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R^i-CONTINUA AND HYPERSPACES

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It is proved that if a continuum X contains an R^i-continuum for some $i \in \{1, 2, 3\}$, then the hyperspaces 2^X and $C(X)$ contain R^i-continua, therefore they are not contractible. Moreover, 2^X has no confluent Whitney map. Some examples concerning this subject are given and some questions are asked.

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hyperspace R^i-continuum contractible
continuum Whitney map

The first example of a continuum with noncontractible hyperspaces has been constructed by Kelley [6, p. 27]. Nadler asked [7, (16.34), p. 557] for some necessary or sufficient conditions on a continuum X in order that 2^X (equivalently $C(X)$) be contractible. Very recently, in an excellent paper, Curtis has given several such conditions [2, Theorem 5.4] that are formulated in terms of fiber functions or the evaluation map.

In this paper another necessary condition of contractibility of 2^X is shown, which is expressed by inner geometric properties of X rather than by functional ones; that is, the continuum must contain no Czuba's R^i-continuum for $i \in \{1, 2, 3\}$. We show even more: the existence of an R^i-continuum in X implies one in 2^X and $C(X)$, and hence it implies noncontractibility of the hyperspaces. We also prove that path-connectedness of the admissible fiber function $A: X \to 2^{C(X)}$ implies that X contains no R^i-continuum, and we show that the implication is not reversible.

In this paper examples are given, showing, among others, that the notion of an R^i-continuum cannot be replaced by one of a zig-zag, in the sense of Graham [5]. Furthermore, it is proved that the property of containing an R^i-continuum is not a Whitney-reversible property.

Last of all, we show that if a continuum X contains an R^i-continuum, then it has no confluent Whitney map for 2^X. Some questions concerning the discussed subject are asked.

All spaces considered in this paper are assumed to be metric. A continuum is a compact and connected space. For a given continuum X and a given sequence $\{K_n\}$

of subsets of X we denote the upper limit by Ls K_n, the lower limit by Li K_n, and the limit of the sequence $\{K_n\}$ by Lim K_n.

Let $x_1, \ldots, x_n$ be points of an Euclidean space. We denote by $x_1 \cdots x_n$ the broken line with vertices $x_1, \ldots, x_n$.

2^X denotes the hyperspace of all nonempty compact subsets of a continuum X, and $C(X)$ the hyperspace of all nonempty subcontinua of X.

Given a continuum X, a Whitney map for 2^X (for $C(X)$, see [7, (0.50), p. 24]) is a continuous function μ from 2^X (from $C(X)$) into $[0, \infty)$ satisfying $\mu(\{x\}) = 0$ for all $x \in X$ and $\mu(A) < \mu(B)$ if A is a proper subset of B.

A topological property $\mathcal{P}$ is called a Whitney property [7, p. 399], provided that if a continuum X has $\mathcal{P}$, then $\mu^{-1}(t)$ has $\mathcal{P}$ for every Whitney map μ for $C(X)$ and for every $t \in [0, \mu(X))$. Further, $\mathcal{P}$ is called a Whitney-reversible property [7, (14.45), p. 453] provided that whenever X is a continuum such that $\mu^{-1}(t)$ has $\mathcal{P}$ for all $t \in (0, \mu(X))$, then X has $\mathcal{P}$.

A space X is called contractible in Y [7, (16.2), p. 532], where $X \subset Y$, if there exists a continuous mapping $H: X \times [0, 1] \to Y$ such that $H(x, 0) = x$ and $H(x, 1)$ is a fixed point for all $x \in X$. If $X = Y$ we say X is contractible.

For a continuum X we denote by $F_1(X)$ [7, (0.48), p. 23] the set of singletons of X, i.e. $F_1(X) = \{\{x\}: x \in X\}$. Note that $F_1(X)$ is homeomorphic (even isometric) to X and that it is a subset of $C(X)$.

Recall a theorem of Kelley [6, 3.1., p. 25; 7, (16.7), p. 535]:

Theorem A. *The following conditions are equivalent for a continuum* X:

(i) 2^X *is contractible,*

(ii) $C(X)$ *is contractible,*

(iii) $F_1(X)$ *is contractible in* $C(X)$.

