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THE LELEK FAN IS UNIQUE

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Abstract. It is proven that a smooth fan with a dense set of end-points is unique. Some other characterizations of the fan are also given.

A. Lelek has shown in [**5**, §9, page 314], an example of a fan with a one-dimensional set of its end-points. The fan is smooth and the set of its end-points is dense in it. In this paper it is proven that each such fan is homeomorphic to the Lelek example. As a consequence, each confluent image of the fan is homeomorphic to it. The only other continuum having this property, that is known to the author, is an arc (see [**1**, Corollary 20, page 32]), and there is a conjecture that the pseudo-arc is another one. Other characterizations of the Lelek fan are also obtained.

Only metric spaces will be considered. A *fan* is an arcwise connected, hereditarily unicoherent continuum with at most one *ramification-point* (i.e., a point which is the unique common point of any two of three distinct arcs). This point is called the *top* of the fan. An *end-point* of a fan is a point that is the end-point of each arc containing it. A fan with a top t is said to be *smooth* (see [**1**, §3, page 7]) if, for each sequence $\{p_n\}_{n=1}^{\infty}$ of its points converging to a point p, the arcs tp_n (joining the top t with the point p_n) converge to the arc tp.

Throughout this paper, the letter X always denotes a smooth fan with a dense set $E(X)$ of its end-points.

We denote by C the Cantor ternary set in the unit interval, and by F the Cantor fan $C \times [0,1]/C \times \{0\}$. We denote its points by $\langle x, y\rangle$ for $x \in C$ and $y \in [0,1]$. So, for each $x \in C$, the point $\langle x, 0\rangle$ means the top of the fan F, and it is denoted by t. Given two sets A and B such that $A \subset C$ and $B \subset [0,1]$, the symbol $A \times B$ is always understood to be a respective subset of F. We denote by d a metric on F satisfying $d(\langle x, y\rangle, \langle x, z\rangle) = |y - z|$ for all $x \in C$ and $y, z \in [0,1]$. We will use a function $\pi : F \to [0,1]$ defined by $\pi(\langle x, y\rangle) = y$.

The symbol N denotes the set of all positive integers, and the symbol Q represents the set of all rationals in (0,1).

It is known that every smooth fan can be embedded into the Cantor fan (see [**2**, Corollary 4, page 90], and [**1**, Theorem 9, page 27]). As a result, it may be assumed that $X \subset F$. Define a function $l_X : C \to [0,1]$ by the formula $l_X(c) = y$ if and only if $\pi((\{c\} \times [0,1]) \cap X) = [0,y]$. The number $l_X(c)$ is the length of the arc of X contained in $\{c\} \times [0,1]$. The function l_X is not continuous; however, the compactness of X implies that it is upper semi-continuous in the sense of real function theory (see, e.g., [**4**, §18, II, Example, 1, page 174]). Moreover, for $A \subset C$, let $L_X(A) = \sup\{l_X(c) : c \in A\}$. Thus, if $A \subset B$ are subsets of C, then $L_X(A) \leq L_X(B)$.

To find a homeomorphism between any two fans with dense sets of end-points we need "good" embeddings of them into the Cantor fan F (Lemma 4 below). In order to construct such embeddings, we first prove some lemmas.

LEMMA 1. *Let $p \in C$ be a point such that $l_X(p) = L_X(C)$ (i.e., the straight-line segment $(\{p\} \times [0,1]) \cap X$ is the longest in X). Then there is a collection $\{B(i) : i \in N\}$ of closed and open disjoint subsets of C such that $C = \{p\} \cup \bigcup_{i=1}^{\infty} B(i)$, and the set $\{L_X(B(i)) : i \in N\}$ is dense in $[0, L_X(C)]$.*

