

## INDUCED MO-MAPPINGS

By

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**Abstract.** A mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is called an MO-mapping provided that it can be represented as the composition of two mappings,  $f_1 : X \rightarrow Z$  and  $f_2 : Z \rightarrow Y$ , such that  $f_1$  is open and  $f_2$  is monotone. Induced MO-mappings,  $2^f$  and  $C(f)$ , between hyperspaces are studied. In particular an example is constructed of an open mapping  $f : [0, 1] \rightarrow [0, 1]$  for which  $C(f)$  is not an MO-mapping. This answers two questions asked by H. Hosokawa.

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. To exclude some trivial statements we assume that all considered mappings are not constant. A *continuum* means a compact connected space. Given a continuum  $X$  with a metric  $d$ , we let  $2^X$  denote the hyperspace of all nonempty closed subsets of  $X$  equipped with the *Hausdorff metric*  $H$  defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see e.g. [9, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by  $C(X)$  the hyperspace of all subcontinua of  $X$ , i.e., of all connected elements of  $2^X$ . The reader is referred to Nadler's book [9] for needed information on the structure of hyperspaces.

Given a mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$ , we consider mappings (called the *induced* ones)

$$2^f : 2^X \rightarrow 2^Y \quad \text{and} \quad C(f) : C(X) \rightarrow C(Y)$$

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1991 *Mathematics Subject Classification.* 54B20, 54E40, 54F15.

*Key words and phrases.* continuum, hyperspace, induced mapping, monotone, MO-mapping, open.

Received April 27, 1998.

Revised October 21, 1998.

defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \text{ and } C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping  $f : X \rightarrow Y$  between spaces  $X$  and  $Y$  is said to be:

- *open*, provided that the image of an open subset of the domain is open in the range;
- *monotone*, provided that it has connected point-inverses;
- *OM-mapping*, provided that it can be represented as the composition of two mappings,  $f = f_2 \circ f_1$ , such that  $f_1$  is monotone and  $f_2$  is open;
- *MO-mapping*, provided that it can be represented as the composition of two mappings,  $f = f_2 \circ f_1$ , such that  $f_1$  is open and  $f_2$  is monotone;
- *confluent*, provided that for each subcontinuum  $Q$  of  $Y$  each component of  $f^{-1}(Q)$  is mapped onto  $Q$  under  $f$ .

Monotone, as well as open mappings of compact spaces are known to be confluent, [12, Theorem 7.5, p. 148]. OM- and MO-mappings were introduced in [7, Section 3, p. 104] and studied in [8]. It is known that OM-mappings coincide with quasi-interior ones, as introduced in [13, p. 9], see [7, Corollary 3.1, p. 104], and that all MO-mappings are OM-mappings, [7, Corollary 3.2, p. 104].

Let  $\mathfrak{M}_i$ , where  $i \in \{1, 2, 3\}$  be some three classes of mappings between continua. A general problem which is related to a given mapping and to the two induced mappings is to find all interrelations between the following three statements:

$$(0.1) \quad f \in \mathfrak{M}_1;$$

$$(0.2) \quad C(f) \in \mathfrak{M}_2;$$

$$(0.3) \quad 2^f \in \mathfrak{M}_3.$$

There are some papers in which particular results concerning this problem are shown for various classes  $\mathfrak{M}_i$  of mappings. In the present paper we will discuss possibly implications between (0.1)–(0.3) for the class of MO-mappings. We start with recalling some related results.

The following results concerning induced mappings for the classes of monotone, of open, and of OM-mappings are known. For monotone mappings see [10, Lemma 2.1, p. 750]; compare [6, Theorem 1.1, p. 121], [3, Lemma 2.3, p. 2], [2, Theorem 3.3, p. 4], and [5, Theorem 3.2, p. 241]. For open mappings see [5, Theorem 4.3, p. 243]; compare also [4, Theorem 3.2]). For OM-mappings see [5, Theorem 5.2, p. 244].

1. THEOREM. *Let a surjective mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  be given. Then the following conditions are equivalent:*

- (1.1)  $f : X \rightarrow Y$  is monotone;
- (1.2)  $C(f) : C(X) \rightarrow C(Y)$  is monotone;
- (1.3)  $2^f : 2^X \rightarrow 2^Y$  is monotone.

