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ON PLANE ARC-SMOOTH STRUCTURES

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Abstract. Arc-structures on subspaces of the plane are studied in the paper. It is shown that each plane arc-smooth continuum admits an embedding in the plane such that its arc-smooth structure can be nicely extended to an arc-smooth structure on the whole plane. Using this it is proved that each plane arc-smooth continuum is a retract of the hyperspace of its closed subsets. Among several applications it is pointed out that each planar smooth dendroid admits a mean.

1. Introduction

The concept of arc-smooth structures is a very powerful tool in continuum theory. It has been introduced in [7] and studied in a large paper [8]. The reader is referred to these papers for a discussion of the relationship of arc-smooth continua to several other generalizations of smooth dendroids, and for basic and important results concerning these continua. Relation between the concept and contractibility was considered in [11] and [14]. Arc-smoothness of hyperspaces was studied in [9] and [10]. For an extension of the concept see [12] and [13].

Therefore it was very natural to consider arc-smooth spaces as a natural generalization of arc-smooth continua (in particular dendrites and smooth dendroids) from one side, and of all convex spaces from the other.

In this paper arc-structures on subspaces of the plane are studied. It is shown that each plane arc-smooth continuum admits an embedding in the plane such that its arc-smooth structure can be extended to an arc-smooth structure (satisfying some extra conditions) on the whole plane (Theorem 4.1). In the fourth

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chapter we also prove that each arc-smooth structure on the plane leads to a decomposition of the plane into simple closed curves which is continuous with respect to homeomorphical convergence (Theorem 4.6).

Next we study the existence of a hyperspace retraction of a space X, i.e., a retraction defined on the hyperspace of all nonempty compact subsets of X onto X. The reader is referred to our paper [4] for a discussion concerning results on hyperspace retractions for continua which have been obtained in recent years. In the present paper we construct a special hyperspace retraction for the plane which agrees with a given arc-structure (Theorem 5.3). By this result, together with Theorem 4.1, a hyperspace retraction for each plane arc-smooth continuum is obtained (Corollary 5.4). In the case when the initial point of the arc-structure is accessible we provide a construction of such a retraction with some extra properties: it is associative and internal (Theorem 5.2). The last chapter contains applications of the obtained results to plane smooth dendroids. The existence of some special retractions on the hyperspace of all closed subsets, as well as the existence of means, for these curves are shown.

In the proofs of the results the concept of an orientation on a plane arc-smooth structure (third chapter) is exploited.

2. Preliminaries

All considered spaces are assumed to be metric. A *continuum* means a compact connected space, and a *mapping* means a continuous function. If any two points of a space can be joined by an arc lying in the space, then the space is said to be *arcwise connected*. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. Given a dendroid X and points x and y of X, we denote by xy the only arc in X with x and y as its end points.

A dendroid X is said to be *smooth at a point* $v \in X$ provided that for each sequence of points $a_n \in X$ which converges to a point $a \in X$ the sequence of arcs $va_n \subset X$ converges to the arc va. A dendroid X is said to be *smooth* provided it is smooth at some point $v \in X$. Then the point v is called an *initial point* of X. It is known that every smooth dendroid is uniformly arcwise connected (see [5, Corollary 16, p. 318]).

A point v of a set X contained in the plane $\mathbb{R}^2$ is said to be *accessible from the complement of* X (or shortly *accessible*) provided that there exists an arc $pv \subset \{v\} \cup (\mathbb{R}^2 \setminus X)$. Note that accessibility of a point of a set is not a topological property, but it depends on a particular embedding of this set in the plane. For a nice example of a plane smooth dendroid X such that the end points are the only points of X which are accessible from $\mathbb{R}^2 \setminus X$ see [1].

As a generalization of smooth dendroids a concept of arc-smooth continua has been introduced in [7] and [8]. We extend this concept to a wider class of spaces, namely to arcwise connected but not necessarily compact ones. To describe our ideas precisely we first recall some known notion related to hyperspaces.

Given a metric space X with a metric d, we denote by $Z(X)$ the hyperspace of all nonempty compact subsets of X equipped with the *Hausdorff metric*

$$\operatorname{dist}(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(equivalently: with the Vietoris (i.e. exponential) topology; see e.g. [16, §42, II, Theorem, p. 47]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e. of all connected members of $Z(X)$, and for each $n \in \mathbb{N}$ we put $F_n(X) = \{A \in Z(X) : A \text{ has at most } n \text{ elements}\}$.

Hence, $F_1(X)$ means the hyperspace of singletons. Thus, for each $n \in \mathbb{N}$, we have

$$F_1(X) \subset F_n(X) \subset Z(X) \quad \text{and} \quad F_1(X) \subset C(X) \subset Z(X).$$

In the case when the space X is compact, in particular a continuum, the hyperspace $Z(X)$ coincides with the hyperspace 2^X of all nonempty closed subsets of X (again with the Hausdorff metric), as considered e.g. in Nadler's book [19, (0.1), p. 1 and (0.12), p. 10].

Note that $F_1(X)$ is homeomorphic to X. We neglect the homeomorphism between X and $F_1(X)$, and we consider X as a subspace of 2^X under its natural embedding. Similarly, we think about $C(X)$ as about a subspace of 2^X.

Given a mapping $f : X \to Y$ between compact spaces X and Y, we consider mappings (called the *induced ones*)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by $2^f(A) = f(A)$ for every $A \in 2^X$ and $C(f)(A) = f(A)$ for every $A \in C(X)$. The reader is referred to [6] for more information on the induced mappings for continua.

By an *arc-structure* on an arbitrary space X we understand a function $\alpha : X \times X \to C(X)$ such that for every two distinct points x and y in X the set $\alpha(x, y)$ is an arc from x to y and that the following metric-like axioms are satisfied for every points x, y and z in X:

(2.1) $\alpha(x, x) = \{x\}$;

(2.2) $\alpha(x, y) = \alpha(y, x)$;
(2.3) $\alpha(x, z) \subset \alpha(x, y) \cup \alpha(y, z)$,
with equality prevailing whenever $y \in \alpha(x, z)$.

Statement 2.1. *To define an arc-structure on a space X it is enough to distinguish a point $v \in X$ and to define a function α on the product $\{v\} \times X$ so that*

(2.4) $\alpha(v, v) = \{v\}$*;*
(2.5) *if $x \in \alpha(v, y)$, then $\alpha(v, x) \subset \alpha(v, y)$ for all points x and y in X.*

Proof. By (2.5), for all points x and y in X, the continuum $\alpha(v, x) \cup \alpha(v, y)$ is a triod, or an arc, or a point. So, it suffices to define $\alpha(x, y)$ to be the unique arc with end points x and y in the considered continuum $\alpha(v, x) \cup \alpha(v, y)$. □

Let α and β be arc-structures on spaces X and Y respectively. We say that (X, α) *is contained in* (Y, β), writing $(X, \alpha) \ll (Y, \beta)$, provided that $X \subset Y$ and $\beta|(X \times X) = \alpha$.

Let an arc-structure α on a space X be given. For each homeomorphism $h : X \to h(X)$ we define an arc-structure $\alpha_h : h(X) \times h(X) \to C(h(X))$ determined by the equality

$$C(h) \circ \alpha = \alpha_h \circ (h \times h), \tag{2.6}$$

i.e., such that the following diagram commutes:

$$\begin{array}{ccc} X \times X & \xrightarrow{\alpha} & C(X) \\ {\scriptstyle h\times h}\big\downarrow & & \big\downarrow{\scriptstyle C(h)} \\ h(X) \times h(X) & \xrightarrow[\alpha_h]{} & C(h(X)) \end{array}$$

The pair (X, α) is said to be *arc-smooth at a point* $v \in X$ provided that the induced function $\alpha_v : X \to C(X)$ defined by $\alpha_v(x) = \alpha(v, x)$ is continuous. Then the point v is called an *initial point* of (X, α). The pair (X, α) is said to be *arc-smooth* provided that there exists a point in X at which (X, α) is arc-smooth. An arbitrary space X is said to be *arc-smooth at a point* $v \in X$ provided that there exists an arc-structure α on X for which (X, α) is arc-smooth at v. The space X is said to be *arc-smooth* if it is arc-smooth at some point. Note that an arc-smooth space is arcwise connected. We will use the symbol (X, α, v) to denote an arc-smooth space X with an arc-structure $\alpha : X \times X \to C(X)$ for which v is the initial point.

