

FANS WITH THE PROPERTY OF KELLEY

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Fans having the property of Kelley are characterized as the limits of inverse sequences of finite fans with confluent bonding mappings and as smooth fans for which the set of their end-points together with the top is closed. Also a characterization is obtained of smooth fans with the set of end-points closed as the limits of inverse sequences of finite fans with open bonding mappings.

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A continuum means a compact connected metric space. It is said to be hereditarily unicoherent provided that the intersection of any two its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a dendroid. A point p of a dendroid X is called an end point of X provided p is an end point of any arc in X that contains p . The set of all end points of a dendroid X will be denoted by $E(X)$. A point p of a dendroid X is called a ramification point of X provided it is the vertex of a simple triod contained in X , i.e., if there are three arcs pa , pb and pc in X having p as the only point of the intersection of any two of them. By a fan we understand a dendroid having exactly one ramification point, called the top of the fan. The cone F_C over the Cantor ternary set C is called the Cantor fan. A locally connected dendroid is called a dendrite. A dendrite X is said to be finite if $E(X)$ is finite. In particular, a finite fan means a fan with finitely many end points. A dendroid X is said to be smooth provided there exists a point $p \in X$ such that for each sequence x_n of points of X converging to a point x the sequence of arcs px_n converges to the arc px [2, p. 298]. It is known that if a fan is smooth, then its top can be taken as a point p in the above definition (see [2, Corollary 9, p. 301]).

All mappings considered in the paper are assumed to be continuous. A mapping $f: X \rightarrow Y$ between continua X and Y is said to be confluent provided for each subcontinuum Q of $f(X) \subset Y$ and for each component K of $f^{-1}(Q)$ we have $f(K) = Q$. It is well known that all open mappings are confluent (see e.g. [10, (0.45.3), p. 21]).

Given a point x in a continuum X , let $C(x, X)$ denote a family of all subcontinua of X to which the point x belongs. The union $C(X) = \bigcup\{C(x, X) : x \in X\}$, called the hyperspace of subcontinua of X , is metrized by the Hausdorff distance $\text{dist}(A, B) = \inf\{\varepsilon > 0 : A \subset N(B, \varepsilon) \text{ and } B \subset N(A, \varepsilon)\}$, where $N(A, \varepsilon)$ is the union of the ε -balls about all points of A . A continuum X is said to have the property of Kelley provided that for each point x in X , for each sequence of points x_n converging to x and for each continuum $K \in C(x, X)$ there exists a sequence of continua $K_n \in C(x_n, X)$ that has K as its limit (see [9, Property (3.2), p. 26]; cf. [11]).

The property, well-known as an important tool in the study of contractibility of hyperspaces, is interesting by its own right, and it has many applications in continua theory. The aim of the paper is to characterize fans enjoying the property. The characterizations are expressed in terms of (and therefore present some relations between) smoothness, structure of the end-points set, special embeddings of the fan into the Cantor fan, and—which is perhaps the most interesting result—a possibility of describing the fan as the limit of an inverse sequence of finite fans with confluent bonding mappings. Some equivalences in Theorem 1 below, viz. ones between conditions (1), (2) and (4), have been obtained earlier in [3]. The present authors have decided to join the proofs of these equivalences here for the sake of completeness of their results and for the reader's convenience.

1. Theorem. *Let a fan X with the top v be given. Then the following conditions are equivalent.*

- (1) X is smooth and the set $E(X)$ of end-points of X is closed;
- (2) there exists an embedding $h: X \rightarrow h(X) \subset F_C$ of X into the Cantor fan F_C such that $h(E(X)) \subset E(F_C)$;
- (3) X is the limit of an inverse sequence of finite fans with open bonding mappings;
- (4) there exists a surjective mapping $f: X \rightarrow [0, 1]$ such that $f(v) = 0$ and for each end-point e of X the partial mapping $f|_{ve}: ve \rightarrow [0, 1]$ is a (surjective) homeomorphism.