2. THEOREM. *Let a surjective mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  be given. Consider the following conditions:*

- (2.1)  $f : X \rightarrow Y$  is open;
- (2.2)  $C(f) : C(X) \rightarrow C(Y)$  is open;
- (2.3)  $2^f : 2^X \rightarrow 2^Y$  is open.

*Then (2.1) and (2.3) are equivalent, and each of them is implied by (2.2).*

3. THEOREM. *Let a surjective mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  be given. Then the following conditions are equivalent:*

- (3.1)  $f : X \rightarrow Y$  is an OM-mapping;
- (3.2)  $C(f) : C(X) \rightarrow C(Y)$  is an OM-mapping;
- (3.3)  $2^f : 2^X \rightarrow 2^Y$  is an OM-mapping.

An example is known [5, Section 4, Example, p. 244] of an open surjective mapping  $f : X \rightarrow Y$  between locally connected continua  $X$  and  $Y$  such that the induced mapping  $C(f) : C(X) \rightarrow C(Y)$  is not open. It is so because of the following result. [1, Theorem 1].

4. THEOREM. *If a continuum  $X$  is locally connected, and for a mapping  $f : X \rightarrow Y$  the induced mapping  $C(f) : C(X) \rightarrow C(Y)$  is open, then  $f$  is monotone.*

As a consequence of this theorem the following corollary has been shown in [1, Corollary 2].

5. COROLLARY. *Let a continuum  $X$  be hereditarily locally connected, and a mapping  $f : X \rightarrow Y$  be such that the induced mapping  $C(f) : C(X) \rightarrow C(Y)$  is open. Then  $f$  is a homeomorphism.*

The following result is a consequence of Theorems 1 and 2, see [5, Theorem 5.3, p. 245].

6. COROLLARY. *If a mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is an MO-mapping, then  $2^f$  is also an MO-mapping.*

Investigating the class  $\mathfrak{M}$  of MO-mappings, H. Hosokawa asked in [4, Remark 3.7] if the condition  $f \in \mathfrak{M}$  implies that  $C(f) \in \mathfrak{M}$ . Later, in [5, Section 8, Problem 2, p. 249] he asked if the implication holds under an additional assumption that the mapping  $f$  is open. Our next result presents a negative answer to both these questions. To formulate it we recall a countable family of open mappings of the closed unit interval onto itself. Let a positive integer  $k$  be given and let  $m \in \{0, 1, \dots, k - 1\}$ . Define a surjection

$$(7) \quad g_k : [0, 1] \rightarrow [0, 1]$$

by the following conditions.

(7.1) If  $m$  is even, then  $g_k(m/k) = 0$ , and if  $m$  is odd, then  $g_k(m/k) = 1$ .

(7.2) For each  $m$  the restriction  $g_k|_{[m/k, (m+1)/k]} : [m/k, (m+1)/k] \rightarrow [0, 1]$  is defined to be linear.

Thus this restriction, and hence the mapping  $g_k$ , is a surjection. Note that  $g_k(0) = 0$  and  $g_k(1)$  is either 1 or 0 according to  $k$  is either odd or even. Observe that  $g_1$  is the identity, and  $g_2$  is the *tent mapping* defined by

$$(7.3) \quad g_2(x) = \begin{cases} 2x. & \text{for } x \in [0, 1/2], \\ 2 - 2x. & \text{for } x \in [1/2, 1]. \end{cases}$$

Recall that two mappings  $f_1 : X_1 \rightarrow Y_1$  and  $f_2 : X_2 \rightarrow Y_2$  are said to be *topologically equivalent* provided that there exist homeomorphisms  $h_X : X_1 \rightarrow X_2$  and  $h_Y : Y_1 \rightarrow Y_2$  such that  $f_2(h_X(x)) = h_Y(f_1(x))$  for each point  $x \in X$ . Observe that this relation is an equivalence in the class of mappings between topological spaces (see [12, p. 127]). It is known (see [12, (1.3), p. 184]) that a mapping of  $[0,1]$  into itself is open if and only if it is topologically equivalent to  $g_k : [0, 1] \rightarrow [0, 1]$  for some positive integer  $k$ .

