CONNECTEDNESS PROPERTIES OF WHITNEY LEVELS

JANUSZ J. CHARATONIK AND WŁODZIMIERZ J. CHARATONIK

ABSTRACT. It is shown that $\delta$-connectedness is not a Whitney reversible property. This answers in the negative a question posed by Sam B. Nadler, Jr. in 1978.

A topological property $\mathcal{P}$ is said to be:

(a) a Whitney property provided that if a continuum $X$ has property $\mathcal{P}$, so does $\mu^{-1}(t)$ for each Whitney map $\mu$ for $C(X)$ and each $t \in [0, \mu(X))$ ([6, p. 165]);

(b) a Whitney reversible property provided that whenever $X$ is a continuum such that $\mu^{-1}(t)$ has property $\mathcal{P}$ for all Whitney maps $\mu$ for $C(X)$ and all $t \in (0, \mu(X))$, then $X$ has property $\mathcal{P}$ ([8, p. 235]).

A continuum $X$ is said to be:

(c) $\delta$-connected provided that for every two points of $X$ there exists an irreducible continuum between them which is hereditarily decomposable ([5, p. 90]);

(d) $\lambda$-connected provided that for every two points of $X$ there exists an irreducible continuum between them which is of type $\lambda$ (that is, each of its indecomposable subcontinua has empty interior) ([5, p. 85]).

2000 Mathematics Subject Classification. 54B20, 54F15, 54F50.

Key words and phrases. continuum, $\delta$-connectedness, hyperspace, Whitney reversible property.

† Sadly, Professor Janusz J. Charatonik passed away on July 11, 2004.
(Note that in some papers, in particular in Sam B. Nadler, Jr.'s monograph [7, (0.30), p. 16], the name “\(\lambda\)-connected” is used in the sense of “\(\delta\)-connected.” See [3, p. 118] for an explanation.)

Answering a question in [6, Section 6, p. 179] (cf. [7, Question 14.36, p. 432]), the second named author has proved in [2, Construction 5.1, p. 387, and Remark 6, p. 389] that \(\delta\)-connectedness is not a Whitney property. A similar assertion for \(\lambda\)-connectedness is not known; see [2, Question 7, p. 390] and compare [4, Question 51.3, p. 280].

In [7, Question 14.57, p. 464], (compare [4, Question 51.4, p. 280]), Nadler asks if (i) \(\delta\)-connectedness is a Whitney reversible property, and if (ii) \(\lambda\)-connectedness is a Whitney reversible property. In this paper we present an example of a continuum showing a negative answer to (i). The question concerning (ii) remains open.

All considered spaces are assumed to be metric. We denote by \(\mathbb{N}\) the set of all positive integers, and by \(\mathbb{R}\) the space of reals. A continuum means a compact connected space, and a mapping means a continuous function. Given two points \(a\) and \(b\) in a Euclidean space, we denote by \(\overline{ab}\) the straight line segment joining these points.

A continuum is said to be decomposable provided that it can be represented as the union of two of its proper subcontinua. Otherwise, it is said to be indecomposable. A continuum is said to be hereditarily decomposable (hereditarily indecomposable) provided that each of its non-degenerate subcontinua is decomposable (indecomposable). A continuum \(X\) is said to be irreducible (between points \(a\) and \(b\) of \(X\)) provided that no proper subcontinuum of \(X\) contains these points. Points \(a\) and \(b\) are then called points of irreducibility of \(X\). An irreducible continuum \(X\) is said to be of type \(\lambda\) if each indecomposable subcontinuum of \(X\) has empty interior.

Given a continuum \(X\), we let \(2^X\) denote the hyperspace of all nonempty closed subsets of \(X\) equipped with the Hausdorff metric \(H\) (see e.g., [7, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by \(C(X)\) the hyperspace of all subcontinua of \(X\), i.e., of all connected elements of \(2^X\). We will write \(A = \text{Lim } A_n\) to denote that the (closed) sets tend to \(A\) with respect to the Hausdorff metric.