We say that a nonempty proper subcontinuum K of a continuum X is
- an R^1-continuum [4, Definition 1.1, p. 75] if there exist an open set U such that $K \subset U$ and two sequences $\{C_n^1\}$, $\{C_n^2\}$ of components of U such that Ls $C_n^1 \cap$ Ls $C_n^2 = K$;
- an R^2-continuum [4, Definition 1.2, p. 75] if there exist an open set U such that $K \subset U$ and two sequences $\{C_n^1\}$, $\{C_n^2\}$ of components of U such that Lim $C_n^1 \cap$ Lim $C_n^2 = K$;
- an R^3-continuum [4, Definition 1.3, p. 75] if there exist an open set U such that $K \subset U$ and a sequence $\{C_n\}$ of components of U such that Li $C_n = K$.

The following proposition has first been observed in [4] for dendroids [4, Proposition 5, p. 77; and, Corollary 11, p. 78], but it holds for all continua.

Proposition 1. *If a continuum* X *contains an* R^i*-continuum for some* $i \in \{1, 2, 3\}$, *then it contains an* R^3*-continuum.*

Really, if X contains an R^1-continuum, then, taking a subsequence if necessary, it contains an R^2-continuum, and each R^2-continuum is an R^3-continuum.

Similarly we have

Theorem 2. *If a continuum X contains an R^i-continuum for some $i \in \{1,2,3\}$, then this R^i-continuum is homotopically fixed and hence X is not contractible.*

Indeed, the theorem is proved by Czuba [3, Theorem 3] under an additional assumption that the continuum X is a dendroid and for R^1-continua only. However, the assumption connected with dendroids is not used in that proof; and argumentation for R^2- and R^3-continua is similar. So the theorem holds true in general, as stated above.

Now we prove some theorems on R^i-continua in hyperspaces. We start with

Theorem 3. *Let a continuum X be given. If $K \subset X$ is an R^1-continuum (R^2-continuum, R^3-continuum), then 2^K is an R^1-continuum (R^2-continuum, R^3-continuum, respectively) in 2^X.*

Proof. We prove the theorem for R^3-continua. The proof for R^1-continua and R^2-continua is almost the same.

Let U be an open set with $K \subset U$ and $\{C_n\}$ be a sequence of components of U satisfying $\mathrm{Li}\, C_n = K$. Then $2^K \subset 2^U$ and 2^U is an open subset of 2^X. Observe that 2^{C_n} are components of 2^U and since $2^*\colon 2^X \to 2^{2^X}$ defined by $2^*(A) = 2^A$ is continuous [7, (15.4), p. 516], it can easily be verified that $\mathrm{Li}\, 2^{C_n} = 2^K$. So 2^K is an R^3-continuum in 2^X. □

Theorems 3, 2 and A imply:

Corollary 4. *If a continuum X contains an R^i-continuum for some $i \in \{1,2,3\}$, then 2^X and $C(X)$ are not contractible.*

The following example of a dendroid (i.e. an arcwise connected and hereditarily unicoherent continuum) shows that the converse of Corollary 4 is not true.

Example 5. There exists a continuum (even a plane dendroid) X with noncontractible hyperspaces 2^X and $C(X)$ that contains no R^i-continuum.

In the Euclidean plane (see Fig. 1) let $a = (0,1)$, $b = (0,0)$, $c = (0,-1)$ and for each positive integer n put $a_n = (2^{-3n}, 1)$, $b_n = (2^{-3n}, 0)$, $c_n = (2^{-(3n+1)}, -1)$, $d_n = (2^{-(3n-1)}, -1)$, and, for $m \in \{1,2,\ldots\}$, let $b_{n,m} = (2^{-3n}(1-2^{-(m+3)}), 0)$, $b'_{n,m} = (2^{-3n}(1+2^{-(m+3)}), 0)$, $c_{n,m} = (2^{-(3n+1)}(1-2^{-(m+3)}), -1)$, $d_{n,m} = (2^{-(3n-1)}(1+2^{-(m+3)}), -1)$. Thus $a_n \to a$, $b_n \to b$, $c_n \to c$ as $n \to \infty$; further, for any fixed n we have $b_{n,m} \to b_n$, $b'_{n,m} \to b_n$, $c_{n,m} \to c_n$ and $d_{n,m} \to d_n$ as $m \to \infty$.