PROOF: Since the Cantor set is homogeneous we may assume that $p = 0$. Let $\{r_n : n \in N\}$ be a sequence of all rationals in $[0, L_X(C)]$. We proceed inductively. Let $B(1) = [2/3, 1] \cap C$. Assume we have defined $B(1), B(2), \ldots, B(2n-1)$ for a natural number n as disjoint closed and open subsets of $C \backslash \{0\}$. By the density of $E(X)$ in X, there is a point

$$e \in E(X) \backslash \bigcup_{i=1}^{2n-1} (B(i) \times [0,1])$$

such that $d(e, \langle 0, r_n \rangle) < 1/2n$. Then $\pi(e) \in (r_n - 1/2n, r_n + 1/2n)$, and we can find a closed and open set $B(2n)$ from the standard base of the Cantor set C such that:

$$(1^\circ) \qquad B(2n) \subset C \backslash \left(\{0\} \cup \bigcup_{i=1}^{2n-1} B(i) \right);$$

$$(2^\circ) \qquad e \in B(2n) \times [0,1];$$

$$(3^\circ) \qquad L_X(B(2n)) \in \left(r_n - \frac{1}{n}, r_n + \frac{1}{n} \right).$$

The set $B(2n+1)$ is defined by

$$B(2n+1) = \left\{ c \in C \setminus \bigcup_{i=1}^{2n-1} B(i) : c > x \text{ for all } x \in B(2n) \right\}.$$

One can easily verify that the defined family satisfies all the required conditions.

The next lemma is a consequence of Lemma 1.

LEMMA 2. *There is a collection $\{B(i_1, \ldots, i_n) : n \in \{0\} \cup N$ and $i_1, \ldots, i_n \in N\}$ of closed and open subset of C and a collection $\{p(i_1, \ldots, i_n) : n \in \{0\}$ and $i_1, \ldots, i_n \in N\}$ of points of C such that:*

(1) *for $n = 0$, we have $B(\ ) = C$;*
(2) *$p(i_1, \ldots, i_n) \in B(i_1, \ldots, i_n)$ and $l_X(p(i_1, \ldots, i_n)) = L_X(B(i_1, \ldots, i_n))$ (i.e., the straight-line segment $(\{p(i_1, \ldots, i_n)\} \times [0,1]) \cap X$ is the longest in the set $(B(i_1, \ldots, i_n) \times [0,1]) \cap X$);*
(3) *$B(i_1, \ldots, i_n) = \{p(i_1, \ldots, i_n)\} \cup \bigcup_{j=1}^{\infty} B(i_1, \ldots, i_n, j)$;*
(4) *$B(i_1, \ldots, i_n, j) \cap B(i_1, \ldots, i_n, j') = \emptyset$ for $j \neq j'$;*
(5) *the set $\{L_X(B(i_1, \ldots i_n, j)) : j \in N\}$ is dense in the interval $[0, L_X(B(i_1, \ldots i_n))]$;*
(6) *for every sequence $i_1, i_2, \ldots$ of natural numbers, the intersection $\bigcap_{n=1}^{\infty} B(i_1, \ldots, i_n)$ is a one-point set.*

PROOF: We proceed inductively. We put $B(\ ) = C$, and we choose $p(\ )$ such that $l_X(p(\ )) = L_X(C)$. If $B(i_1, \ldots, i_n)$ and $p(i_1, \ldots, i_n)$ are defined, we use Lemma 1 to construct $B(i_1, \ldots, i_n, j)$ for $j \in N$, and we take $p(i_1, \ldots, i_n, j)$ to be that point of $B(i_1, \ldots, i_n, j)$ which satisfies (2).

LEMMA 3. *Let $p \in C$ satisfy $l_X(p) = L_X(C)$ and let $\{B(i) : i \in N\}$ be as in Lemma 1. Then, for every $\epsilon > 0$ and $\alpha < 1$, there is an ϵ-homeomorphism $f : X \to Y \subset F$ of X onto Y such that:*

(7) *$f(t) = t$ and $f(\langle p, l_X(p) \rangle) = \langle p, l_X(p) \rangle$;*
(8) *$f((\{c\} \times [0,1]) \cap X) \subset \{c\} \times [0,1]$ for each $c \in C$ (i.e., f does not interchange arcs in F);*
(9) *$d(f(\langle c, x \rangle), f(\langle c, y \rangle)) \geq \alpha \cdot d(\langle c, x \rangle, \langle c, y \rangle)$ for each $c \in C$ and $x, y \in [0,1]$;*
(10) *$L_Y(B(i)) \neq L_Y(B(j))$ for $i \neq j$;*
(11) *$\{L_Y(B(i)) : i \in N\} = Q \cap (0, L_Y(C))$.*