Given an arc-smooth space (X, α, v), we define now some auxiliary functions and relations on X which will be useful in proofs of the next results.

Denote by $\omega : Z(X) \to [0, \infty)$ any *Whitney map*, i.e., a mapping such that (see e.g. Nadler's book [19, p. 24 ff])

$$\omega(\{x\}) = 0 \quad \text{for each point} \quad x \in X, \quad \text{and if} \quad A \subsetneq B, \quad \text{then} \quad \omega(A) < \omega(B).$$

Define a mapping $\lambda : X \to [0, \infty)$ by

$$\lambda(x) = \omega(\alpha(v, x)) \quad \text{for each point} \quad x \in X \tag{2.7}$$

and note that continuity of λ follows from that of ω and arc-smoothness of X. Therefore, for each point $x \in X$ the partial mapping $\lambda|\alpha(v, x)$ is a homeomorphism of $\alpha(v, x)$ onto $[0, \lambda(x)]$. For each number $t \in [0, \lambda(x)]$ we put

$$H(x, t) = (\lambda|\alpha(v, x))^{-1}(t), \tag{2.8}$$

i.e., we define $H(x, t)$ as the only point of the arc $\alpha(v, x)$ such that $\lambda(H(x, t)) = t$.

The following observation is a consequence of the above definition of an arc-smooth space.

Observation 2.2. *If a sequence of points $\{a_n\}$ in an arc-smooth space (X, α, v) converges to a limit point $a \neq v$, then there is a sequence of homeomorphisms $h_n : [0, 1] \to \alpha(v, a_n)$ which converges (uniformly) to a homeomorphism $h : [0, 1] \to \alpha(v, a)$.*

Proof. Really, it is enough to put, for $n \in \mathbb{N}$ and $t \in [0, 1]$:

$$h_n(t) = H(a_n, t \cdot \lambda(a_n)) \quad \text{and} \quad h(t) = H(a, t \cdot \lambda(a)).$$

□

We define the concept of an end point of a space X equipped with an arc-structure $\alpha : X \times X \to C(X)$ as follows (compare [8, Section I-9, p. 555]). A point $e \in X$ is called an *end point* of X (writing $e \in E(X)$) provided that if $e \in \alpha(x, y)$, then $e = x$ or $e = y$.

Recall (see [8, Lemma I-9-A, p. 555]) the following property of arc-smooth continua.

Proposition 2.3. *Let an arc-smooth continuum (X, α, v) be given. For each point $x \in X$ the arc $\alpha(v, x)$ is contained in an arc $\alpha(v, e)$ for some $e \in E(X)$. Thus $E(X) \neq \emptyset$.*

Remark 2.1. The concept of smoothness of dendroids has been generalized to smoothness of arbitrary continua by T. Maćkowiak in [17]. However, it should be underlined that even in the case when a continuum X is smooth (in the sense of Maćkowiak), is arcwise connected and admits an arc-structure $\alpha : X \times X \to C(X)$, it need not be arc-smooth at no one of its points. A suitable example (namely a simple closed curve) is discussed in [8, I.3, Example, p. 550]. Thus the class of arc-smooth continua does not coincide with the class of smooth ones in the sense of [17].

A mapping $f : X \to Y$ from a space X onto a space Y is called *monotone* provided that it has connected point-inverses. If $Y \subset X$, then Y is a *retract* of X means that there exists a mapping $r : X \to Y$ (called a *retraction*) such that the restriction $r|Y$ is the identity on Y. Since, for a given space X the hyperspace $F_1(X)$ is homeomorphic to X, we shall consider a retraction $r : Z(X) \to X$ rather than a retraction $r : Z(X) \to F_1(X)$ (and similarly for $C(X)$), although the former notation is perhaps less formal.

We use the symbols Li, Ls and Lim to denote the lower limit, the upper limit and the limit of a sequence of subsets of a metric space, according to the notation given in Kuratowski's monograph [15, §29, pp. 335-340]. Further, we write $f = \lim f_n$ to denote that a sequence of mappings f_n converges uniformly to the limit mapping f.

3. Orientation on plane arc-smooth structures

The present chapter plays an auxiliary role. It contains a sequence of results concerning orientation of a triple of points in a subspace of the plane equipped with an arc-structure, leading to Proposition 3.5 on preserving the orientation under limit operation. The theorem will be used in the next chapters to prove Theorem 4.1 on an extension of an arc-smooth structure from a subcontinuum of the plane to the whole plane, as well as to construct some special hyperspace retractions for arc-smooth subspaces of the plane (Theorem 5.2).

We denote by d the usual metric in the Euclidean plane $\mathbb{R}^2$, and by ρ the supremum metric between mappings from a continuum C into $\mathbb{R}^2$, i.e., for $f_1, f_2 : C \to \mathbb{R}^2$ we write

$$\rho(f_1, f_2) = \sup\{d(f_1(x), f_2(x)) : x \in C\}.$$

Let $s_0 = (0,0)$, $s_1 = (1,0)$, $s_2 = (0,1)$, $s_3 = (-1,0)$ be points in the plane $\mathbb{R}^2$ equipped with a rectangular Cartesian coordinate system, and let

$$T = s_0s_1 \cup s_0s_2 \cup s_0s_3$$

be the standard simple triod. For any homeomorphism $h : T \to T' \subset \mathbb{R}^2$ we say that h is *positive* (*negative*) provided that going from $h(s_1)$ to $h(s_2)$ in T' the arc $h(s_0)h(s_3)$ lies left (right) of the arc $h(s_1)h(s_2)$.

For a space X with an arc-structure $\alpha : X \times X \to C(X)$ and a point $v \in X$ we denote by $\mathcal{G}(X, v)$ the set of all mappings $g : T \to X$ such that $g(s_0) = v$, and for $i \in \{1, 2, 3\}$ the partial mappings $g|s_0s_i$ are either constant or homeomorphisms, with $g(s_0s_i) = \alpha(g(s_0), g(s_i))$.

Now we are going to define a concept of the orientation of a triple of points in a plane set X with an arc-structure α. To this aim we need a lemma.

Lemma 3.1. *Let a subspace X of the plane $\mathbb{R}^2$ with an arc-structure $\alpha : X \times X \to C(X)$ be given, and let a, b, c, v be points of X. Then for any mapping $g \in \mathcal{G}(X, v)$ with*

$$g(s_1) = a, \quad g(s_2) = b, \quad g(s_3) = c \tag{3.1}$$

there are a sequence of triods $T_n \subset \mathbb{R}^2$ and of homeomorphisms $h_n : T \to T_n$ such that

$$g = \lim h_n. \tag{3.2}$$

Proof. We will sketch the construction of T_n's and of h_n's as follows. If $g(s_1) = v$ (i.e., if $g|s_0s_1$ is a constant mapping), then we define $h_n(s_1)$ as an arbitrary point in $\mathbb{R}^2$ near v, and $h_n|s_0s_1$ as a homeomorphism onto some small arc with end points v and $h_n(s_1)$. If $g|s_0s_1$ is not constant, then we put $h_n|s_0s_1 = g|s_0s_1$.

If $g(s_2) = v$, then as previously we define $h_n|s_0s_2$ as a homeomorphism onto a small arc near the point v satisfying $h_n(s_0s_1) \cap h_n(s_0s_2) = \{v\}$. If $g|s_0s_2$ is a homeomorphism, then we take $h_n|s_0s_2$ as a homeomorphism onto an arc $h_n(s_0s_2)$ such that $h_n(s_0s_1) \cap h_n(s_0s_2) = \{v\}$ and $h_n|s_0s_2$ is close to $g|s_0s_2$. We repeat the construction for the arc s_0s_3, i.e., we define a homeomorphism $h_n|s_0s_3$ such that $h_n|s_0s_3$ is close to $g|s_0s_3$ and $h_n(s_0s_3) \cap (h_n(s_0s_1) \cup h_n(s_0s_2)) = \{v\}$. This ends the proof. □

Now we are ready to define the orientation of a triple of points in a subspace of the plane with an arc-structure. For a given subspace X of the plane with an arc-structure $\alpha : X \times X \to C(X)$ and a point v in X we say that a triple (a, b, c) of points of X is oriented *nonnegatively (nonpositively)* in (X, α) with respect to v provided that there are a mapping $g \in \mathcal{G}(X, v)$ with (3.1) and a sequence of positive (negative) homeomorphisms $h_n : T \to T_n \subset \mathbb{R}^2$ such that (3.2) holds; we say that the triple (a, b, c) is oriented *positively (negatively)* in (X, α) with respect to v provided that it is oriented nonnegatively (nonpositively) with respect to v.