Proof. (1) \Rightarrow (2). Since X is smooth, it is embeddable into the Cantor fan F_C (see [8, Corollary 4, p. 90] and [1, Theorem 9, p. 27]); let $h_1: X \rightarrow h_1(X) \subset F_C$ be an embedding. Obviously $h_1(v)$ is the top v' of F_C . Since the set $h_1(E(X))$ is a closed subset of F_C not containing the top, there exists $\varepsilon > 0$ such that each point of $h_1(E(X))$ is more than ε apart from v' . Hence it is possible to expand each segment $v'h_1(e) \subset v'e'$, where $e \in E(X)$ and $e' \in E(F_C)$, to the whole segment $v'e'$ in such a way that the induced (global) mapping from $h_1(X)$ into F_C is continuous. More precisely, there is a homeomorphism $h_2: h_1(X) \rightarrow h_2(h_1(X)) \subset F_C$ such that $h_2(v') = v'$ and that for each end-point $e \in E(X)$ and for each point $e' \in E(F_C)$ with the property $h_1(e) \in v'e'$ the partial mapping $h_2|_{v'h_1(e)}: v'h_1(e) \rightarrow v'e'$ is a linear homeomorphism. Obviously $h_2(h_1(E(X))) \subset E(F_C)$. Since $h_1(E(X))$ is closed, it is straight-forward to conclude continuity of h_2 . The composition $h = h_2h_1: X \rightarrow h(X) \subset F_C$ is the needed embedding.

(2) \Rightarrow (3). For each natural number i let F^i be the cone over the set $A^i = \{0, 1\}^i$, and let F_C be the cone over the Cantor set $C = \{0, 1\}^\infty$, i.e., $F^i = A^i \times [0, 1] / A^i \times \{0\}$ and $F_C = C \times [0, 1] / C \times \{0\}$ are the 2^i -fan and the Cantor fan respectively. The projections $C \rightarrow A^i$ and $A^{i+1} \rightarrow A^i$ induce open mappings $p^i: F_C \rightarrow F^i$ and $u^i: F^{i+1} \rightarrow F^i$. Observe that $\{F^i, u^i\}_{i=1}^\infty$ is an inverse sequence having F_C as its limit (see [5, p. 165]). Now let $h: X \rightarrow h(X) \subset F_C$ be an embedding as in (2). Put $X^i = p^i(h(X))$ and $f^i = u^i|X^{i+1}: X^{i+1} \rightarrow X^i$. Note that $\{X^i, f^i\}_{i=1}^\infty$ is an inverse sequence of finite fans with open bonding mappings and that $h(X)$ is its limit.

(3) \Rightarrow (4). Let $X = \lim\{X^i, f^i\}$, where each X^i is a finite fan with the top v^i , and each $f^i: X^{i+1} \rightarrow X^i$ is open. First observe that by openness of the mappings f^i , for each end-point $e \in E(X^{i+1})$ the restriction $f^i|v^{i+1}e: v^{i+1}e \rightarrow f^i(v^{i+1}e) \subset X^i$ is a homeomorphism of the arc $v^{i+1}e$ onto the arc $v^i f^i(e)$, and $f^i(e) \in E(X^i)$. Really, this is a consequence of Theorem (7.31) of [12, p. 147], saying that the Menger-Urysohn order of a point in a continuum is never increased when the continuum undergoes an open mapping. It follows from this observation that for each end point $e \in E(X)$ the partial mapping $p^i|ve: ve \rightarrow v^i e'$ is a homeomorphism (as the inverse limit of homeomorphisms), where $e' \in E(X^i)$ with $p^i(v) = v^i$ and $p^i(e) = e'$. Now let us join to the inverse sequence a zero-term $X^0 = [0, 1]$ and let $f^0: X^1 \rightarrow X^0$ be defined by the conditions $f^0(v^1) = 0$, $f^0(E(X^1)) = \{1\}$ and $f^0|v^1e$ is a homeomorphism for each $e \in E(X^1)$. Then the projection $f = p^0: X \rightarrow X^0 = [0, 1]$ is the mapping of condition (4).