PROOF: Assume for shortness that $l_X(p) = 1$. Let $\epsilon > 0$ and $\alpha < 1$ be given. Note that there exists an $\epsilon/2$-homeomorphism $g : X \to g(X) \subset F$, linear on each arc $(\{c\} \times [0,1]) \cap X$, such that conditions (7) — (10) are satisfied for g in place of f, for $g(X)$ in place of Y, and for $\sqrt{\alpha}$ in place of α, and, in addition, $L_{g(X)}(B(i)) \neq 1$ for all $i \in N$. Since the set $\{L_X(B(i)) : i \in N\}$ is dense in [0,1] and since g is a homeomorphism, the set $\{L_{g(X)}(B(i)) : i \in N\}$ is dense in [0,1]. Thus, by [**3**, Corollary, page 96], there is a differentiable $\epsilon/2$-homeomorphism $h : [0,1] \to [0,1]$ with $h(0) = 0$ and $h(1) = 1$ such that the derivative $h' : [0,1] \to [0,\infty]$ satisfies $h'(x) \geq \sqrt{\alpha}$ for all $x \in [0,1]$ and, moreover, $h(\{L_{g(X)}(B(i)) : i \in N\}) = Q \cap (0,1)$. The function $f = (id \times h) \circ g$ is the needed ϵ-homeomorphism.

The sequence of auxiliary results is now completed by proving the strongest lemma.

LEMMA 4. *Let* $\{B(i_1, \ldots, i_n) : n \in \{0\} \cup N$ *and* $i_1, \ldots, i_n \in N\}$ *and* $\{p(i_1, \ldots, i_n) : n \in \{0\} \cup N$ *and* $i_1, \ldots, i_n \in N\}$ *be as in Lemma* 2. *Then there is a homeomorphism* $f : X \to Y \subset F$ *of* X *onto* Y *satisfying condition* (8) *and*

$$(12) \quad L_Y(C) = L_X(C),$$

and such that, for each $n \in N$ *and for each system* $(i_1, \ldots, i_n)$, *we have:*

$$(13) \quad l_Y(p(i_1, \ldots, i_n)) = L_Y(B(i_1, \ldots, i_n));$$

$$(14) \quad L_Y(B(i_1, \ldots, i_n, j)) \neq L_Y(B(i_1, \ldots, i_n, j')) \text{ for } j \neq j';$$

$$(15) \quad \{L_Y(B(i_1, \ldots, i_n, j)) : j \in N\} = Q \cap (0, L_Y(B(i_1, \ldots, i_n))).$$

PROOF: Let $\alpha_0, \alpha_1, \ldots$ be a sequence of numbers from (0,1) such that the infinite product $\alpha_0 \cdot \alpha_1 \cdot \ldots$ is positive. We inductively define homeomorhisms $f(i_1, \ldots, i_n)$ and f_n for every $n \in \{0\} \cup N$. First, let $f(\ )$ be a homeomorphism as in Lemma 3 for α_0 in place of α and for $p(\ )$ in place of p, and we put $f_0 = f(\ )$. Thus $\{L_{f_0(X)}(B(i)) : i \in N\}$ is the set of all rationals in $(0, L_X(C))$. Next, for each $i \in N$, using Lemma 3 for $f_0((B(i) \times [0,1]) \cap X)$ in place of X, for $p(i)$ in place of p, for 2^{-i} in place of ϵ, and for α_1 in place of α, we construct an embedding

$$f(i) : f_0\left((B(i) \times [0,1]) \cap X\right) \to B(i) \times [0,1]$$

satisfying analogous conditions to (7) — (11). We define a homeomorphism $f_1 : X \to f_1(X) \subset F$ by $f_1(x) = (f(i) \circ f_0)(x)$ for $x \in (B(i) \times [0,1]) \cap X$

and $f_1(x) = f_0(x)$ for $x \in (\{p(\,)\} \times [0,1]) \cap X$. Observe that f_1 satisfies (12) — (15) for $n \leq 1$; moreover, it satisfies (9) for $\alpha_0 \cdot \alpha_1$ in place of α.