A Whitney map for \(C(X)\) is a mapping \(\mu : C(X) \to [0, \infty)\) such that:
(0.1) $\mu(A) < \mu(B)$ for every two $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;

(0.2) $\mu(A) = 0$ if and only if $A$ is a singleton.

For the concept and existence of a Whitney map, see [4, Section 13, p. 105-110]. For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a Whitney level. It is known that each Whitney level is a continuum, see [4, p. 159].

The reader is referred to monographs [4] and [7] for definitions and basic properties of other notions used in the paper.

To present the needed example of continuum $X$ showing that $\delta$-connectedness is not a Whitney reversible property, we start with some auxiliary constructions.

**Construction 1.** In the Cartesian coordinates $(x, y)$ in the plane, let $S_0$ be the standard $\sin \frac{1}{x}$-curve, that is,

\[(1.1) \quad S_0 = \{(0) \times [-1, 1] \cup \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 : x \in (0, 1]\}.

We will call $(1, 0)$ the end point of $S_0$, and $\{0\} \times [-1, 1]$ the limit segment of $S_0$.

Take a sequence of local maxima of $S_0$, that is, a sequence of points $p_n = (x_n, 1) \in S_0$ with $x_{n+1} < x_n$ for each $n \in \mathbb{N}$. Thus, $(0, 1) = \lim p_n$. Further, in the segment $\{0\} \times [1, 2]$ take a sequence of points $q_n = (0, y_n)$ such that the numbers $y_n$ form a decreasing sequence tending to 1

\[(1.2) \quad \lim y_n = 1 < \cdots < y_{n+1} < y_n < \cdots < y_1 = 2.

Thus, $(0, 1) = \lim q_n$.

For each $n \in \mathbb{N}$, let $S_n$ be a homeomorphic copy of $S_0$ situated in the rectangle $[0, 1] \times [1, 2]$ in such a way that:

\[(1.3) \quad p_n \text{ is the end point, and } L_n = \{0\} \times [y_{2n}, y_{2n-1}] \text{ is the limit segment, of } S_n, \text{ for each } n;
\]

\[(1.4) \quad S_n \cap S_0 = \{p_n\} \text{ for each } n;
\]

\[(1.5) \quad S_m \cap S_n = \emptyset \text{ for } m, n \in \mathbb{N} \text{ with } m \neq n;
\]

\[(1.6) \quad \text{Lim } S_n = \{(0, 1)\}.

Define

\[(1.7) \quad X_1 = (\{0\} \times [1, 2]) \cup S_0 \cup \bigcup \{S_n : n \in \mathbb{N}\}
\]

and observe that $X_1$ is a continuum having two arc components, and that the limit segments of $S_0$ and of all $S_n$ are components of the set of non-local connectedness of $X_1$. 
Construction 2. Recall that the pseudo-arc means an arc-like hereditarily indecomposable continuum (see e.g., [9, 1.23, p. 13]). Let $X_1$ be the continuum defined by (1.7). For each $n \in \{0\} \cup \mathbb{N}$, let $L_n$ stand for the limit segment, and $M_n$ stand for the non-compact arc component of $S_n$. Thus, $L_0 = (0, -1)(0, 1)$ and $L_n = \overline{q_{2n}q_{2n-1}}$ for any $n \in \mathbb{N}$. Hence, $S_n = L_n \cup M_n$ for each integer $n \geq 0$, and $L_0, L_1, L_2, \ldots$ are components of the set of points at which $X_1$ is not locally connected.