For each $n \in \{1,2,\ldots\}$ put $Q_n = a_n b_n \cup b_n c_n \cup b_n d_n \cup \bigcup_{m=1}^{\infty} a_n b_{n,m} c_{n,m} \cup \bigcup_{m=1}^{\infty} a_n b'_{n,m} d_{n,m}$ and note that Q_n is a dendroid. Next we define $Y = aa_1 \cup ac \cup \bigcup_{n=1}^{\infty} Q_n$ and let Y' be the image of Y under the central symmetry with respect to the origin $b = (0,0)$. Finally we put $X = Y \cup Y'$. So X is a plane dendroid. One can verify that X contains no R^i-continuum for $i \in \{1,2,3\}$.

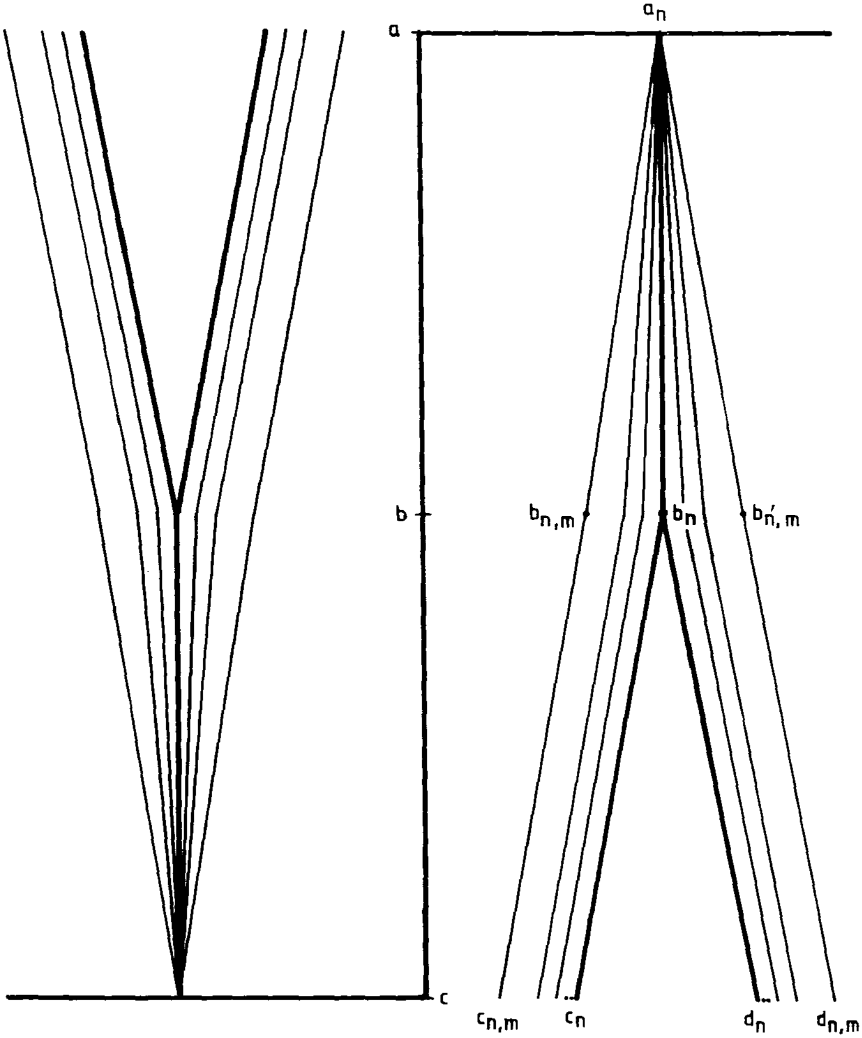

Fig. 1

Now we show $C(X)$ is not contractible. Assume, on the contrary, that there is a homotopy $H: X\times[0,1]\to C(X)$ satisfying $H(x,0)=\{x\}$, $H(x,1)=X$ and $H(x,t_1)\subset H(x,t_2)$ whenever $t_1\leq t_2$ (see [7, (16.5), p. 534]). Observe that there is a $t\in[0,1]$ such that $H(b,t)$ is a nondegenerate continuum in the arc ac and containing the point b. By the symmetry of X with respect to the point b we can assume without loss of generality that $H(b,t)\cap bc\neq\{b\}$. Then for almost all n we have $H(b_n,t)\cap c_nb_nd_n$ is a nondegenerate continuum containing b_n. On the other hand, for each fixed n, we have $H(b_n,t)=\operatorname{Lim} H(b_{n,m},t)\subset a_nb_nc_n$ and $H(b_n,t)=\operatorname{Lim} H(b'_{n,m},t)\subset a_nb_nd_n$, whence $H(b_n,t)\subset a_nb_n$, a contradiction.

Theorem 6. *If $K \subset X$ is an R^1-continuum (R^2-continuum, R^3-continuum) in a continuum X, then $C(X)$ contains an R^1-continuum (R^2-continuum, R^3-continuum, respectively) in $C(X)$.*