Remark 3.1. It can be seen from Lemma 3.1 that for every plane arc-smooth space (X, α, v) and for every triple (a, b, c) of points of X the triple is oriented nonnegatively or nonpositively in (X, α) with respect to the point v.

Lemma 3.2. *Let a subspace X of the plane $\mathbb{R}^2$ with an arc-structure $\alpha : X \times X \to C(X)$ be given, and let a, x, b, v be points of X such that*

$$a \in X \setminus (\alpha(v, x) \cup \alpha(v, b)),$$
$$b \in X \setminus (\alpha(v, x) \cup \alpha(v, a)),$$
$$x \in X \setminus (\alpha(v, a) \cup \alpha(v, b)).$$

Assume that, going from a to b, the arc to the point x in (X, α) lies right (left) of the arc $\alpha(a, b)$. Then the triple (a, x, b) is oriented positively (negatively) with respect to v.

Proof. Assume that, going from a to b, the arc to the point x in (X, α) lies right the arc $\alpha(a, b)$ (if it lies left, the argument is the same). Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ stand for the unit circle. Let A, B and C be successive subarcs in S^1 (i.e., their order coincides with the positive orientation of S^1), whose union is S^1. Take an arc J in $\mathbb{R}^2$ such that

$$J \cap (\alpha(v, a) \cup \alpha(v, x) \cup \alpha(v, b))) = \{a, b\}$$

and that the point x lies in the bounded component of $\mathbb{R}^2 \setminus (J \cup \alpha(b, a))$. Let $h_1 : S^1 \to J \cup \alpha(b, a)$ be a homeomorphism that preserves orientation. Define a mapping $h_2 : S^1 \to J \cup \alpha(b, v) \cup \alpha(v, a) \subset \mathbb{R}^2 \setminus \{x\}$ so that the partial mappings

$$h_2|A : A \to J, \quad h_2|B : B \to \alpha(b, v), \quad h_2|C : C \to \alpha(v, a)$$

are homeomorphisms. Then h_2 is homotopic to h_1 in $\mathbb{R}^2 \setminus \{x\}$. Consider a mapping $g : T \to \alpha(v, a) \cup \alpha(v, x) \cup \alpha(v, b)$ that belongs to $\mathcal{G}(X, v)$ with

$$g(s_1) = a, \quad g(s_2) = x, \quad g(s_3) = b.$$

Let g' be an embedding of the standard simple triod T into $\mathbb{R}^2$ that is close to g. Define an embedding $h_3 : S^1 \to \mathbb{R}^2 \setminus \{x\}$ in such a way that the partial mappings

$$h_3|B : B \to g'(s_3 s_0) \quad \text{and} \quad h_3|C : C \to g'(s_0 s_1)$$

are homeomorphisms, while $h_3|A$ is a slight modification of $h_2|A$ such that $h_3(A)$ meets $h_3(B \cup C)$ only at the images of end points of A, i.e., at points $g'(s_1)$ and $g'(s_3)$. Then h_3 can be chosen as close to h_2 as we wish; in particular we may assume that h_3 is homotopic to h_2 in $\mathbb{R}^2 \setminus \{x\}$. Therefore we infer that h_1 is homotopic to h_3 in $\mathbb{R}^2 \setminus \{x\}$, so the point x lies in the bounded component of

$\mathbb{R}^2 \setminus h_3(S^1)$, and thus, going from $g'(s_1)$ to $g'(s_3)$ in $g'(T)$, the arc $\alpha(g'(s_0), g'(s_2))$ lies right of the arc $\alpha(g'(s_1), g'(s_3))$. This means that g' is positive. By the definition of the orientation of triples of points in (X, α) with respect to v the proof is complete. □

As an immediate consequence of Lemma 3.2 we have the following observation.

Observation 3.3. *Let a triple (a, x, b) of points in a subspace X of the plane with an arc-structure α on X be given, and let a point $c \in X$ be defined by*

$$\alpha(a, b) \cap \alpha(x, c) = \{c\}. \tag{3.3}$$

If the triple (a, x, b) is oriented nonnegatively (nonpositively) in (X, α) with respect to a point $v \in X$, then for every point $y \in \alpha(x, c)$ the triple (a, y, b) is also oriented nonnegatively (nonpositively) in (X, α) with respect to v.

Using Observation 2.2 we get the next lemma. Details of its proof are left to the reader.

Lemma 3.4. *Let an arc-smooth space (X, α, v) with $X \subset \mathbb{R}^2$ be given. For every three convergent sequences of points a_n, b_n, c_n of X with*

$$\lim a_n = a, \quad \lim b_n = b, \quad \lim c_n = c, \tag{3.4}$$

there exist a sequence of mappings $g_n \in \mathcal{G}(X, v)$ such that for every $n \in \mathbb{N}$ we have

$$g_n(s_1) = a_n, \quad g_n(s_2) = b_n, \quad g_n(s_3) = c_n, \tag{3.5}$$

and a mapping $g \in \mathcal{G}(X, v)$ satisfying (3.1) and

$$g = \lim g_n. \tag{3.6}$$

The following proposition is a basic result of this chapter. We shall use it in the next one.

Proposition 3.5. *Let an arc-smooth space (X, α, v) with $X \subset \mathbb{R}^2$ be given. Then the set of all triples $(a, b, c) \in X^3$ oriented nonnegatively (nonpositively) in (X, α, v) with respect to v is closed in X^3.*

Proof. For both versions the proof is the same. Take a sequence of nonnegatively oriented triples $(a_n, b_n, c_n) \in X^3$ which is convergent to a triple (a, b, c). Then (3.4) is satisfied. Thus by Lemma 3.4 there are a sequence of mappings $g_n \in \mathcal{G}(X, v)$ and a mapping $g \in \mathcal{G}(X, v)$ which satisfy (3.5), (3.1) and (3.6). Since each triple (a_n, b_n, c_n) is oriented nonnegatively in (X, α) with respect to v, then by the definition each mapping g_n can be approximated by a sequence of positive homeomorphisms $\{h_{n,m} : m \in \mathbb{N}\}$. Thus (3.6) implies that g is approximated by some sequence $\{h_{n,m(n)} : n \in \mathbb{N}\}$ of positive homeomorphisms, and thus the triple (a, b, c) is oriented nonnegatively in (X, α) with respect to v. The proof is complete. □

4. Extension of arc-structures in the plane

Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

stand for the unit circle in the plane $\mathbb{R}^2$. We will use the following concepts. Let $a, b \in S^1$ with $a \neq b$. Put

$$P(a, b) = \{z \in S^1 : \text{the triple } (a, z, b) \text{ is oriented counter clockwise}\}.$$

We say that a sequence $\{z_n\}$ of points of S^1 is oriented *counter clockwise (clockwise)* provided that for any triple of indices $n_1, n_2, n_3 \in \mathbb{N}$ with $n_1 < n_2 < n_3$ the triple $(z_{n_1}, z_{n_2}, z_{n_3})$ is oriented counter clockwise (clockwise, respectively). Note that

(4.1) each sequence $\{z_n\}$ of points of S^1 contains a subsequence which is oriented counter clockwise, clockwise, or is constant;

(4.2) each sequence $\{z_n\}$ of points of S^1 which is oriented counter clockwise or clockwise is convergent.

Theorem 4.1. *Let an arc-smooth continuum (X, a, v) in the plane $\mathbb{R}^2$ be given. Then there are an arc-structure $\beta : \mathbb{R}^2 \times \mathbb{R}^2 \to C(\mathbb{R}^2)$ and a homeomorphism $h : X \to h(X) \subset \mathbb{R}^2$ such that*

$$(h(X), \alpha_h) \ll (\mathbb{R}^2, \beta), \tag{4.3}$$

and

$$(\mathbb{R}^2, \beta, h(v)) \quad \text{is arc-smooth.} \tag{4.4}$$

Moreover, the arc-structure β satisfies the following implication for every two points $x, y \in \mathbb{R}^2$:

$$\text{if} \quad x \in \mathbb{R}^2 \setminus \beta(h(v), y) \quad \text{and} \quad y \in \mathbb{R}^2 \setminus \beta(h(v), x), \quad \text{then} \quad \beta(h(v), x) \cap \beta(h(v), y) \subset h(X). \tag{4.5}$$

PROOF. We will show the theorem under an additional assumption that no arc in X with an end point v is a neighborhood of v. The general case will be a simple consequence of this particular situation.