(4) \Rightarrow (1). The fan X is smooth by Corollary 15 of [1, p. 28] and Corollary 4 of [8, p. 90]. The set $E(X)$ is closed since $E(X) = f^{-1}(1)$. \square

2. Remark. The property of Kelley is not mentioned in Theorem 1. But it is related to conditions (1)–(4), as it will be formulated later, in a corollary after the next theorem.

We denote by \mathbb{N} the set of all positive integers. In the sequel H will mean the union of straight line segments in the plane joining the origin $v = (0, 0)$ with points of the set $\{(2, 0)\} \cup \{(1, 1/n): n \in \mathbb{N}\}$. Thus H is a smooth fan with the top v that has not the property of Kelley.

3. Theorem. *Let a fan X with the top v be given. Then the following conditions are equivalent.*

- (a) X has the property of Kelley;
- (b) X is smooth and for no countable subset of end points $\{e_n: n \in \mathbb{N}\}$ of X the union $\bigcup\{ve_n: n \in \mathbb{N}\}$ is homeomorphic with H ;
- (c) X is smooth and the set $\{v\} \cup E(X)$ is closed;
- (d) X is the limit of an inverse sequence of finite fans with confluent bonding mappings.

Proof. (a) \Rightarrow (b). If a dendroid X has the property of Kelley, then it is smooth according to Corollary 5 of [6]. Suppose there exists a countable subset of end

points of X , $\{e_n: n \in \mathbb{N}\} \subset E(X)$, such that the union $U = \bigcup\{ve_n: n \in \mathbb{N}\}$ is homeomorphic with H . Let a point $x \in U$ correspond to the point $(1, 0) \in H$ under the considered homeomorphism. So $x = \lim e_n$. Denote by e an end point of X such that $x \in ve$, and let $K = xe$. Since X has the property of Kelley, we conclude there is a sequence of continua K_n with $e_n \in K_n$ and $K_n \rightarrow K$. Consider two cases. If infinitely many K_n 's contain v , then v is in the limit continuum K , a contradiction with the definition of K . If v is not in K_n for sufficiently large n , then each such K_n is contained in ve_n , and since $ve_n \rightarrow vx$ by smoothness of X , we see that $K \subset vx$, a contradiction again.

(b) \Rightarrow (c). Since the fan X is smooth, it can be embedded into the Cantor fan F_C , thus for each $e \in E(X)$ the arc ve can be understood as a straight line segment (see [8, Corollary 4, p. 90] and [1, Theorem 9, p. 27]). Suppose $\{v\} \cup E(X)$ is not closed. Thus there exist a point x in $X \setminus (\{v\} \cup E(X))$ and a sequence of end points e_n of X with $x = \lim e_n$. Denote by e_0 the end point of X such that $x \in ve_0$. Thus in particular $x \neq e_0$, and we can define a homeomorphism between the union $\bigcup\{ve_n: n \in \{0\} \cup \mathbb{N}\}$ and the continuum H in a routine way.