Proof. If $K \subset X$ is an R^1-continuum, let U and $\{C_n^1\}$, $\{C_n^2\}$ be as in the definition. Then $C(C_n^1)$ and $C(C_n^2)$ are components of $C(U)$. Put $\mathcal{K} = \text{Ls } C(C_n^1) \cap \text{Ls } C(C_n^2)$. Then $K \in \mathcal{K}$, so $\mathcal{K}$ is nonempty, $\mathcal{K} \subset C(X)$ and it is an R^1-continuum in $C(X)$.

If $K \subset X$ is an R^2-continuum, then $\mathcal{K}$ defined above is an R^1-continuum in $C(X)$ and thus it contains an R^2-continuum.

If $K \subset X$ is an R^3-continuum and U, $\{C_n\}$ satisfy the definition, then $\mathcal{K} = \text{Li } C(C_n)$ is an R^3-continuum in $C(X)$. □

Corollary 7. *If a continuum X contains an R^i-continuum for some $i \in \{1, 2, 3\}$, then 2^X and $C(X)$ contain R^3-continua.*

Note that in Theorem 6 we cannot say '$C(X)$ is an R^i-continuum' (see Example 9), because the function $C^*: C(X) \to C(C(X))$ defined by $C^*(A) = C(A)$ need not be continuous. If it is continuous, then the continuum X is said to be C^*-smooth [7, (15.5), p. 517]. So, arguing as in Theorem 3, we can prove

Proposition 8. *If a continuum X is C^*-smooth and if it contains an R^1-continuum (R^2-continuum, R^3-continuum) K, then $C(K)$ is an R^1-continuum (R^2-continuum, R^3-continuum, respectively) in $C(X)$.*

The following example shows that the assumption of C^*-smoothness of X is necessary in Proposition 8.

Example 9. There exist a continuum X and a subcontinuum S of X such that S is an R^2-continuum in X (and hence it is an R^1-continuum and R^3-continuum) and such that $C(S)$ is an R^i-continuum in $C(X)$ for no $i \in \{1, 2, 3\}$.