8. PROPOSITION. *If  $g_2 : [0, 1] \rightarrow [0, 1]$  is the tent mapping, then the induced mapping  $C(g_2)$  is an MO-mapping which is neither open nor monotone.*

PROOF. Since any nonempty subcontinuum of  $[0, 1]$  is a closed interval  $[x, y]$  with  $0 \leq x \leq y \leq 1$ , where  $[x, x] = \{x\}$ , hence one can assign to  $[x, y] \subset [0, 1]$  a

point  $(x, y)$  of the triangle

$$T = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq y \leq 1\},$$

and, under this correspondence, the topology induced by the Hausdorff metric on  $C([0, 1])$  coincides with the Euclidean topology inherited from the plane  $\mathbf{R}^2$  on  $T$  (see e.g. [11, p. 62]). To simplify notations we omit the homeomorphism between  $C([0, 1])$  and  $T$ . Thus the formula (7.3) for  $g_2$  implies the following one for the induced mapping  $C(g_2) : T \rightarrow T$ :

$$C(g_2)((x, y)) = \begin{cases} (2x, 2y) & \text{if } 0 \leq y \leq 1/2, \\ (\min\{2x, 2 - 2y\}, 1) & \text{if } 0 \leq x \leq 1/2 \leq y \leq 1, \\ (2 - 2y, 2 - 2x) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

To see that  $C(g_2)$  so defined is an MO-mapping let us consider two additional triangles:  $T' = \{(x, y) \in T : x + y \leq 1\}$  with vertices  $(0, 0)$ ,  $(0, 1)$ ,  $(1/2, 1/2)$ , and  $T'' = \{(x, y) \in T' : 0 \leq y \leq 1/2\}$  with vertices  $(0, 0)$ ,  $(0, 1/2)$ ,  $(1/2, 1/2)$ . Define a surjection  $f_1 : T \rightarrow T'$  such that  $f_1|_{T'}$  is the identity and  $f_1|(T \setminus T')$  is the symmetry with respect to the straight line  $x + y = 1$ . Thus  $f_1$  is open, and we have

$$f_1((x, y)) = \begin{cases} (x, y) & \text{if } 0 \leq x + y \leq 1, \\ (1 - y, 1 - x) & \text{if } x + y \geq 1. \end{cases}$$

Next define a surjection  $f_2 : T' \rightarrow T''$  such that  $f_2|_{T''}$  is the identity, and  $f_2|(T' \setminus T'')$  projects points on the side of  $T''$  that joins  $(0, 1/2)$  with  $(1/2, 1/2)$ . Thus  $f_2$  is monotone, and its formula is

$$f_2((x, y)) = \begin{cases} (x, y) & \text{if } 0 \leq y \leq 1/2, \\ (x, 1/2) & \text{if } y \geq 1/2. \end{cases}$$

Finally, let  $h : T'' \rightarrow T$  be a homeomorphism defined by  $h((x, y)) = (2x, 2y)$  for all  $(x, y) \in T''$ . It can be verified (details are left to the reader) that  $C(g_2) = (h \circ f_2) \circ f_1$ . Thus  $C(g_2)$  is an MO-mapping. The proof is complete.

**9. PROPOSITION.** *Let a mapping  $g_k : [0, 1] \rightarrow [0, 1]$  be as in (7). Then for each integer  $k \geq 3$  the induced mapping  $C(g_k)$  is not an MO-mapping.*

**PROOF.** Suppose on the contrary that for some  $k \geq 3$  the induced mapping  $C(g_k) : C([0, 1]) \rightarrow C([0, 1])$  can be represented as the composition of two

mappings,  $C(g_k) = f_2 \circ f_1$ , where  $f_1$  is open and  $f_2$  is monotone. Let  $Y = f_1(C([0, 1]))$ , and put

$$A = \left[0, \frac{1}{2k}\right], \quad B = \left[\frac{2}{k} - \frac{1}{2k}, \frac{2}{k}\right], \quad C = \left[\frac{2}{k}, \frac{2}{k} + \frac{1}{2k}\right].$$

Observe that  $C(g_k)(A) = C(g_k)(B) = C(g_k)(C) = [0, 1/2]$ . Let  $\mathcal{U} = \{P \in C([0, 1]) : H(P, A) < 1/4k\}$ . Then

(9.1) the restriction  $C(g_k)|_{\mathcal{U}}$  is one-to-one,

whence  $f_1|_{\mathcal{U}}$  is one-to-one. We claim that

(9.2)  $f_1(A) = f_1(B)$ .