Note that each $M_n$ is locally compact. Since for each locally compact, noncompact metric space $M$ an arbitrary continuum $P$ can be a remainder of a compactification of $M$, (see [1, Theorem, p. 35]), it is possible to replace $L_n$ by a copy $P_n$ of the pseudo-arc in such a way that $\text{diam}(P_n) = \text{diam}(L_n)$ and that, if for each $n \in \{0\} \cup \mathbb{N}$, the symbol $M'_n$ denotes a one-to-one copy of the non-compact arc component $M_n$ of $S_n$, then $P_n = \text{cl}(M'_n) \setminus M'_n$ for $n \in \{0, 1, 2, \ldots \}$. Since the pseudo-arc is a plane continuum, the construction can be made in such a way that all the inserted pseudo-arcs $P_0, P_1, P_2, \ldots$ lie in the plane $\{(x, y, z) \in \mathbb{R}^3 : x = 0\}$. It follows that

\begin{equation}
(2.1) \quad \text{for each } n \in \mathbb{N}, \text{ the end points } q_{2n} \text{ and } q_{2n-1} \text{ of } L_n \text{ belong to the pseudo-arc } P_n.
\end{equation}

Denote by $X_2$ the continuum obtained from $X_1$ by the replacement described above, that is,

\begin{equation}
(2.2) \quad X_2 = \bigcup \{ (M'_n \cup P_n) : n \in \{0\} \cup \mathbb{N} \} \cup \bigcup \{ q_{2n+1}q_{2n} : n \in \mathbb{N} \},
\end{equation}

where $q_{2n+1}q_{2n} \subset \{0\} \times [1, 2]$ is the segment joining the point $q_{2n+1} \in P_{n+1}$ with $q_{2n} \in P_n$. Observe that $X_2$ is located in the half-space $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$ and that, if $S'_n = M'_n \cup P_n$ denotes the compactification of the ray $M'_n$ having $P_n$ as its remainder, then

\begin{equation}
\text{Lim } P_n = \text{Lim } S'_n = \{(0, 1, 0)\},
\end{equation}

whence it follows that

\begin{equation}
(2.3) \quad \text{for each } \varepsilon > 0 \text{ there exists an } n(\varepsilon) \in \mathbb{N} \text{ such that } S'_n \subset B(\varepsilon) \text{ for each } n > n(\varepsilon)
\end{equation}

where $B(\varepsilon)$ means the ball of radius $\varepsilon$ centered at $(0, 1, 0)$.

Construction 3. Recall that, given a continuum $B$, the cone over $B$ is defined as the quotient space $\text{Cone}(B) = (B \times [0, 1])/(B \times \{1\})$. 

The set $B \times \{0\}$ is called the base of the cone, and the point which corresponds to $B \times \{1\}$ is called its vertex.

Let $X_2$ be the continuum defined above in Construction 2. For each $n \in \{0\} \cup \mathbb{N}$ let $C_n$ be a cone over the pseudo-arc $P_n$ such that

1. $C_n \cap X_2 = P_n$ is the base of $C_n$;
2. $\text{diam}(C_n) = \text{diam}(P_n)$;
3. $C_m \cap C_n = \emptyset$ for $m \neq n$.

The needed continuum $X$ will be obtained by attaching all the cones $C_n$ to the continuum $X_2$. It can be geometrically realized in the space $\mathbb{R}^3$ as follows. Put $v_0 = (-2, 0, 0)$ and let $C_0$ be the geometric cone with the vertex $v_0$ and the base $P_0$. For each $n \in \mathbb{N}$ let $d_n = \text{diam}(P_n)$ and put $v_n = (-d_n, \frac{1}{2}(y_{2n} + y_{2n-1}), 0)$, where $y_{2n}$ and $y_{2n-1}$ are the $y$-coordinates of the end points of $L_n$ (see (1.2)). Then consider $C_n$ as the geometric cone with the vertex $v_n$ and the base $P_n$. Note that each $C_n$ such defined is located in the half-space $\{(x, y, z) \in \mathbb{R}^3 : x \leq 0\}$ and observe that the cones $C_n$ satisfy conditions (3.1)-(3.3). So, define

\[ X = X_2 \cup \bigcup \{C_n : n \in \{0\} \cup \mathbb{N}\}, \]

and note that $X$ is a continuum.

Recall that a continuum $W$ is said to be continuum-chainable provided that for each $\varepsilon > 0$ and every two distinct points $p, q \in X$ there is a finite sequence of subcontinua $\{A_1, \ldots, A_k\}$ of $X$ such that $\text{diam}(A_i) < \varepsilon$, $p \in A_1$, $q \in A_k$ and $A_i \cap A_{i+1} \neq \emptyset$ for each index $i < k$.