In the Euclidean three-space let $S = \{(x, y, 0): x^2 + y^2 = 1\}$ and $A_n = \{(x, y, 0): x^2 + y^2 = (1 + 1/n)^2 \text{ and } x \geq -1\}$ for $n \in \{1, 2, \ldots\}$. Thus $\lim A_n = S$. Put $a = (0, 0, 1)$, $b = (0, 0, -1)$, $p = (1, 0, 0)$, and $p_n = (1 + 1/n, 0, 0)$. Hence $p \in S$, $p_n \in A_n$ and $p_n \to p$ as $n \to \infty$. Define

$$X = S \cup ap \cup bp \cup \bigcup_{n=1}^{\infty} (A_{2n} \cup ap_{2n}) \cup \bigcup_{n=1}^{\infty} (A_{2n-1} \cup bp_{2n-1}).$$

One can easily verify that S is an R^2-continuum in X. To see that $C(S)$ is not an R^3-continuum in $C(X)$ consider an open set $\mathcal{U} \subset C(X)$ containing $C(S)$ and assume there exists a sequence $\mathcal{C}_n$ of components of $\mathcal{U}$ such that $\text{Li } \mathcal{C}_n = C(S)$. Let $K = \{(x, y, 0): x^2 + y^2 = 1 \text{ and } x \leq 0\} \subset S$. By the definition of the lower limit there are continua $K_n \in \mathcal{C}_n$ such that $K = \text{Lim } K_n$, but then $K_n \in C(S)$ for almost all n which means that, for almost all n, $\mathcal{C}_n$ is the component of $\mathcal{U}$ containing $C(S)$, and so $C(S)$ is a proper subcontinuum of $\text{Li } \mathcal{C}_n$, a contradiction. The argumentation showing that $C(S)$ is neither R^1- nor R^2-continuum is very similar.

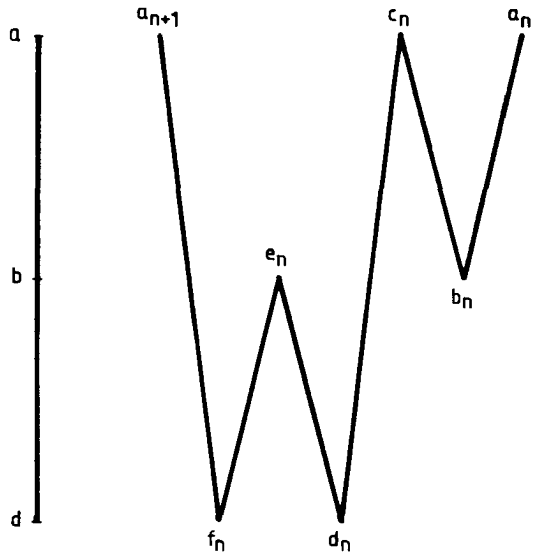

Fig. 2.

Note that by Theorem 6 the continuum $C(S)$ contains an R^2-continuum $\mathscr{K}$. In our example $\mathscr{K} = \{K \in C(S)$: $K = S$, or $K \neq S$ and the point $(-1, 0, 0)$ is not an interior point of the arc $K\}$.

Example 10. There exists a continuum X which is a compactification of the half-line $[0, \infty)$ by an arc and which contains a one-point R^2-continuum; hence $C(X)$ is not contractible.

To describe X define in the plane, for $n \in \{1, 2, \dots\}$, points (see Fig. 2) $a_n = (1/6n, 1)$, $b_n = (1/(6n+1), 0)$, $c_n = (1/(6n+2), 1)$, $d_n = (1/(6n+3), -1)$, $e_n = (1/(6n+4), 0)$, and $f_n = (1/(6n+5), -1)$, and let $I = \{(0, y)$: $-1 \leq y \leq 1\}$. Put $X = I \cup \bigcup_{n=1}^{\infty} a_n b_n c_n d_n e_n f_n a_{n+1}$. Then the one-point continuum $\{(0,0)\}$ is an R^2-continuum in X.

Proposition 11. *The property of containing an R^i-continuum is not a Whitney property.*

Indeed, if the continuum X is defined as in Example 10 and μ: $C(X) \to [0, \infty)$ is any Whitney map, then $\mu^{-1}(\mu(I))$ is an arc.

Proposition 12. *The property of containing no R^i-continuum is not a Whitney-reversible property.*

Indeed, take as X the one-point union of two harmonic fans such that their intersection is the common end point (distinct from the top) of the limit segment of each of them (see [8, Example 3.7, p. 243]), and let μ: $C(X) \to [0, \infty)$ be any Whitney map. Then the intersection is a one-point R^2-continuum, while for each $t > 0$ the continuum $\mu^{-1}(t)$ is even contractible, so it contains no R^i-continuum.