(c) \Rightarrow (d). By smoothness of X we can again assume that X is a subset of the Cantor fan. If $E(X)$ is closed, the conclusion follows from Theorem 1. So assume $E(X)$ is not closed. However, since $\{v\} \cup E(X)$ is closed (thus compact), its components are continua, and so they are singletons. Consequently $\{v\} \cup E(X)$ is zero-dimensional, and therefore $E(X)$ can be represented as the union of countably many disjoint closed sets E_n (where $n \in \mathbb{N}$) such that $\lim E_n = \{v\}$. Putting $X_n = \bigcup\{ve: e \in E_n\}$ we see that each X_n is closed, so it is a (smooth) fan, and that $X = \bigcup\{X_n: n \in \mathbb{N}\}$ with $X_m \cap X_n = \{v\}$ for $m \neq n$, and with $\lim X_n = \{v\}$. Note that $E(X_n) = E_n$ is closed, so Theorem 1 can be applied to each X_n separately. Thus, let $X_n = \lim\{X_n^i, f_n^i\}_{i=1}^\infty$, where X_n^i are some finite fans and f_n^i are open. Further, for each $i \in \mathbb{N}$ define $X^i = X_1^i \cup X_2^i \cup \dots \cup X_n^i$, and note that each X^i is a finite fan. Finally, define $f^i: X^{i+1} \rightarrow X^i$ as follows. For $j \in \{1, 2, \dots, i\}$ let $f^i|X_j^{i+1}: X_j^{i+1} \rightarrow X_j^i$ be understood as $f^i|X_j^{i+1} = f_j^i$ (so it is open), while for $j = i+1$ we put $f^i|X_{i+1}^{i+1}: X_{i+1}^{i+1} \rightarrow \{v\} \subset X^i$ (so it is a constant mapping). Observe that the mapping f^i just defined can easily be considered as a composition of a monotone and of an open mapping, and therefore it is confluent. One can verify in a routine way that $\{X^i, f^i\}_{i=1}^\infty$ is an inverse sequence having the fan X as its limit.

(d) \Rightarrow (a). This is an immediate consequence of the fact that the property of Kelley is preserved under the inverse limit operation if bonding mappings are confluent (see [4, Theorem 2, p. 190]). \square

4. Corollary. *Let a fan X with the top v be given. Each of conditions (1)–(4) of Theorem 1 is equivalent to*

(5) *X has the property of Kelley and v is not in the closure of $E(X)$.*

5. Remark. The common assumption of Theorems 1 and 3 and Corollary 4 (saying that the considered continuum X is a fan) is essential. Namely let X_0 be the union

of the harmonic fan F_H being the cone with the top $v = (0, 0)$ over the harmonic sequence of points $\{(1, 0)\} \cup \{(1, 1/n) : n \in \mathbb{N}\}$ and of the straight line segment ab , where $a = (\frac{1}{2}, 0)$ and $b = (1, -\frac{1}{2})$. Then X_0 satisfies (1) and (4) and it has not the property of Kelley. Similarly, it satisfies (b) and (c) but not (a). And there is no inverse sequence of continua having the property of Kelley (in particular finite dendrites) with confluent bonding mappings having X_0 as its limit (see Theorem 2 of [4, p. 190]).

In the light of the above remark we have a problem.

6. Problem. Characterize dendroids having the property of Kelley.

In particular, the following question is open.

7. Question Let a dendroid X have the property of Kelley. Is then X the limit of an inverse sequence of finite dendrites with confluent bonding mappings?

8. Remark. Let a point x of a continuum X be given. Recall that a continuum neighborhood of x in X is a subcontinuum of X containing x in its interior. Denote by $T(x)$ the set of points of X which have no continuum neighborhood missing x . It is known that $T(x)$ is a continuum for each x in X (see [7, Corollary 1.1, p. 115]). The following property (*) is known to be intermediate between the property of Kelley and smoothness for dendroids.

(*) For each point x of a dendroid X and for each $e \in E(X)$ the condition $xe \cap T(x) \neq \{x\}$ implies $e \in T(x)$.

Namely the property of Kelley of a dendroid X implies (*) [6, Lemma 2], and (*) implies smoothness of X [6, Theorem 4]. None of these implications can be reversed even if X is a fan. Indeed, the fan H defined just before Theorem 3 above is smooth without having the property (*); and if X denotes the union of straight line segments in the plane joining the origin $v = (0, 0)$ with the points of the set $\{(2, 0)\} \cup \{(1, 1/n) : n \in \mathbb{N}\} \cup \{(2, -1/n) : n \in \mathbb{N}\}$, then X is a fan with the top v satisfying (*) without having the property of Kelley.

Using the two implications mentioned above one can easily see the following corollary.

9. Corollary. *Smoothness of the fan X in condition (1) of Theorem 1 (and in Corollary 4) as well as in conditions (b) and (c) of Theorem 3 can be replaced by property (*)*

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