Indeed, if not, we have  $f_1(A) \neq f_1(B)$ , but  $f_2(f_1(A)) = f_2(f_1(B)) = [0, 1/2]$ , and since  $f_2$  is monotone, there is a continuum  $M \subset Y$  with  $f_1(A), f_1(B) \in M$  and  $f_2(M) = \{[0, 1/2]\}$ . Let  $\mathcal{C} \subset C([0, 1])$  be the component of  $f_1^{-1}(M)$  which contains  $A$ . Since  $f_1$  is open, it is confluent, [12, Theorem 7.5, p. 148], so  $f_1(\mathcal{C}) = M$ , and thus  $\mathcal{C}$  is a nondegenerate continuum containing  $A$ . Then  $C(g_k)(\mathcal{C} \cap \mathcal{U})$  is a one-point set  $\{[0, 1/2]\}$ , contrary to (9.1). Thus (9.2) is established.

Let  $\mathcal{V} = \{P \in C(B) : H(P, B) < 1/4k\}$ . Then  $C(g_k)|_{\mathcal{V}}$  is one-to-one, whence  $f_1|_{\mathcal{V}}$  is one-to-one as well. Note that  $\mathcal{V}$  is not a neighborhood of  $B$ .

Let  $\{B_m\}$  be a sequence of continua in  $[0, 1]$  satisfying  $B_m \subset B$  and  $2/k \notin B_m$  for each  $m \in \mathbb{N}$ , and  $B = \text{Lim } B_m$ . Observe that  $(C(g_k))^{-1}(C(g_k)(B_m))$  has exactly  $k$  points. Therefore  $f_1^{-1}(f_1(B_m))$  is a subset of the finite set  $(C(g_k))^{-1}(C(g_k)(B_m))$ , so it is finite. Openness of  $f_1$  implies that  $f_1^{-1}$  is continuous, see [12, Theorem 4.32, p. 130], so

(9.3)  $f_1^{-1}(f_1(B))$  is finite.

Let  $\mathcal{A}$  be the (unique) order arc in  $C([0, 1])$  from  $B$  to  $B \cup C$ . By (9.3) the set  $f_1(\mathcal{A})$  is a nondegenerate subcontinuum of  $Y$ . By (9.2) we see that  $A \in f_1^{-1}(f_1(\mathcal{A}))$ . Then the component  $\mathcal{X}$  of  $f_1^{-1}(f_1(\mathcal{A}))$  which contains  $A$  is a nondegenerate subcontinuum of  $C([0, 1])$  by confluence of  $f_1$ . Note that  $C(g_k)(\mathcal{A}) = \{[0, 1/2]\}$ , whence  $C(g_k)(\mathcal{X}) = C(g_k)(\mathcal{A}) = \{[0, 1/2]\}$ , contrary to (9.1). Thus the proof is finished.

As a consequence of Propositions 8 and 9 we have the following result.

10. THEOREM. *The identity  $g_1$  and the tent mapping  $g_2$  are the only two (up to equivalence) open mappings  $f : [0, 1] \rightarrow [0, 1]$  for which the induced mapping  $C(f)$  is an MO-mapping.*

11. REMARKS. (11.1) Taking as a mapping  $f : X \rightarrow Y$  the mapping  $g_k$  for some integer  $k \geq 3$  we see, by Proposition 9, that even in the case when  $f$  is open, the induced mapping  $C(f)$  need not be an MO-mapping.

(11.2) Since openness of  $f$  is equivalent to that of  $2^f$  (see Theorem 2), it follows from (11.1) that even if  $2^f$  is an open mapping (an MO-mapping, in particular), then  $C(f)$  need not be an MO-mapping.

The following three questions remain open. The first two of them were asked in [5, Section 8, Problem 2, p. 249].

12. QUESTIONS. (12.1) If  $2^f$  is an MO-mapping, must  $f$  be an MO-mapping?  
 (12.2) If  $C(f)$  is an MO-mapping, must  $f$  be an MO-mapping?  
 (12.3) If  $C(f)$  is an MO-mapping, must  $2^f$  be an MO-mapping?

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