The same counterexample shows, by Corollary 4, the following.

Proposition 13. *Contractibility of the hyperspace is not a Whitney-reversible property.*

Remark 14. Let us return to the continuum X defined in Example 10. One can verify that for any Whitney map μ: $C(X) \to [0, \infty)$ and for any $t > 0$ the continuum $\mu^{-1}(t)$ has contractible hyperspaces. Thus X of Example 10 can be used as another counterexample showing Propositions 12 and 13 are true.

Given a continuum X, we denote by d a metric on X and by 'dist' the Hausdorff distance on $C(X)$. Recall that a subcontinuum M of X is said to be admissible at a point $x \in M$ (see [2, 5.1]) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each $y \in X$ with $d(x, y) < \delta$, there exists a continuum N containing y with $\text{dist}(M, N) < \varepsilon$. Let A: $X \to 2^{C(X)}$ denote the admissible fiber function defined by $A(x) = \{M \in C(X)$: $x \in M$ and M is admissible at $x\}$. We say that A is path-connected if each fiber $A(x)$ is path-connected.

Curtis has shown [2, Theorem 5.4], among other things, that path-connectedness of A is equivalent to almost openness of the evaluation map (see [2, § 1] for the definition) and it is implied by contractibility of $C(X)$. We prove that containing no R^i-continuum is a weaker condition than path-connectedness of A.

Proposition 15. *Let X be a continuum and let A: $X \to 2^{C(X)}$ be the admissible fiber function. If, for some point $x \in X$, there is an arc in $A(x)$ joining $\{x\}$ and X, then there is no R^3-continuum containing x.*

Proof. Assume on the contrary that $K \in C(X)$ with $x \in K$ is an R^3-continuum in X. Let U be an open set containing K as in the definition of an R^3-continuum. Since $A(x)$ is path-connected between $\{x\}$ and X, there is a continuum $L \in A(x)$ such that $L \subset U$ and $L \setminus K \neq \emptyset$. Take $y \in L \setminus K$ and consider a sequence $\{x_n\}_{n=1}^{\infty}$ of points of X tending to x such that, if $K(x_n)$ is the component of U containing x_n, we have $y \notin \text{Ls}\, K(x_n)$ (such a sequence does exist because K is an R^3-continuum). Since $L \in A(x)$, there are continua $L_n \to L$ with $x_n \in L_n$ and we have $y \in \text{Lim}\, L_n \subset \text{Ls}\, K(x_n)$, a contradiction. □

Corollary 16. *If the admissible fiber function A: $X \to 2^{C(X)}$ is path-connected, then X contains no R^i-continuum for $i \in \{1, 2, 3\}$.*

Really, it follows from Propositions 15 and 1.

Proposition 17. *The converse of Corollary 16 is not true.*

Proof. Take the Cantor ternary set C in the unit closed interval I and construct a sequence of disjoint circles having the components of $I \setminus C$ as their diameters. The union of the circles together with the Cantor set is a (one-dimensional) plane continuum. Next approximate the upper and the lower halves of the continuum by sequences of arcs from above and below correspondingly starting from the point $0 \in I$ and disjoint out of it. The plane continuum obtained in this way contains no R^i-continuum, but the fibers $A(x)$ are not connected for $x \in C$. □

The next four questions are related to the topic.

Question 18. Is contractibility of the hyperspace a Whitney property?

Question 19. Is the property of containing no R^3-continuum a Whitney property?

Question 20. If $C(X)$ (2^X) contains an R^3-continuum, then does X?

Question 21. Does there exist a continuum X such that $C(X)$ (equivalently 2^X) is not contractible and $C(X)$ and/or 2^X contain no R^3-continuum? In particular does the continuum X described in Example 5 possess this property?

Now we give an example showing that another property—containing a zig-zag (see [5, p. 78] for the definition) that implies noncontractibility of dendroids does not imply noncontractibility of their hyperspaces.

