



**CONNECTEDNESS PROPERTIES OF  
WHITNEY LEVELS**

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**ABSTRACT.** It is shown that  $\delta$ -connectedness is not a Whitney reversible property. This answers in the negative a question posed by Sam B. Nadler, Jr. in 1978.

A topological property  $\mathcal{P}$  is said to be:

- (a) a *Whitney property* provided that if a continuum  $X$  has property  $\mathcal{P}$ , so does  $\mu^{-1}(t)$  for each Whitney map  $\mu$  for  $C(X)$  and each  $t \in [0, \mu(X))$  ([6, p. 165]);
- (b) a *Whitney reversible property* provided that whenever  $X$  is a continuum such that  $\mu^{-1}(t)$  has property  $\mathcal{P}$  for all Whitney maps  $\mu$  for  $C(X)$  and all  $t \in (0, \mu(X))$ , then  $X$  has property  $\mathcal{P}$  ([8, p. 235]).

A continuum  $X$  is said to be:

- (c)  $\delta$ -*connected* provided that for every two points of  $X$  there exists an irreducible continuum between them which is hereditarily decomposable ([5, p. 90]);
- (d)  $\lambda$ -*connected* provided that for every two points of  $X$  there exists an irreducible continuum between them which is of type  $\lambda$  (that is, each of its indecomposable subcontinua has empty interior) ([5, p. 85]).

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<sup>†</sup> Sadly, Professor Janusz J. Charatonik passed away on July 11, 2004.

(Note that in some papers, in particular in Sam B. Nadler, Jr.'s monograph [7, (0.30), p. 16], the name “ $\lambda$ -connected” is used in the sense of “ $\delta$ -connected.” See [3, p. 118] for an explanation.)

Answering a question in [6, Section 6, p. 179] (cf. [7, Question 14.36, p. 432]), the second named author has proved in [2, Construction 5.1, p. 387, and Remark 6, p. 389] that  $\delta$ -connectedness is not a Whitney property. A similar assertion for  $\lambda$ -connectedness is not known; see [2, Question 7, p. 390] and compare [4, Question 51.3, p. 280].

In [7, Question 14.57, p. 464], (compare [4, Question 51.4, p. 280]), Nadler asks if (i)  $\delta$ -connectedness is a Whitney reversible property, and if (ii)  $\lambda$ -connectedness is a Whitney reversible property. In this paper we present an example of a continuum showing a negative answer to (i). The question concerning (ii) remains open.

All considered spaces are assumed to be metric. We denote by  $\mathbb{N}$  the set of all positive integers, and by  $\mathbb{R}$  the space of reals. A *continuum* means a compact connected space, and a *mapping* means a continuous function. Given two points  $a$  and  $b$  in a Euclidean space, we denote by  $\overline{ab}$  the straight line segment joining these points.

A continuum is said to be *decomposable* provided that it can be represented as the union of two of its proper subcontinua. Otherwise, it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its non-degenerate subcontinua is decomposable (indecomposable). A continuum  $X$  is said to be *irreducible* (between points  $a$  and  $b$  of  $X$ ) provided that no proper subcontinuum of  $X$  contains these points. Points  $a$  and  $b$  are then called *points of irreducibility of  $X$* . An irreducible continuum  $X$  is said to be of *type  $\lambda$*  if each indecomposable subcontinuum of  $X$  has empty interior.

Given a continuum  $X$ , we let  $2^X$  denote the hyperspace of all nonempty closed subsets of  $X$  equipped with the Hausdorff metric  $H$  (see e.g., [7, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by  $C(X)$  the hyperspace of all subcontinua of  $X$ , i.e., of all connected elements of  $2^X$ . We will write  $A = \text{Lim } A_n$  to denote that the (closed) sets tend to  $A$  with respect to the Hausdorff metric.

A *Whitney map* for  $C(X)$  is a mapping  $\mu : C(X) \rightarrow [0, \infty)$  such that:

(0.1)  $\mu(A) < \mu(B)$  for every two  $A, B \in C(X)$  such that  $A \subset B$  and  $A \neq B$ ;

(0.2)  $\mu(A) = 0$  if and only if  $A$  is a singleton.

