said to be C^* -smooth provided that the function C^* is continuous on $C(X)$, i.e., at each $A \in C(X)$ (see [56, Definition 5.15, p. 517]).

A.19. A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $x \in X$ and for each sequence of points $\{x_n\}$ in X tending to x , the sequence of arcs px_n tends to the arc px . A dendroid X is said to be *smooth* provided that it is smooth at some point $p \in X$ (see [32, p. 298]).

A.20. A dendroid X is said to be *weakly smooth at a point* $p \in X$ provided that the subspace of 2^X consisting of all subarcs of X of the form px for $x \in X$ is compact (see [50, p. 113]).

A.21. A point p of a dendroid X is called a *shore point* of X if there exists a sequence of subdendroids X_n of $X \setminus \{p\}$ such that $X = \text{Lim } X_n$. A shore point of X that is not an end point of X is called an *improper shore point* of X (see [63, p. 939]).

A.22. A dendroid X is said to be *neat* provided that each one of its subdendroids has no improper shore point (see [63, p. 939]).

A.23. A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [56, Definition 16.10, p. 538] for an equivalent formulation).

A.24. Let X be a space and let A and B be subspaces of X with $A \subset B$. Then A is called a *deformation retract of B over X* provided that the identity mapping $i_B : B \rightarrow B$ is homotopic in X to a retraction $r : B \rightarrow A$. Further, A is called a *strong deformation retract of B over X* provided that it is a deformation retract of B over X and the homotopy keeps the points of A fixed throughout the entire deformation of B into A (see e.g. [35, Definition 6.3, p. 324]).

A.25. Let a continuum X , a compact space Y and a function $F : X \rightarrow 2^Y$ be given. Put

$$F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

The function F is said to be *lower (upper) semi-continuous* provided that $F^{-1}(B)$ is open (closed) for each open (closed) subset $B \subset Y$. It is said to be *continuous* provided that it is both lower and upper semi-continuous. This notion of continuity agrees with the one for mappings between metric spaces.

We say that a function $F : X \rightarrow 2^X$ (or a function $F : X \rightarrow C(X)$) *has a fixed point* provided that there is a point $x \in X$ such that $x \in F(x)$.

A.26. Let a continuum X , a compact space Y and a function $F : X \rightarrow 2^Y$ be given. A function $f : X \rightarrow Y$ is called a *continuous selection of F* provided that it is continuous and $f(x) \in F(x)$ for each $x \in X$.

A.27. For each point p of a continuum X equipped with an arc-structure $A : X \times X \rightarrow C(X)$ as defined in A.9 we define a partial order \leq_p by letting $x \leq_p y$ whenever

$A(p, x) \subset A(p, y)$. Let X and Y be continua with fixed arc-structures A and B , respectively. We say that a surjective mapping $f : X \rightarrow Y$ is a \leq_p -mapping provided that $x \leq_p y$ in X implies that $f(x) \leq_{f(p)} f(y)$ in Y . If, in addition, $Y \subset X$, $B = A|(Y \times Y)$, and f is a retraction, then f is called a \leq_p -retraction (or \leq_p -preserving retraction). The concept of a $<_p$ -mapping is defined in a similar manner (with $f(x) \neq f(y)$ implied by $x \neq y$). For order preserving mappings see e.g. [36, I.7, p. 553].

A.28. An *absolute retract* is a space X such that if X is embedded as a closed subset X' of a space Y , then X' is a retract of Y .

A.29. A continuum Y is said to be *hereditarily indecomposable* provided that each of its subcontinua is indecomposable, that is, for each subcontinuum $C \subset Y$ and for every continua A and B such that $A \cup B = C$ we have either $A = C$ or $B = C$.

A.30. A continuum X is said to be *selectible* provided that there exists a mapping $\sigma : C(X) \rightarrow X$ (called a *selection* for $C(X)$) such that $\sigma(A) \in A$ for each continuum $A \subset X$ (see e.g. [56, p. 253]).