Example 22. There exists a dendroid X containing a zig-zag such that $C(X)$ (equivalently 2^X) is contractible.

Proof. For $n \in \{1, 2, \dots\}$, let $a_n = (1/n, 1)$, $b_n = (-1/n, -1)$, $a = (0, 1)$, $b = (0, -1)$, and $I = ab$. Put $X = I \cup \bigcup_{n=1}^{\infty} ba_n \cup \bigcup_{n=1}^{\infty} ab_n$. To see $C(X)$ is contractible define a function $\alpha: X \to C(C(X))$ by

$$\alpha(x) = \begin{cases} \{M \in C(X): x \in M \subset I \text{ or } I \subset M\} & \text{if } x \in I, \\ \{M \in C(X): x \in M\} & \text{if } x \notin I. \end{cases}$$

It is easy to verify that the function α is a c-function as defined in [2, 5.2] and, because the existence of a c-function is equivalent to contractibility of $C(X)$ [2, Theorem 5.4], we are done. □

Recall that a continuous mapping $f: X \to Y$ of a continuum X onto Y is said to be confluent provided that for each subcontinuum Q of Y and each component C of $f^{-1}(Q)$ we have $f(C) = Q$. It is known that monotone mappings and open mappings are confluent (see e.g. [7, (0.45.3), p. 21]).

Now we are interested in the existence of a confluent Whitney map for 2^X for an arbitrary continuum X. The existence of such a mapping is related to two questions of Nadler [7, (14.63), p. 468, and (14.64), p. 469] who asks if for each continuum X there is a Whitney map for 2^X which is either monotone or open. In [1] the author has given a negative answer to both these questions by showing an example of a continuum X having no confluent Whitney map for 2^X. Our next result is a theorem from which the existence of many such counterexamples follows. To prove the result we need a lemma.

Lemma 23. *Let a compact space Z be given, and let $P, Q \in 2^Z$. Then P and Q belong to the same component of 2^Z if and only if for each component C of Z the condition $P \cap C \neq \emptyset$ is equivalent to the condition $Q \cap C \neq \emptyset$. Moreover, each component of 2^Z is arcwise connected.*

Proof. First assume there is a component C of Z with $P \cap C = \emptyset$ and $Q \cap C \neq \emptyset$. Let U be an open and closed subset of Z such that $C \subset U$ and $P \cap U = \emptyset$. Then $\{A \in 2^Z: A \subset Z \setminus U\}$ and $\{A \in 2^Z: A \cap U \neq \emptyset\}$ are both open subsets of 2^Z containing P and Q respectively, and their union equals the whole space. So 2^Z is not connected between P and Q.

Secondly assume $P \cap C \neq \emptyset$ if and only if $Q \cap C \neq \emptyset$ for each component C of Z. Let R denote the union of all components of Z intersecting P (equivalently intersecting Q). Then, by [7, (1.8), p. 59] there exist in 2^Z ordered arcs from P to R and from Q to R. Thus P and Q belong to the same component of 2^Z and the component is arcwise connected. □

Theorem 24. *If a continuum X contains an R^i-continuum for some $i \in \{1, 2, 3\}$, then it has no confluent Whitney map for 2^X.*

Proof. By Proposition 1 it is enough to prove the theorem for R^3-continua only. So assume X is a given continuum, K is a proper subcontinuum of X, U is an open subset of X with $K \subset U$, and C_n, for $n \in \{1, 2, \ldots\}$, are components of U such that $\operatorname{Li} C_n = K$. Let p be any point of K and choose $p_n \in C_n$ in such a way that $\lim p_n = p$. Define $A_n = \{p, p_n, p_{n+1}, \ldots\}$. So each A_n is a compact subset of X and $\operatorname{Lim} A_n = \{p\}$.

Let μ be any fixed Whitney map for 2^X and let $\varepsilon > 0$ be such a number that an ε-ball V about K satisfies $\bar{V} \subset U$.