By Proposition 2.3 the set $E(X)$ of end points of X is nonempty. So, choose an end point e_0 of X. For any two end points $e_1, e_2 \in E(X)$ we write $e_1 \preceq e_2$ provided that the triple (e_0, e_1, e_2) is oriented nonnegatively and $e_0 \neq e_2$. Thus $\preceq$ is a linear order on $E(X)$ with e_0 being the smallest element. As a consequence of Proposition 3.5 we conclude that

$$\text{the relation } \preceq \text{ is closed in } E(X) \times E(X). \tag{4.6}$$

Define functions

$$L : E(X) \to C(X) \quad \text{and} \quad R : E(X) \to C(X)$$

putting

$$L(e) = \bigcup\{\alpha(v, e') : e' \preceq e\} \quad \text{and} \quad R(e) = \bigcup\{\alpha(v, e') : e \preceq e'\},$$

and note that each member of the unions is an arc containing the point v, so $L(e)$ and $R(e)$ are connected. We will show that they are closed. An argument presented below for $L(e)$ can also be applied for $R(e)$ as well. So, take a convergent sequence of points $x_n \in L(e)$ and put $x = \lim x_n$. Thus $x_n \in \alpha(v, e_n)$ with $e_n \preceq e$. It follows that the triple (e_0, e_n, e) is oriented nonnegatively with respect to v, and $e_0 \neq e$. Choose an arbitrary convergent subsequence $\{e_{n_k}\}$ of the sequence $\{e_n\}$ and denote $a = \lim e_{n_k}$. Applying Proposition 3.5 we infer that the triple (e_0, a, e) is oriented nonnegatively with respect to v. By Proposition 2.3 there is a point $e^* \in E(X)$ such that $a \in \alpha(v, e^*)$ and that the triple (e_0, e^*, e) is also oriented nonnegatively with respect to v, whence we infer that $e^* \preceq e$. By arc-smoothness of X we have $x \in \alpha(v, a) \subset \alpha(v, e^*)$, so $x \in L(e)$. The argument for closedness of $L(e)$ is complete.

Continuity of each of the functions L and R at each point $e \in E(X) \setminus \{e_0\}$ is a consequence of arc-smoothness of X. Note that $L(e_0) = R(e_0) = ve_0$. We fix a Whitney map $\omega : 2^X \to [0, 1]$, and define two mappings

$$g_1, g_2 : E(X) \to S^1$$

putting, for each $e \in E(X)$,

$$g_1(e) = \exp(\pi i(\omega(L(e)) - \omega(L(e_0))))$$

and

$$g_2(e) = \exp(-\pi i(\omega(R(e)) - \omega(R(e_0)))).$$

Thus $g_1(e_0) = g_2(e_0) = 1$. For each $z \in \mathbb{C}$ let $\operatorname{Arg} z$ stand for the argument of z in the interval $[0, 2\pi)$. Note that

(4.7) $e_1 \preceq e_2 \iff \operatorname{Arg} g_1(e_1) \le \operatorname{Arg} g_1(e_2) \iff \operatorname{Arg} g_2(e_1) \le \operatorname{Arg} g_2(e_2)$,

and

(4.8) both g_1 and g_2 are one-to-one.

Now we are going to construct a one-to-one mapping $g : E(X) \to S^1$ satisfying the following conditions

(4.9) $e_1 \preceq e_2 \iff \operatorname{Arg} g(e_1) \le \operatorname{Arg} g(e_2)$,

(4.10) if either of the two sequences $\{g_1(e_n)\}$ and $\{g_2(e_n)\}$ is convergent, then the sequence $\{g(e_n)\}$ is convergent, too.

To this aim we will define a monotone mapping $m_1 : S^1 \to S^1$. Up to homeomorphisms on S^1 it is enough to decide, for any two points $x, y \in S^1$, whether or not $m_1(x) = m_1(y)$. Let U be a component of the set $S^1 \setminus \operatorname{cl} g_1(E(X))$. Thus U is an open arc; denote its end points by a and b. Let $\{e_n^a\}$ and $\{e_n^b\}$ be two sequences of end points of X such that

$$a = \lim g_1(e_n^a) \quad \text{and} \quad b = \lim g_1(e_n^b).$$

If $\lim g_2(e_n^a) = \lim g_2(e_n^b)$, then we shrink $\operatorname{cl} U$ to a point, i.e., $m_1(x) = m_1(y)$ for any $x, y \in \operatorname{cl} U$. In this manner we have defined a monotone mapping $m_1 : S^1 \to S^1$. Observe that if both end points a and b of U are in $g_1(E(X))$, then (since g_2 is one-to-one), $\operatorname{cl} U$ is not shrunk to a point, and thus $m_1|g_1(E(X))$ is one-to-one. Observe further that

(4.11) for each sequence of end points $\{e_n\} \subset E(X)$ and for each end point $e \in E(X)$, if $\lim g_1(e_n) = g_1(e)$ or $\lim g_2(e_n) = g_2(e)$, then $\operatorname{Ls} ve_n \subset ve$.

The composition $g = m_1 \circ g_1 : E(X) \to S^1$ is the needed mapping. Recall that two surjective mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between topological spaces are said to be *topologically equivalent* provided that there are homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2 \circ h_X = h_Y \circ f_1$. Observe that the mapping g is topologically equivalent to a mapping $g' = m_2 \circ g_2 : E(X) \to S^1$, where $m_2 : S^1 \to S^1$ is defined in a similar manner as m_1 was, interchanging indices 1 and 2.

It follows from (4.8) by the definition of g that g is also one-to-one. As another consequence of the definition of g we see that

(4.12) if a sequence of end points $\{e_n\} \subset E(X)$ is monotone with respect to $\preceq$ and the sequence $\{g(e_n)\} \subset S^1$ is convergent to $g(e)$ for some $e \in E(X)$, then either $\lim g_1(e_n) = g_1(e)$ or $\lim g_2(e_n) = g_2(e)$.

We will show several facts about the mapping g.

FACT 1. For each sequence of end points $\{e_n\} \subset E(X)$ and for each end point $e \in E(X)$, if $\lim g(e_n) = g(e)$, then $\operatorname{Ls} \alpha(v, e_n) \subset \alpha(v, e)$.

To show Fact 1 suppose on the contrary that the conclusion does not hold. Then there is a subsequence $\{e'_n\}$ of the sequence $\{e_n\}$ such that $\{e'_n\}$ converges to some point $x \in X$ (and then $\operatorname{Lim} \alpha(v, e'_n) = \alpha(v, x)$ by arc-smoothness of X) with $x \notin \alpha(v, e)$ (and, consequently, $e \notin \alpha(v, x)$), and the sequence $\{e'_n\}$ is monotone with respect to $\preceq$. Since $\lim g(e'_n) = g(e)$, by (4.12) we have either $\lim g_1(e'_n) = g_1(e)$ or $\lim g_2(e'_n) = g_2(e)$. Hence $\operatorname{Ls} \alpha(v, e'_n) \subset \alpha(v, e)$ by (4.11), a contradiction.

FACT 2. For each sequence of end points $\{e_n\} \subset E(X)$, for each $x \in X$ and $z \in S^1$ if $x = \lim e_n$ and $z = \lim g(e_n)$, then

$$z \in \operatorname{cl} \{g(e) : e \in E(X) \text{ and } x \in \alpha(v, e)\}. \tag{4.13}$$

Indeed, if $x = v$, then the conclusion (4.13) is obvious, so we can assume that $x \neq v$.

First consider the case when the sequence $\{e_n\}$ is increasing and bounded with respect to the relation $\preceq$. Then the sequence $\{R(e_n)\}$ is decreasing, and $x \in \bigcap\{R(e_n) : n \in \mathbb{N}\} = \operatorname{Lim} R(e_n)$. Further, for each $e \in E(X)$ such that $x \in \alpha(v, e)$ we have $R(e) \subset \bigcap\{R(e_n) : n \in \mathbb{N}\}$, whence it follows that

$$\operatorname{cl}\Big(\bigcup\{R(e) : e \in E(X) \quad \text{and} \quad x \in \alpha(v, e)\}\Big) \subset \bigcap\{R(e_n) : n \in \mathbb{N}\}.$$

We will prove the equality

$$\operatorname{cl}\Big(\bigcup\{R(e) : e \in E(X) \text{ and } x \in \alpha(v, e)\}\Big) = \bigcap\{R(e_n) : n \in \mathbb{N}\}. \tag{4.14}$$

To this aim suppose there is a point

$$y \in \bigcap\{R(e_n) : n \in \mathbb{N}\} \setminus \operatorname{cl}\Big(\bigcup\{R(e) : e \in E(X) \text{ and } x \in \alpha(v, e)\}\Big).$$

Then the triple (e_0, y, x) is oriented positively, so there is an end point e_{n_0} (for some $n_0 \in \mathbb{N}$) such that the triple (e_0, y, e_{n_0}) is oriented positively, too, and thus $y \notin R(e_{n_0})$, contrary to the assumption. Hence (4.14) is proved.