For the concept and existence of a Whitney map, see [4, Section 13, p. 105-110]. For each  $t \in [0, \mu(X)]$  the preimage  $\mu^{-1}(t)$  is called a *Whitney level*. It is known that each Whitney level is a continuum, see [4, p. 159].

The reader is referred to monographs [4] and [7] for definitions and basic properties of other notions used in the paper.

To present the needed example of continuum  $X$  showing that  $\delta$ -connectedness is not a Whitney reversible property, we start with some auxiliary constructions.

**Construction 1.** In the Cartesian coordinates  $(x, y)$  in the plane, let  $S_0$  be the standard  $\sin \frac{1}{x}$ -curve, that is,

$$(1.1) \quad S_0 = (\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 : x \in (0, 1]\}.$$

We will call  $(1, 0)$  the *end point* of  $S_0$ , and  $\{0\} \times [-1, 1]$  the *limit segment* of  $S_0$ .

Take a sequence of local maxima of  $S_0$ , that is, a sequence of points  $p_n$  determined by  $p_n = (x_n, 1) \in S_0$  with  $x_{n+1} < x_n$  for each  $n \in \mathbb{N}$ . Thus,  $(0, 1) = \lim p_n$ . Further, in the segment  $\{0\} \times [1, 2]$  take a sequence of points  $q_n = (0, y_n)$  such that the numbers  $y_n$  form a decreasing sequence tending to 1

$$(1.2) \quad \lim y_n = 1 < \dots < y_{n+1} < y_n < \dots < y_1 = 2.$$

Thus,  $(0, 1) = \lim q_n$ .

For each  $n \in \mathbb{N}$ , let  $S_n$  be a homeomorphic copy of  $S_0$  situated in the rectangle  $[0, 1] \times [1, 2]$  in such a way that:

(1.3)  $p_n$  is the end point, and  $L_n = \{0\} \times [y_{2n}, y_{2n-1}]$  is the limit segment, of  $S_n$ , for each  $n$ ;

(1.4)  $S_n \cap S_0 = \{p_n\}$  for each  $n$ ;

(1.5)  $S_m \cap S_n = \emptyset$  for  $m, n \in \mathbb{N}$  with  $m \neq n$ ;

(1.6)  $\text{Lim } S_n = \{(0, 1)\}$ .

Define

$$(1.7) \quad X_1 = (\{0\} \times [1, 2]) \cup S_0 \cup \bigcup \{S_n : n \in \mathbb{N}\}$$

and observe that  $X_1$  is a continuum having two arc components, and that the limit segments of  $S_0$  and of all  $S_n$  are components of the set of non-local connectedness of  $X_1$ .

**Construction 2.** Recall that the *pseudo-arc* means an arc-like hereditarily indecomposable continuum (see e.g., [9, 1.23, p. 13]). Let  $X_1$  be the continuum defined by (1.7). For each  $n \in \{0\} \cup \mathbb{N}$ , let  $L_n$  stand for the limit segment, and  $M_n$  stand for the non-compact arc component of  $S_n$ . Thus,  $L_0 = \overline{(0, -1)(0, 1)}$  and  $L_n = \overline{q_{2n}q_{2n-1}}$  for any  $n \in \mathbb{N}$ . Hence,  $S_n = L_n \cup M_n$  for each integer  $n \geq 0$ , and  $L_0, L_1, L_2, \dots$  are components of the set of points at which  $X_1$  is not locally connected.