Denote by D_n the component of $\bar{V}$ which contains the point p_n. For a given point $x \in \operatorname{Bd} \bar{V}$ let $\{D_n(x)\}_{n=1}^{\infty}$ be a subsequence of $\{D_n\}_{n=1}^{\infty}$ such that $x \notin \operatorname{Ls} D_n(x)$ (the existence of $\{D_n(x)\}_{n=1}^{\infty}$ is guaranteed by the fact that K is an R^3-continuum). Put $K(x) = \operatorname{Ls} D_n(x)$. Since $K(x)$ is a continuum and $x \notin K(x)$, there is a number $\varepsilon(x)$ such that the $\varepsilon(x)$-ball about x does not intersect $K(x)$. Let $V(x)$, for $x \in \operatorname{Bd} \bar{V}$ be the $\varepsilon(x)/2$-ball about x. Thus $\{V(x): x \in \operatorname{Bd} \bar{V}\}$ is a covering of $\operatorname{Bd} \bar{V}$, so we can choose a finite subcovering $\{V(x_1), \ldots, V(x_i)\}$. Define $\varepsilon_0 = \min\{\varepsilon, \varepsilon(x_1), \ldots, \varepsilon(x_i)\}$ and let $\mathscr{B} = \{A \in 2^X: \operatorname{diam}(A) \geq \varepsilon_0/2\}$ ($\operatorname{diam}(A)$ denotes the diameter of A).

Thus $t = \inf\{\mu(A): A \in \mathscr{B}\}$ is a positive number. Consider the segment $[0, t/2]$ and suppose on the contrary that each component of $\mu^{-1}([0, t/2])$ is mapped by μ onto the whole segment. Note that $\lim \mu(A_n) = 0$, so there exists such a number k that $\mu(A_k) \in [0, t/2]$. Let $\mathscr{C}$ be the component of $\mu^{-1}([0, t/2])$ which contains A_k. Recall that, by the assumption, $0 \in \mu(\mathscr{C})$, i.e., $\mathscr{C}$ contains a one-point set, and $\mathscr{B} \cap \mathscr{C} = \emptyset$.

Let $\mathscr{L}$ denote the component of $2^{\bar{V}}$ such that $A_k \in \mathscr{L}$. By Lemma 23 we have $\mathscr{L} = \{A \in 2^{\bar{V}}: A \cap D_n \neq \emptyset$ if and only if $n \geq k\}$. Since $A_k \in \mathscr{C} \cap \mathscr{L}$ and $\mathscr{C} \setminus \mathscr{L}$ is non-empty (it contains a one-point set), there exists a set B belonging to the boundary of $\mathscr{C} \cap \mathscr{L}$ in $\mathscr{C}$. Then there exists a point $q \in B$ with $q \in \operatorname{Bd} \bar{V}$, hence there is an index $j \in \{1, 2, \ldots, i\}$ such that $q \in V(x_j)$. Now $B \in \mathscr{L}$ implies $B \cap D_n(x_j) \neq \emptyset$ for almost all n, so $B \cap \operatorname{Ls} D_n(x_j) = B \cap K(x_j) \neq \emptyset$. Let $r \in B \cap K(x_j)$. So we have found two points of B—namely q and r—one of them is in the $\varepsilon(x_j)/2$-ball about x_j and the other outside of $\varepsilon(x_j)$-ball about x_j. So $\operatorname{diam}(B) > \varepsilon(x_j)/2 \geqslant \varepsilon_0/2$ and hence $B \in \mathscr{B}$, a contradiction to $B \in \mathscr{C}$. $\square$

We end the paper asking some questions on the subject.

Question 25. Does there exist a continuum X which has a confluent (a monotone, an open) Whitney map for 2^X and such that 2^X is not contractible? In particular, has the continuum X of Example 5 a confluent (a monotone, an open) Whitney map for 2^X?

Remark that the continuum X described in [1] has the property of Kelley, so it has a contractible hyperspace, while it does not admit a confluent Whitney map for 2^X.

Question 26. Give a necessary and/or sufficient conditions for a continuum X to have a confluent (a monotone, an open) Whitney map for 2^X.

Partial answers to this question are in Theorem 24 above and in [7, (14.66), p. 471].

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