Since each of the two families of continua

$$\{R(e) : e \in E(X) \text{ and } x \in \alpha(v, e)\} \quad \text{and} \quad \{R(e_n) : n \in \mathbb{N}\}$$

is monotone, we infer from (4.14) that

$$\text{if} \quad z' = \lim g_2(e_n), \quad \text{then} \quad z' \in \operatorname{cl} \{g_2(e) : e \in E(X) \text{ and } x \in \alpha(v, e)\},$$

and therefore (4.13), in the considered case, is a consequence of the definition of g.

If the sequence $\{e_n\}$ is decreasing (thus bounded by e_0) with respect to the relation $\preceq$, then, substituting g_1 for g_2 and L for R, the argument remains the same. If $\{e_n\}$ is increasing and unbounded, then $\alpha(v,e_0) = \operatorname{Lim} R(e_n)$, whence $\lim g_2(e_n) = g_2(e_0) = 1$, and thus $z = 1$ is an element of the right member of (4.13).

Finally, if $\{e_n\}$ is any sequence of end points of X, then it contains an increasing or a decreasing subsequence, and the conclusion (4.13) of Fact 2 is a consequence of the conclusion for one of the already considered cases. So, the proof of Fact 2 is complete.

For each point $p \in X \setminus \{v\}$ we assign a subset

$$E(X,p) = \{e \in E(X) : p \in \alpha(v,e)\}$$

and a minimal connected subset $A(p)$ of S^1 containing $g(E(X,p))$ such that if neither $\alpha(v,x)$ is contained in $\alpha(v,y)$ nor $\alpha(v,y)$ is contained in $\alpha(v,x)$, then $A(x) \cap A(y) = \emptyset$ for all points $x, y \in X \setminus \{v\}$. Namely, if $E(X,p)$ consists of exactly one end point e of X, we define $A(p) = \{g(p)\}$. Otherwise for an arbitrary $p \in X \setminus \{v\}$ we define $q \in A(p)$ provided that there are two end points $e_1, e_2 \in E(X)$ and an arc $A \subset S^1$ containing q with end points $g(e_1)$ and $g(e_2)$ such that $A \cap g(E(X)) \subset g(E(X,p))$.

Fact 3. For every two points x and y of $X \setminus \{v\}$ such that $x \notin \alpha(v,y)$ and $y \notin \alpha(v,x)$ we have $\operatorname{cl} A(x) \cap \operatorname{cl} A(y) = \emptyset$.

Indeed, suppose on the contrary that there is $p \in \operatorname{cl} A(x) \cap \operatorname{cl} A(y)$.

Case 1. $p \in g(E(X))$. Then there are sequences $\{e_n\} \subset E(X,x)$ and $\{e'_n\} \subset E(X,y)$ such that

$$\lim g(e_n) = \lim g(e'_n) = p = g(e) \quad \text{for some} \quad e \in E(X).$$

Thus $\operatorname{Ls} \alpha(v,e_n) \subset \alpha(v,e)$ and $\operatorname{Ls} \alpha(v,e'_n) \subset \alpha(v,e)$ by Fact 1. Since $x \in \alpha(v,e_n)$ and $y \in \alpha(v,e'_n)$, we have $\alpha(v,x) \subset \alpha(v,e)$ and $\alpha(v,y) \subset \alpha(v,e)$, hence either $\alpha(v,x) \subset \alpha(v,y)$ or $\alpha(v,y) \subset \alpha(v,x)$, contrary to the assumption on x and y.

Case 2. $p \notin g(E(X))$. Then there are two sequences $\{e^1_n\}$ and $\{e^2_n\}$, both monotone with respect to $\preceq$, such that

$$p = \lim g(e^1_n) = \lim g(e^2_n), \quad e^1_n \in E(X,x) \quad \text{and} \quad e^2_n \in E(X,y).$$

By symmetry we may assume that $\{e^1_n\}$ is increasing and $\{e^1_n\}$ is decreasing. Since the sequence $\{g_1(e^1_n)\}$ is oriented counter clockwise, it is convergent, and similarly $\{g_1(e^2_n)\}$ is convergent. Then $y \notin \operatorname{Ls} L(e^1_n)$ and $y \in \operatorname{Li} L(e^2_n)$, so $a_1 = \lim g_1(e^1_n) \neq \lim g_1(e^2_n) = b_1$. Similarly, $x \notin \operatorname{Ls} R(e^2_n)$ and $x \in \operatorname{Li} R(e^1_n)$, so

$a_2 = \lim g_2(e_n^1) \neq \lim g_2(e_n^2) = b_2$. According to the construction of g, since

$$P(a_1, b_1) \cap g_1(E(X)) = \emptyset = P(a_2, b_2) \cap g_2(E(X)),$$

we have

$$\begin{aligned}\lim g(e_n^1) &= \lim m_1(g_1(e_n^1)) = m_1(\lim g_1(e_n^1)) = m_1(a_1)\\ &\neq m_1(b_1) = m_1(\lim g_1(e_n^2)) = \lim m_1(g_1(e_n^2)) = \lim g(e_n^2),\end{aligned}$$

a contradiction. So, Fact 3 is established.

Fact 4. For each sequence of end points $\{e_n\} \subset E(X)$ if the sequence $\{g(e_n)\}$ is convergent in S^1, then there exists a point $x \in X$ such that $\operatorname{Ls} \alpha(v, e_n) = \alpha(v, x)$.

Because if not, then there are points x_1 and x_2 in $X \setminus \{v\}$ such that $x_1 \notin \alpha(v, x_2)$ and $x_2 \notin \alpha(v, x_1)$, and there are two subsequences $\{e_n^1\}$ and $\{e_n^2\}$ of the sequence $\{e_n\}$ with $\lim e_n^1 = x_1$ and $\lim e_n^2 = x_2$. Let $z = \lim g(e_n) = \lim g(e_n^1) = \lim g(e_n^2)$. Applying Fact 3 we see that

$$\operatorname{cl} E(X, x_1) \cap \operatorname{cl} E(X, x_2) \subset \operatorname{cl} A(x_1) \cap \operatorname{cl} A(x_2) = \emptyset,$$

and, applying Fact 2, we get $z \in \operatorname{cl} E(X, x_1)$ and $z \in \operatorname{cl} E(X, x_2)$, a contradiction. Thus Fact 4 has been demonstrated.

The next fact is an immediate consequence of the previous one.

Fact 5. For each sequence of end points $\{e_n\} \subset E(X)$ if each of the two sequences $\{g(e_n)\}$ and $\{\lambda(e_n)\}$ is convergent, then the sequence $\{e_n\}$ is convergent, too.

Recall that the mapping $\lambda : X \to [0, \infty)$ is defined by (2.7), and put, in the complex plane $\mathbb{C}$,

$$F = \{t\lambda(e) \cdot g(e) \in \mathbb{C} : t \in [0, 1] \text{ and } e \in E(X)\}.$$

We define a mapping $f : F \to X$ by

$$f(t\lambda(e) \cdot g(e)) = x \quad \text{if and only if} \quad x \in \alpha(v, e) \text{ and } \lambda(x) = t\lambda(e).$$

Moreover, we can extend f to a function $\tilde{f} : \operatorname{cl} F \to X$ satisfying the equality $\tilde{f}(\lim p_n) = \lim \tilde{f}(p_n)$ for each sequence of points p_n of F. We will show that $\tilde{f}$ is well defined and continuous. Indeed, for each convergent sequence $p_n = t_n\lambda(e_n)g(e_n) \in F$ with $t_n \in [0, 1]$ and $e_n \in E(X)$ such that $\lim g(e_n) = z$ and $\lim t_n\lambda(e_n) = \lambda_0 \in \mathbb{R}$ we define $\tilde{f}(\lim p_n)$ as a point q in the arc $\operatorname{Ls} \alpha(v, e_n)$ (see Fact 4) with $\lambda(q) = \lambda_0$, so q is uniquely determined, and $\tilde{f}$ is continuous.