Note that each  $M_n$  is locally compact. Since for each locally compact, noncompact metric space  $M$  an arbitrary continuum  $P$  can be a remainder of a compactification of  $M$ , (see [1, Theorem, p. 35]), it is possible to replace  $L_n$  by a copy  $P_n$  of the pseudo-arc in such a way that  $\text{diam}(P_n) = \text{diam}(L_n)$  and that, if for each  $n \in \{0\} \cup \mathbb{N}$ , the symbol  $M'_n$  denotes a one-to-one copy of the non-compact arc component  $M_n$  of  $S_n$ , then  $P_n = \text{cl}(M'_n) \setminus M'_n$  for  $n \in \{0, 1, 2, \dots\}$ . Since the pseudo-arc is a plane continuum, the construction can be made in such a way that all the inserted pseudo-arcs  $P_0, P_1, P_2, \dots$  lie in the plane  $\{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ . It follows that

(2.1) for each  $n \in \mathbb{N}$ , the end points  $q_{2n}$  and  $q_{2n-1}$  of  $L_n$  belong to the pseudo-arc  $P_n$ .

Denote by  $X_2$  the continuum obtained from  $X_1$  by the replacement described above, that is,

(2.2)  $X_2 = \bigcup \{(M'_n \cup P_n) : n \in \{0\} \cup \mathbb{N}\} \cup \bigcup \{\overline{q_{2n+1}q_{2n}} : n \in \mathbb{N}\}$ ,

where  $\overline{q_{2n+1}q_{2n}} \subset \{0\} \times [1, 2]$  is the segment joining the point  $q_{2n+1} \in P_{n+1}$  with  $q_{2n} \in P_n$ . Observe that  $X_2$  is located in the half-space  $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$  and that, if  $S'_n = M'_n \cup P_n$  denotes the compactification of the ray  $M'_n$  having  $P_n$  as its remainder, then

$$\text{Lim } P_n = \text{Lim } S'_n = \{(0, 1, 0)\},$$

whence it follows that

(2.3) for each  $\varepsilon > 0$  there exists an  $n(\varepsilon) \in \mathbb{N}$  such that  $S'_n \subset B(\varepsilon)$  for each  $n > n(\varepsilon)$

where  $B(\varepsilon)$  means the ball of radius  $\frac{\varepsilon}{2}$  centered at  $(0, 1, 0)$ .

**Construction 3.** Recall that, given a continuum  $B$ , the *cone over*  $B$  is defined as the quotient space  $\text{Cone}(B) = (B \times [0, 1]) / (B \times \{1\})$ .

The set  $B \times \{0\}$  is called the *base* of the cone, and the point which corresponds to  $B \times \{1\}$  is called its *vertex*.

Let  $X_2$  be the continuum defined above in Construction 2. For each  $n \in \{0\} \cup \mathbb{N}$  let  $C_n$  be a cone over the pseudo-arc  $P_n$  such that

$$(3.1) \quad C_n \cap X_2 = P_n \text{ is the base of } C_n;$$

$$(3.2) \quad \text{diam}(C_n) = \text{diam}(P_n);$$

$$(3.3) \quad C_m \cap C_n = \emptyset \text{ for } m \neq n.$$

The needed continuum  $X$  will be obtained by attaching all the cones  $C_n$  to the continuum  $X_2$ . It can be geometrically realized in the space  $\mathbb{R}^3$  as follows. Put  $v_0 = (-2, 0, 0)$  and let  $C_0$  be the geometric cone with the vertex  $v_0$  and the base  $P_0$ . For each  $n \in \mathbb{N}$  let  $d_n = \text{diam}(P_n)$  and put  $v_n = (-d_n, \frac{1}{2}(y_{2n} + y_{2n-1}), 0)$ , where  $y_{2n}$  and  $y_{2n-1}$  are the  $y$ -coordinates of the end points of  $L_n$  (see (1.2)). Then consider  $C_n$  as the geometric cone with the vertex  $v_n$  and the base  $P_n$ . Note that each  $C_n$  such defined is located in the half-space  $\{(x, y, z) \in \mathbb{R}^3 : x \leq 0\}$  and observe that the cones  $C_n$  satisfy conditions (3.1)-(3.3). So, define

$$(3.4) \quad X = X_2 \cup \bigcup \{C_n : n \in \{0\} \cup \mathbb{N}\},$$

and note that  $X$  is a continuum.