The next steps are similar. Assume we have defined embeddings $f_0, \dots, f_n : X \to F$ such that:

1° $f_m(t) = t$ for all $m \leq n$;

2° $f_m(\langle p(i_1, \dots, i_m), x\rangle) = f_{m'}(\langle p(i_1, \dots, i_m), x\rangle)$ for all m, m' and x satisfying $m \leq m' \leq n$ and $x \in [0, l_X(p(i_1, \dots, i_m))]$;

3° for each $m \leq n$, the function f_m satisfies (8), satisfies (9) for $\alpha_0 \cdot \ldots \cdot \alpha_m$ in place of α, and satisfies (12) — (15) for all $k \leq m$ in place of n.

For each $j \in N$, we use Lemma 3 for $f_n((B(i_1, \dots, i_n) \times [0,1] \cap X)$ in place of X, for $p(i_1, \dots, i_n)$ in place of p, for $2^{-(i_1+\cdots+i_n+j)}$ in place of ϵ and for α_n in place of α to find an embedding

$$f(i_1, \dots, i_n, j) : f_n\left((B(i_1, \dots, i_n, j) \times [0,1]) \cap X\right) \to B(i_1, \dots, i_n, j) \times [0,1]$$

satisfying conditions analogous to (7) — (11). For every $i_1, \dots, i_n \in N$, we put $f_{n+1}(x) = (f(i_1, \dots, i_n, j) \circ f_n)(x)$ if $x \in (B(i_1, \dots, i_n, j) \times [0,1]) \cap X$ and $f_{n+1}(x) = f(x)$ if $x \in (\{p(i_1, \dots, i_m)\} \times [0,1]) \cap X$ for some $m \leq n$.

Now, observe that, for each $n \in N$ and for all $x \in X$, we have

$$\begin{aligned} d\left(f_n(x), f_{n+1}(x)\right) &\leq d\left(f_n(x), \left(f\left(i_1, \dots, i_n, j\right) \circ f_n\right)(x)\right) \\ &\leq 2^{-(i_1+\cdots+i_n+j)} \\ &\leq 2^{-(n+1)} \end{aligned}$$

for the corresponding numbers $i_1, \dots, i_n, j$. Therefore the defined sequence of embeddings $f_0, f_1, \dots, f_n, \dots$ converges uniformly to a function f. The limit function f satisfies (8), satisfies (12) — (15) for $Y = f(X)$, and satisfies (9) for the infinite product $\alpha_0 \cdot \alpha_1 \dots$ in place of α, and thus it is one-to-one.

THEOREM. *Each two smooth fans with a dense set of end-points are homeomorphic.*

PROOF: Let X_1 and X_2 be two such fans. We may assume that they are embedded in F in such a way that $\pi(X_1) = \pi(X_2) = [0,1]$. Let, for $s \in$

$\{1,2\}$, the sets $\{B_s(i_1,\ldots,i_n) : n \in \{0\} \cup N \text{ and } i_1,\ldots,i_n \in N\}$ and $\{p_s(i_1,\ldots,i_n) : n \in \{0\} \cup N \text{ and } i_1,\ldots,i_n \in N\}$ be as in Lemma 2 for X_s in place of X. By Lemma 4, we may assume that conditions (13) — (15) are satisfied for X_s, $B_s(i_1,\ldots,i_n)$ and $p_s(i_1,\ldots,i_n)$ in place of Y, $B(i_1,\ldots,i_n)$ and $p(i_1,\ldots,i_n)$, respectively.

Now we define a homeomorphism $h : C \to C$ such that the induced homeomorphism $h \times id : F \to F$ sends the fan X_1 onto X_2. We proceed inductively. First, let $h(p_1(\ )) = p_2(\ )$. Next, we define $h(B(i))$ [we do not define the function h on $B_1(i)$, but only the set $h(B_1(i))$] which will be the image of $B_1(i)$ under h. Namely, we put $h(B_1(i)) = B_2(j)$ if and only if $L_{X_1}(B_1(X)) = L_{X_2}(j))$. Note that (14) and (15) imply that the definition is correct. We also put $h(p_1(i)) = p_2(j)$.