Now we extend the upper semicontinuous decomposition $\{\tilde{f}^{-1}(x) : x \in X\}$ of $\operatorname{cl} F$ to an upper semicontinuous decomposition $\mathcal{D}$ of the complex plane $\mathbb{C}$ whose elements are either singletons or arcs contained in the circles $\{z \in \mathbb{C} : |z| = \text{const}\}$, so the quotient space $\mathbb{C}/\mathcal{D}$ is homeomorphic to the plane $\mathbb{R}^2$ (by the theorem of R. L. Moore, see e.g. [16, §61, IV, Theorem 8, p. 533]). Indeed, we identify all maximal arcs A with end points a_1 and a_2 in the circles $\{z \in \mathbb{C} : |z| = \text{const}\}$ such that $\tilde{f}(ta_1) = \tilde{f}(ta_2) = \tilde{f}(ta)$ for each $t \in [0,1]$ and $a \in A$ with $ta \in \operatorname{cl} F$. In other words, we define a relation $\simeq$ on $\mathbb{C}$ putting

$z_1 \simeq z_2 \iff$ there is a nonnegative real number λ_0, two points $a_1, a_2 \in \operatorname{cl} F$, and an arc A contained in the circle $\{z \in \mathbb{C} : |z| = \lambda_0\}$ such that:

1^o a_1 and a_2 are end points of A;

2^o $z_1, z_2 \in A$;

3^o $\tilde{f}(ta_1) = \tilde{f}(ta_2) = \tilde{f}(ta)$ for each $t \in [0,1]$ and each point $a \in A$ such that $ta \in \operatorname{cl} F$.

The reader can verify in a straightforward way that $\simeq$ is a closed equivalence relation, which implies that equivalence classes of $\simeq$ are closed and that the induced decomposition of $\mathbb{C}$ is upper semicontinuous. Moreover, for every two points z_1 and z_2 of $\mathbb{C}$ with $z_1 \simeq z_2$, if A, a_1 and a_2 are as in the definition of the relation $\simeq$, then we see that for every $u_1, u_2 \in A$ we also have $u_1 \simeq u_2$. This implies monotoneity of the decomposition.

Since no arc with an end point v is a neighborhood of v, condition 3^o implies that the equivalence classes of $\simeq$ are not whole circles, so they are arcs or singletons. Denote by $\mathcal{D}$ the decomposition of the plane $\mathbb{C}$ into equivalence classes of $\simeq$. By the Moore theorem (see e.g. [16, §61, IV, Theorem 8, p. 533]) the decomposition space is homeomorphic to the plane $\mathbb{R}^2$. Let $\pi : \mathbb{C} \to \mathbb{R}^2$ be the quotient mapping. Since

$$\{\tilde{f}^{-1}(x) : x \in X\} = \{\pi^{-1}(x) \cap \operatorname{cl} F : x \in \mathbb{R}^2\},$$

the image $\pi(\operatorname{cl} F)$ is homeomorphic to X. Denote by $h : X \to \pi(\operatorname{cl} F) \subset \mathbb{R}^2$ the homeomorphism, and note that $h(v) = \pi(0)$. Consider the natural arc-structure γ on the complex plane $\mathbb{C}$ putting

$$\gamma(0, z) = \{zt : t \in [0,1]\} \quad \text{for each} \quad z \in \mathbb{C}$$

and extending it according to Statement 2.1. Then $(\mathbb{C}, \gamma, 0)$ is arc-smooth. Now we are going to define the needed arc-structure β on $\mathbb{R}^2$ that satisfies (4.3), (4.4) and (4.5). Namely put

$$\beta(\pi(0), \pi(z)) = \pi(\gamma(0, z)),$$

and extend β in accordance with Statement 2.1. Condition (4.3) is satisfied by the construction, $(\mathbb{R}^2, \beta, h(v))$ is arc-smooth by Theorem I-7-A of [8, p. 553], so

(4.4) holds true, and (4.5) follows from the fact the sets $\pi^{-1}(x)$ for $x \in \mathbb{R}^2 \setminus h(X)$ are singletons.

Finally we discuss the case when v is an end point of an arc $\alpha(v, a)$ which is a neighborhood of v in X. Then we take an arc $ve \subset \mathbb{R}^2$ such that $ve \cap X = \{v\}$ and we consider an arc-structure (X', α') putting $X' = X \cup ve$, with v as the initial point of (X', α') and such that $\alpha' | X \times X = \alpha$. Then we apply the whole construction of the previous case with (X', α') in place of (X, α). One can verify that the homeomorphism h and the arc-structure β satisfy the conclusion of the theorem. The proof is complete. □

In Theorem 4.1 we have constructed, for a given plane arc-smooth continuum X, such an embedding $h : X \to h(X) \subset \mathbb{R}^2$ that the arc-structure on $h(X)$ can be extended to an arc-structure on $\mathbb{R}^2$. It seems to be interesting to know whether such an extension could be obtained for an arbitrary embedding of X in the plane. Precisely, we have the following question.

Question 4.1. Let an arc-smooth continuum (X, α, v) in the plane $\mathbb{R}^2$ be given. Does there exist an arc-structure $\beta : \mathbb{R}^2 \times \mathbb{R}^2 \to C(\mathbb{R}^2)$ such that $(X, \alpha) \ll (\mathbb{R}^2, \beta)$ and $(\mathbb{R}^2, \beta, v)$ is arc-smooth?

The next question is closely related to the previous one. The authors expect both of them to be answered in the affirmative.

Question 4.2. Let $h : (X_1, \alpha_1, v_1) \to (X_2, \alpha_2, v_2)$ be a homeomorphism between plane arc-smooth continua (X_1, α_1, v_1) and (X_2, α_2, v_2). Is it true that if h preserves the orientation of triples of end points, then h can be extended to an orientation preserving homeomorphism of the plane?

Note that the converse implication to that of Question 4.2 is obvious.

Let $\omega : Z(\mathbb{R}^2) \to [0, \infty)$ be a Whitney map satisfying the condition

$$(4.15) \quad \text{for each sequence } \{A_n\} \subset Z(\mathbb{R}^2) \text{ if } \lim \operatorname{diam} A_n = \infty, \quad \text{then } \lim \omega(A_n) = \infty.$$

The reader can observe that the Whitney map constructed from the Euclidean metric on $\mathbb{R}^2$ according to [19, (0.50.1), p. 25], is such.

Now let an arc-structure $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to C(\mathbb{R}^2)$ be given in the plane $\mathbb{R}^2$ such that $(\mathbb{R}^2, \alpha, v)$ is arc-smooth. For a Whitney mapping ω satisfying (4.15) define a mapping $\lambda : \mathbb{R}^2 \to [0, \infty)$ by (2.7). We shall prove several properties of the mapping λ. They will be used in the proof of the next theorem.

Statement 4.2. *For each $t > 0$ the set $\lambda^{-1}(t) \subset \mathbb{R}^2$ is a simple closed curve.*

Proof. Since $(\mathbb{R}^2, \alpha, v)$ is arc-smooth, we infer from continuity of λ and from condition (4.15) that for each $t > 0$ the set $\lambda^{-1}(t)$ is closed and bounded, so compact. It separates the plane into two components: one, $\lambda^{-1}([0,t))$, which is bounded, and the other, $\lambda^{-1}((t,\infty))$, unbounded. This implies that $\lambda^{-1}(t)$ is an irreducible separator of the plane (see [16, §46, VII, Theorem 4, p. 155]), so it is connected, [16, §62, V, Theorem 7, p. 559]. Thus $\lambda^{-1}(t)$ is a continuum. Further, each point $x \in \lambda^{-1}(t)$ is accessible from both these components. To see this it suffices to take any point $y \in \mathbb{R}^2$ with $\lambda(y) > t$ and $x \in \alpha(v,y)$. Applying Schönflies' theorem [16, §61, II, Theorem 12, p. 518] we conclude that $\lambda^{-1}(t)$ is a simple closed curve. □

Statement 4.3. *For every $a, b \in [0,\infty)$ with $a \leq b$ there is a monotone surjective mapping*

$$f(a,b) : \lambda^{-1}(b) \to \lambda^{-1}(a)$$

such that for each $y \in \lambda^{-1}(b)$ we have $f(a,b)(y) \in \alpha(v,y)$. Moreover, for every $a, b, c \in [0,\infty)$ with $a \leq b \leq c$ the equality

$$f(a,b) \circ f(b,c) = f(a,c)$$

holds.