A.31. A continuum X is said to be *contractible* provided there is a homotopy $h : X \times [0, 1] \rightarrow X$ such that for some point $p \in X$ we have $h(x, 0) = x$ and $h(x, 1) = p$ for each $x \in X$ (see e.g. [56, (16.2), p. 532]).

A.32. Let \mathfrak{M} be a class of mappings between continua. A mapping $f : X \rightarrow Y$ between continua is said to be *hereditarily \mathfrak{M}* provided that its restriction to any subcontinuum of the domain X is in \mathfrak{M} (see [52, Chapter 4, Section B, p. 16]).

A.33. If P and Q are subspaces of a metric space X with a metric d , and ε is a positive number, then a mapping $g : P \rightarrow Q$ is called an ε -translation provided that $d(p, g(p)) < \varepsilon$ for each point $p \in P$.

A.34. Let continua X and Y be given. A mapping $f : X \times Y \rightarrow X$ is called the *natural projection* provided that it is defined by $f((x, y)) = x$.

A.35. Let $f : X \rightarrow Y$ be a mapping between continua. Then the *induced mapping* $C(f) : C(X) \rightarrow C(Y)$ is defined by $C(f)(A) = f(A)$, where A in the left member of the equality means an element of $C(X)$, while in the right one it is understood as a subcontinuum of X (see [56, (0.49), p. 23]).

A.36. Let \mathfrak{m} be a cardinal number. By a *simple \mathfrak{m} -od* with the center p we mean the union of \mathfrak{m} arcs every two of which have p as the only common point. Let a dendroid X and a point $p \in X$ be given. Then p is said to be a *point of order at least \mathfrak{m} in the classical sense* provided that p is the center of an \mathfrak{m} -od contained in X . We say that p is a *point of order \mathfrak{m} in the classical sense* provided that \mathfrak{m} is the minimum cardinality for which the above condition is satisfied (see [8, p. 229]).

A.37. A surjective mapping $f : X \rightarrow Y$ between compact spaces is said to be (see [52, Chapter 3 and 4, p. 12-28]):

– a *local homeomorphism* provided that for each point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$ and that f restricted to U is a homeomorphism between U and $f(U)$;

- *atomic* provided that for each subcontinuum K of X such that the set $f(K)$ is nondegenerate we have $K = f^{-1}(f(K))$;
- an *OM-mapping* (an *MO-mapping*) provided that it can be represented as the composition of two mappings, $f = f_2 \circ f_1$ such that f_1 is monotone and f_2 is open (such that f_1 is open and f_2 is monotone, respectively);
- *quasi-monotone* provided that for each subcontinuum Q of Y with the nonempty interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q ;
- *weakly monotone* provided that for each subcontinuum Q of Y with the nonempty interior each component the set $f^{-1}(Q)$ is mapped under f onto Q ;
- *weakly confluent* provided that for each subcontinuum Q of Y there is a component of the set $f^{-1}(Q)$ which is mapped under f onto Q ;
- *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

A.38. Let \mathfrak{M} be a class of mappings between compact spaces. A surjective mapping $f : X \rightarrow Y$ between continua is said to be *locally* \mathfrak{M} provided that for each point $x \in X$ there is a closed neighborhood V of x such that $f(V)$ is a closed neighborhood of $f(x)$ and that the restriction $f|_V$ is in \mathfrak{M} (see [52, Chapter 4, Section C, p. 18]).

A.39. Given a space X let $\mathcal{H}(X)$ stand for the group of autohomeomorphisms of X . If a point $p \in X$ is fixed, then $\{h(p) \in X : h \in \mathcal{H}(X)\}$ is called an *orbit of p* . Orbits of points of X either are mutually disjoint or coincide, and their union is the whole X .

A.40. The reader can find necessary information on the inverse limits of inverse sequences e.g. in Section 2 of the second chapter of [57], as well as in [48].

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