Assume we have defined $h(B_1(i_1,\ldots,i_n))$ such that $h(B_1(i_1,\ldots,i_n)) = B_2(j_1,\ldots,j_n)$ satisfies

$$L_{X_1}\left(B_1(i_1,\ldots,i_n)\right) = L_{X_2}\left(B_2(j_1,\ldots,j_n)\right). \tag{16}$$

Define $h(B_1(i_1,\ldots,i_n,k)) = B_2(j_1,\ldots,j_n,k')$ if and only if $L_{X_1}(B_1(i_1,\ldots,i_n,k)) = L_{X_2}(B_2(j_1,\ldots,j_n,k'))$, and observe, as previously, that (16), (14) and (15) imply the correctedness of the definition. Note that, by (6), this procedure uniquely defines $h(c)$ for every point c of the Cantor set. The function h is continuous and one-to-one by its definition, and $(h \times id)|X_1$ is a homeomorphism of X_1 onto X_2.

Now we prove that another property of the Lelek fan also characterizes it:

PROPOSITION 1. *Each smooth fan such that the set of end-points together with the top is connected is homeomorphic to the Lelek fan.*

PROOF: By [5, page 314], the Lelek fan has the considered property. To prove the converse, assume, on the contrary, there is a smooth fan Y with the set $E(Y)$ of end-points such that $E(Y) \cup \{t\}$ is connected, but $E(Y)$ is not dense. Embed Y in F, and choose open and closed sets $B \subset C$ and $t_0, t_1 \in [0,1]$ such that $B \times [t_0, t_1]$ contains some points of Y, but does not contain points of $E(Y)$. The set $E(Y)$ is the union of two of its open subsets, $E(Y) \cap (B \times [t_1, 1])$ and $E(Y) \cap ((C \setminus B) \times [0,1])$ which is a contradiction.

PROPOSITION 2. *Each non-degenerate confluent image of the Lelek fan is homeomorphic to it.*

PROOF: First observe that every non-degenerate confluent image of a smooth fan is a smooth fan (see [**1**, Theorem 13, page 33]). Next, by [**1**, Lemma 4, page 32], every confluent mapping of a fan is monotone relative to its top (see [**6**, page 720] for the definition) and hence, by [**6**, Corollary (2.10), page 722], the image of an end-point is either an end-point or the top of the fan. Consequently, the confluent images of a smooth fan with dense sets of end-points are again smooth fans with dense sets of end-points and thus, by the theorem, they are homoemorphic to the Lelek fan.

We show that the converse to Proposition 2 is true in a more general case.

PROPOSITION 3. *If Y is a smooth fan, different from an arc, and every monotone image of Y is homeomorphic to it, then Y is the Lelek fan.*

PROOF: We show that if the set $E(Y)$ is not dense, then Y has two non-homeomorphic monotone images. Assume $Y \subset F$. Observe that, for no end-point e of Y, the set $te \setminus \{t\}$ is open in Y (otherwise Y can be monotoneously retracted onto the arc te). So, shrinking one of the arcs te, for $e \in E(Y)$, to the top t, we obtain a monotone mapping of Y onto a fan with a non-compact set of end-points. Now we construct a monotone image of Y having a compact set of end-points. By the assumption, there are open and closed sets $B \subset C$ and two points $t_0, t_1 \in [0,1]$ such that $B \times [t_0, t_1]$ contains some points of Y, but does not contain end-points of Y. Then shrink $Y \cap ((C \setminus B) \times [0,1] \cup B \times [0, t_0])$ to the top, and every arc $Y \cap (\{c\} \times [t_1, 1])$ for $c \in B$ to a point to get a required fan. The proof is complete.

Collecting the characterizations of the Lelek fan, the following corollary is obtained.

COROLLARY. *If Y is a smooth fan, different from an arc, then the following statements are equivalent to each other:*

(a) *the fan Y is homeomorphic to the Lelek fan X;*
(b) *the set of end-points of Y is dense;*
(c) *the set of end-points of Y together with the top is connected;*
(d) *every confluent image of Y is homeomorphic to it;*
(e) *every monotone image of Y is homeomorphic to it.*

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