Proof. For each $y \in \lambda^{-1}(b)$, define $f(a,b)(y)$ as the unique point x of the arc $\alpha(v,y)$ such that $\lambda(x) = a$. Thus $f(a,b)$ is a continuous surjection. We will prove that it is monotone. To this aim fix a point $x \in \lambda^{-1}(a)$ and let $y_1, y_2 \in (f(a,b))^{-1}(x)$. There exists exactly one arc $B \subset \lambda^{-1}(b)$ with end points y_1, y_2 such that the union $U = \alpha(x,y_1) \cup \alpha(x,y_2)$ separates the disk $\lambda^{-1}([a,b])$ between v and any point of $B \setminus \{y_1, y_2\}$. Then for each point $y \in B$ the arc $\alpha(x,y)$ intersects U and thus $x \in \alpha(v,y)$, i.e., we have $f(a,b)(y) = x$. Consequently, $B \subset (f(a,b))^{-1}(x)$, so $(f(a,b))^{-1}(x)$ is connected, whence $f(a,b)$ is monotone. The rest of the conclusion follows from the definition of $f(a,b)$ by arc-smoothness of $(\mathbb{R}^2, \alpha, v)$. □

Corollary 4.4. *For each point $x \in \mathbb{R}^2 \setminus \{v\}$ and for each real number $t \geq \lambda(x)$ the set*

$$A = \{y \in \mathbb{R}^2 : \lambda(y) = t \quad \textit{and} \quad x \in \alpha(v,y)\}$$

is either an arc or a singleton.

Proof. Taking $a = \lambda(x)$ and $b = c = t$ in Statement 4.3 we see that $A = (f(\lambda(x), t))^{-1}(x)$. Thus A is a subcontinuum of the simple closed curve $\lambda^{-1}(\lambda(x))$. Since the singleton $\{x\}$ is a proper subset of the range $\lambda^{-1}(\lambda(x))$, hence its inverse image A is a proper subset of the domain $\lambda^{-1}(t)$. Thus the conclusion follows. □

The following statement concerning monotone mappings between simple closed curves is known (see e.g. [2, p. 478]).

Statement 4.5. *Let a surjective mapping $f : P \to Q$ between simple closed curves P and Q be given. Then f is monotone if and only if for each $\varepsilon > 0$ there exists a homeomorphism $h : P \to Q$ such that $\rho(f, h) < \varepsilon$ (consequently, the set of homeomorphisms is dense in the space of monotone mappings between P and Q with the supremum metric ρ).*

Let X be a metric space equipped with a metric d. A decomposition of X is defined to be a family $\mathcal{D}$ of mutually disjoint nonempty closed subsets of X whose union is X. Given a point $x \in X$, we denote by $D(x)$ the element of $\mathcal{D}$ to which x belongs. A decomposition $\mathcal{D}$ of X is said to be *continuous with respect to homeomorphical convergence* provided that for each sequence $\{x_n\}$ of points of X converging to a point x there is a sequence of homeomorphisms $h_n : D(x_n) \to D(x)$ such that

$$\lim(\sup\{d(p, h_n(p)) : p \in D(x_n)\}) = 0.$$

Theorem 4.6. *Let an arc-structure $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to C(\mathbb{R}^2)$ be given in the plane $\mathbb{R}^2$ such that $(\mathbb{R}^2, \alpha, v)$ is arc-smooth, and let a mapping $\lambda : \mathbb{R}^2 \to [0, \infty)$ be defined by* (2.7)*, where the Whitney map $\omega : Z(\mathbb{R}^2) \to [0, \infty)$ satisfies condition* (4.15)*. Then the family $\mathcal{D} = \{\lambda^{-1}(t) : t > 0\}$ is a decomposition of $\mathbb{R}^2 \setminus \{v\}$ into simple closed curves which is continuous with respect to homeomorphical convergence.*

Proof. Indeed, elements of the decomposition are simple closed curves by Statement 4.2. Further, one can see that for each sequence $\{\lambda^{-1}(t_n)\}$ of elements of $\mathcal{D}$ the condition

$$\lim(\inf\{d(x, y) : x \in \lambda^{-1}(t) \quad \text{and} \quad y \in \lambda^{-1}(t_n)\}) = 0$$

holds if and only if $\lim t_n = t$ (where d stands for the Euclidean metric on $\mathbb{R}^2$). For each sequence of positive numbers $\{t_n\}$ we define a sequence of mappings $\{g_n\}$ putting

$$g_n = \begin{cases} f(t, t_n), & \text{if} \quad t < t_n, \\ f(t_n, t), & \text{if} \quad t_n < t. \end{cases}$$

Let $\operatorname{dom} g_n$ mean the domain of the mapping g_n, i.e., either $\lambda^{-1}(t_n)$ if $t < t_n$, or $\lambda^{-1}(t)$, if $t_n < t$. Then

$$\lim(\sup\{d(x, g_n(x)) : x \in \operatorname{dom} g_n\}) = 0.$$

So, by Statement 4.5, for each g_n there is a homeomorphism between the simple closed curves $\lambda^{-1}(t)$ and $\lambda^{-1}(t_n)$ which is sufficiently close to g_n, and thus the conclusion follows. □

5. Hyperspace retractions

Let α be an arc-structure on a space X. We say that a set $A \subset X$ is *convex* provided that for every two points p and q of A we have $\alpha(p,q) \subset A$. Let a point $v \in X$ be distinguished. A set $A \subset X$ is called *star-convex with respect to* v provided that $v \in A$ and A is convex. If a point $v \in X$ is fixed, in particular if (X, α, v) is an arc-smooth space, then *star-convex* means star-convex with respect to v. For a subset $A \subset X$ the *convex hull about* A (the *star-convex hull about* A) is defined as the intersection of all convex (star-convex) sets containing A.

Given a space X with an arc-structure $\alpha : X \times X \to C(X)$ and a point $v \in X$, we say that a retraction $r : Z(X) \to X$ is *internal* (*star-internal*) provided that for each set $A \in Z(X)$ its image $r(A)$ is an element of the convex hull (star-convex hull) about A.

The following proposition is straightforward.

Proposition 5.1. *If $(X, \alpha) \ll (Y, \beta)$ and a retraction $r : Z(Y) \to Y$ is internal, then the restriction $r|Z(X) : Z(X) \to X$ is an internal retraction. Moreover, if r is star-internal and the distinguished point $v \in Y$ is an element of X, then $r|Z(X)$ is a star-internal retraction.*

Let a continuum X be given. A retraction $r : 2^X \to X$ is said to be *associative* provided that

$$r(A \cup B) = r(r(A) \cup B) \quad \text{for every} \quad A, B \in 2^X. \tag{5.1}$$

Theorem 5.2. *For each arc-smooth space (X, α, v) with $X \subset \mathbb{R}^2$ such that the initial point v is accessible from the complement $\mathbb{R}^2 \setminus X$ there is an associative and internal retraction $r : Z(X) \to X$.*

Proof. We present an idea of the proof first.

For a given compact set $A \in Z(X)$ if $v \in A$ we put $r(A) = v$. Otherwise we define the needed retraction as follows.

We take the smallest number $\tau(A)$ such that $\lambda^{-1}(\tau(A))$ intersects A; next we project the set A (along the arc-structure α) onto the set $r_0(A) \subset \lambda^{-1}(\tau(A))$. Thanks to the existence of the arc pv out of X (except v) the set $r_0(A)$ can be linearly ordered using the orientation of triples of the form (p, x, y), where $x, y \in r_0(A)$. Then the value $r(A)$ is the minimum of $r_0(A)$ in the order.

Details run as follows. We need two auxiliary mappings

$$\tau : Z(X) \to [0, \infty) \quad \text{and} \quad r_0 : Z(X) \to Z(X)$$

defined below. Recall that for a Whitney mapping $\omega : Z(X) \to [0, \infty)$ we have defined the mapping $\lambda : \mathbb{R}^2 \to [0, \infty)$ by (2.7). Now put

$$\tau(A) = \min\{\lambda(x) : x \in A\}, \tag{5.2}$$

and

$$r_0(A) = \{H(x, \tau(A)) : x \in A\}, \tag{5.3}$$

where the homotopy H is defined in (2.8). Note that the just defined functions τ and r_0 are continuous, and that $r_0(A)$ equals the intersection of the convex hull about A and $\lambda^{-1}(\tau(A))$.