The main result of this paper is the following theorem.

**Theorem 4.** There exists a continuum $X$ having the following properties.

1. $X$ is not $\delta$-connected;
2. $X$ is $\lambda$-connected;
3. $X$ has two arc-components;
4. $X$ is continuum-chainable;
5. for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in (0, \mu(X))$, the Whitney level $\mu^{-1}(t)$ is arcwise connected.

**Proof:** The continuum $X$ is defined by (3.4). We have to show that it has the properties (4.1)-(4.5).
1) Let \( p = v_0 \) and \( q = p'_0 \) be the vertex of the cone \( C_0 \) and the end point of \( S'_0 \), respectively. Then each irreducible continuum between \( p \) and \( q \) contains indecomposable subcontinua (contained in the union \( \bigcup \{ P_n : n \in \{ 0 \} \cup \mathbb{N} \} \)), so \( X \) is not \( \delta \)-connected.

2) For each \( n \in \{ 0 \} \cup \mathbb{N} \), the ray \( M'_n \) approximates the pseudo-arc \( P_n = \overline{M'_n} \setminus M'_n \) according to Construction 2. Thus, each \( P_n \) (and therefore each indecomposable subcontinuum of \( X \)) has empty interior. Hence, \( X \) is \( \lambda \)-connected.

3) Indeed, it follows from the constructions 1-3 that the two arc components of \( X \) are

\[
A^+ = \bigcup \{ M'_n : n \in \{ 0 \} \cup \mathbb{N} \} \subset \{(x, y, z) \in \mathbb{R}^3 : x > 0 \},
\]

\[
A^- = C_0 \cup \bigcup \{(C_n \cup q_{2n+1}q_{2n}) : n \in \mathbb{N} \} \subset \{(x, y, z) \in \mathbb{R}^3 : x \leq 0 \}.
\]

4) To see that \( X \) is continuum-chainable, take two distinct points \( p, q \in X \) and \( \varepsilon > 0 \). If both \( p \) and \( q \) belong to the same arc component of \( X \), the argument is obvious. So, let \( p \in A^- \) and \( q \in A^+ \). Choose \( n > n(\varepsilon) \) according to assertion (2.3) of Construction 2 and note that, by (2.1) and (2.3),

\[
q_{2n-1} \in P_n \subset S'_n \subset B(\varepsilon).
\]

Let \( p'_n \in S'_n \) be the end point of \( S'_n \) and let \( D^+ \) be an arc from \( p'_n \) to \( q \) contained in the arc component \( A^+ \) of \( X \). Further, let \( D^- \) be an arc from \( p \) to \( q_{2n-1} \) contained in \( A^- \). Then the union \( U = D^- \cup S'_n \cup D^+ \) is a continuum joining \( p \) and \( q \). Since \( \text{diam}(S'_n) < \varepsilon \) and since each of the arcs \( D^- \) and \( D^+ \) can be represented as a finite union of subarcs of diameter less than \( \varepsilon \), we conclude that all the conditions of the definition of continuum chainability are satisfied. Hence, (4.4) is shown.

5) Conditions (4.4) and (4.5) coincide with conditions (a) and (d) of [4, Theorem 33.4, p. 248], and thereby they are equivalent.

The proof is complete. \( \square \)

Theorem 4 implies, by the definition of a Whitney reversible property, the following corollary.

**Corollary 5.** \( \delta \)-connectedness is not a Whitney reversible property.

**Remark 6.** Since by (4.2) the continuum \( X \) is \( \lambda \)-connected, the described example does not answer the other part of [4, Question 54.4, p. 280].
REFERENCES


DEPARTMENT OF MATHEMATICS AND STATISTICS; UNIVERSITY OF MISSOURI-ROLLA; ROLLA, MO 65409-0020

E-mail address: wjcharat@umr.edu