Recall that a continuum  $W$  is said to be *continuum-chainable* provided that for each  $\varepsilon > 0$  and every two distinct points  $p, q \in X$  there is a finite sequence of subcontinua  $\{A_1, \dots, A_k\}$  of  $X$  such that  $\text{diam}(A_i) < \varepsilon$ ,  $p \in A_1$ ,  $q \in A_k$  and  $A_i \cap A_{i+1} \neq \emptyset$  for each index  $i < k$ .

The main result of this paper is the following theorem.

**Theorem 4.** *There exists a continuum  $X$  having the following properties.*

$$(4.1) \quad X \text{ is not } \delta\text{-connected};$$

$$(4.2) \quad X \text{ is } \lambda\text{-connected};$$

$$(4.3) \quad X \text{ has two arc-components};$$

$$(4.4) \quad X \text{ is continuum-chainable};$$

$$(4.5) \quad \text{for each Whitney map } \mu : C(X) \rightarrow [0, \infty) \text{ and for each } t \in (0, \mu(X)), \text{ the Whitney level } \mu^{-1}(t) \text{ is arcwise connected.}$$

*Proof:* The continuum  $X$  is defined by (3.4). We have to show that it has the properties (4.1)-(4.5).

1) Let  $p = v_0$  and  $q = p'_0$  be the vertex of the cone  $C_0$  and the end point of  $S'_0$ , respectively. Then each irreducible continuum between  $p$  and  $q$  contains indecomposable subcontinua (contained in the union  $\bigcup\{P_n : n \in \{0\} \cup \mathbb{N}\}$ ), so  $X$  is not  $\delta$ -connected.

2) For each  $n \in \{0\} \cup \mathbb{N}$ , the ray  $M'_n$  approximates the pseudo-arc  $P_n = \text{cl}(M'_n) \setminus M'_n$  according to Construction 2. Thus, each  $P_n$  (and therefore each indecomposable subcontinuum of  $X$ ) has empty interior. Hence,  $X$  is  $\lambda$ -connected.

3) Indeed, it follows from the constructions 1-3 that the two arc components of  $X$  are

$$A^+ = \bigcup\{M'_n : n \in \{0\} \cup \mathbb{N}\} \subset \{(x, y, z) \in \mathbb{R}^3 : x > 0\},$$

$$A^- = C_0 \cup \bigcup\{(C_n \cup \overline{q_{2n+1}q_{2n}}) : n \in \mathbb{N}\} \subset \{(x, y, z) \in \mathbb{R}^3 : x \leq 0\}.$$

4) To see that  $X$  is continuum-chainable, take two distinct points  $p, q \in X$  and  $\varepsilon > 0$ . If both  $p$  and  $q$  belong to the same arc component of  $X$ , the argument is obvious. So, let  $p \in A^-$  and  $q \in A^+$ . Choose  $n > n(\varepsilon)$  according to assertion (2.3) of Construction 2 and note that, by (2.1) and (2.3),

$$q_{2n-1} \in P_n \subset S'_n \subset B(\varepsilon).$$

Let  $p'_n \in S'_n$  be the end point of  $S'_n$  and let  $D^+$  be an arc from  $p'_n$  to  $q$  contained in the arc component  $A^+$  of  $X$ . Further, let  $D^-$  be an arc from  $p$  to  $q_{2n-1}$  contained in  $A^-$ . Then the union  $U = D^- \cup S'_n \cup D^+$  is a continuum joining  $p$  and  $q$ . Since  $\text{diam}(S'_n) < \varepsilon$  and since each of the arcs  $D^-$  and  $D^+$  can be represented as a finite union of subarcs of diameter less than  $\varepsilon$ , we conclude that all the conditions of the definition of continuum chainability are satisfied. Hence, (4.4) is shown.

5) Conditions (4.4) and (4.5) coincide with conditions (a) and (d) of [4, Theorem 33.4, p. 248], and thereby they are equivalent.

The proof is complete.  $\square$

Theorem 4 implies, by the definition of a Whitney reversible property, the following corollary.

**Corollary 5.**  *$\delta$ -connectedness is not a Whitney reversible property.*

**Remark 6.** Since by (4.2) the continuum  $X$  is  $\lambda$ -connected, the described example does not answer the other part of [4, Question 54.4, p. 280].

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