Since the initial point v is accessible from $\mathbb{R}^2 \setminus X$, there is an arc $pv \subset (\mathbb{R}^2 \setminus X) \cup \{v\}$. Put $X' = X \cup pv$. Let (X', α', v) be the natural extension of (X, α, v).

Define an order $\leq^*$ on the set $\lambda^{-1}(\tau(A))$ by

$$x \leq^* y \iff \text{the triple } (p, x, y) \text{ is oriented nonnegatively in } X',$$

and note that this order is linear on $\lambda^{-1}(\tau(A))$ because the arc pv lies outside of X (except v). Now define $r(A)$ as the minimum of $r_0(A)$ in the order $\leq^*$. Since the projection along the arc-structure α between the sets $\lambda^{-1}(t_1)$ and $\lambda^{-1}(t_2)$ for $t_1, t_2 \in [0, \infty)$ preserves the order $\leq^*$, the retraction r is continuous by Proposition 3.5. It is associative and internal by its construction. □

The authors do not know if accessibility of v is necessary in Theorem 5.2 for the existence of a retraction r in matter which is associative or internal. So we have a question.

Question 5.1. Is it true that for every arc-smooth space (X, α, v) with $X \subset \mathbb{R}^2$ there exists an associative or internal retraction $r : Z(X) \to X$?

If condition (5.1) is not required, then a retraction in matter does exist, as it is shown in the theorem below, but the retraction presented in the proof is neither associative nor internal. Only a weaker property of being star-internal is achieved.

Theorem 5.3. *For each arc-smooth structure* $(\mathbb{R}^2, \alpha, v)$ *on the plane* $\mathbb{R}^2$ *there is a star-internal retraction* $r : Z(\mathbb{R}^2) \to \mathbb{R}^2$.

Proof. Again we start with a sketch of the proof. Its beginning is exactly the same as in the proof of Theorem 5.2. If v is not in A, then $r_0(A)$ lies in the circle $S = \lambda^{-1}(\tau(A))$. The value of the retraction r depends on the "angle" $\theta(A)$ at which the set $r_0(A)$ is seen from the point v. For large ones we keep $r(A) = v$. For the "angles" less than π there exists exactly one minimal arc in S containing $r_0(A)$. Let $r_1(A)$ be the left end point $a(A)$ of this arc in the counter clockwise orientation of S. The value $r(A)$ is then defined as the point of $\alpha(v, a(A))$ that divides $\alpha(v, a(A))$ proportionally to the "angle" $\theta(A)$. The precise definitions of this "angle" and uniqueness of the considered arc are the keys of the proof.

Now we present details of the argument. As in the beginning of the proof of Theorem 5.2 we define two auxiliary mappings

$$\tau : Z(\mathbb{R}^2) \to [0, \infty) \quad \text{and} \quad r_0 : Z(\mathbb{R}^2) \to Z(\mathbb{R}^2)$$

by (5.2) and (5.3), where, as previously, the homotopy H is defined by (2.8). Note that the functions τ and r_0 are continuous, and that $r_0(A)$ is equal to the intersection of the convex hull about A and $\lambda^{-1}(\tau(A))$. We assume that the Whitney map $\omega : Z(\mathbb{R}^2) \to [0, \infty)$ in the definition (2.7) of λ satisfies condition (4.15), so the set $S = \lambda^{-1}(\tau(A))$ is a simple closed curve (or the singleton $\{v\}$) according to Statement 4.2.

If $r_0(A)$ is a singleton, we define $r(A)$ as the only point of $r_0(A)$ (in particular, it is the case if $v \in A$). If $r_0(A)$ is nondegenerate, to define the needed retraction we introduce the concept of the "angle" $\theta(A)$ first. For any two distinct points $a, b \in S$ we put

$$P(a, b) = \{x \in S : \text{ the triple } (a, x, b) \text{ is oriented counter clockwise}\},$$

and $P(a, a) = S$. Define

$$\pi(S) = \min\{\omega(P(a, b)) : a, b \in S \text{ and } \omega(P(a, b)) = \omega(P(b, a))\},$$

and note that $\pi(S) > 0$. Next put

$$\theta(A) = \min\{\omega(P(a, b)) : a, b \in S \text{ and } r_0(A) \subset P(a, b)\}/\pi(S).$$

Claim. If $\theta(A) < 1$, then there is *exactly one* pair $a(A), b(A) \in S$ realizing the minimum in the definition of $\theta(A)$, i.e., such that

$$r_0(A) \subset P(a(A), b(A)) \tag{5.4}$$

and

$$\theta(A) = \omega(P(a(A), b(A)))/\pi(S). \tag{5.5}$$

Proof of the claim. Suppose on the contrary that there are two distinct arcs $P(a_1, b_1)$ and $P(a_2, b_2)$ in S satisfying (5.4) and (5.5). By (5.4) we infer that $a_1, b_1 \in r_0(A)$, whence $a_1, b_1 \in P(a_2, b_2)$ by (5.4). Because $P(a_1, b_1)$ is not a subset of $P(a_2, b_2)$ (they have the same value of the Whitney map ω) it follows that

$$P(a_1, b_1) \cup P(a_2, b_2) = S.$$

Denote by a_1^* the point of S which is "antipodal" to a_1 i.e., the point of $S \setminus \{a_1\}$ satisfying $\omega(P(a_1, a_1^*)) = \omega(P(a_1^*, a_1))$. If $a_1^* \in P(a_1, b_1)$, then $\theta(A) \cdot \pi(S) = \omega(P(a_1, b_1)) \geq \omega(P(a_1, a_1^*)) \geq \pi(S)$, a contradiction. If $a_1^* \in P(a_2, b_2)$, then $\theta(A) \cdot \pi(S) = \omega(P(a_2, b_2)) \geq \omega(P(a_1^*, a_1)) \geq \pi(S)$, a contradiction again. This finishes the proof of the claim.

Observe that, thanks to continuity of the decomposition $\mathcal{D} = \{\lambda^{-1}(t) : t > 0\}$ with respect to homeomorphical convergence (see Theorem 4.6), the points $a(A)$ and $b(A)$ depend continuously on A. Obviously $a(A) \in r_0(A)$.

If $\theta(A) \geq 1$, then we put $r(A) = v$. Otherwise let $a(A)$ and $b(A)$ be points of S satisfying conditions (5.4) and (5.5) of the claim. We define $r(A)$ as a point of the arc $\alpha(v, a(A))$ by

$$r(A) = H(a(A), \tau(A) \cdot (1 - \theta(A))).$$

It can be seen by the construction that r is a star-internal retraction. To show its continuity one should verify continuity of all auxiliary functions. The details are left to the reader. The proof is finished. □

As a consequence of Theorems 4.1 and 5.3 and Proposition 5.1 we get the following corollary.

Corollary 5.4. *For each arc-smooth continuum (X, α, v) in the plane there is a star-internal retraction $r : 2^X \to X$. Moreover, there are an embedding $h : X \to h(X) \subset \mathbb{R}^2$, an arc-structure β on the plane $\mathbb{R}^2$, and a star-internal retraction $\tilde{r} : Z(\mathbb{R}^2) \to \mathbb{R}^2$ such that (see* (2.6)*)*

$$\alpha_h = \beta | h(X) \times h(X)$$

and the diagram

$$\begin{array}{ccc} 2^X & \xrightarrow{\ r\ } & X \\ {\scriptstyle 2^h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Z(\mathbb{R}^2) & \xrightarrow[\tilde{r}]{} & \mathbb{R}^2 \end{array}$$

commutes, i.e.,

$$h \circ r = \tilde{r} \circ 2^h.$$

6. Applications to plane smooth dendroids

Recall the following result that is closely related to our applications (see [8, Theorem II-4-B, p. 559]).

Theorem 6.1. *A one-dimensional continuum is arc-smooth if and only if it is a smooth dendroid.*

If S is an arbitrary set in a continuum X, we denote by $I(S)$ a continuum in X containing S such that no its proper subcontinuum contains S, i.e., $I(S)$ means a continuum in X which is irreducible about the set $S \subset X$. It is known (see [3, Theorem T1, p. 187]) that in hereditarily unicoherent continua X the continuum $I(S)$ is unique (equal to the intersection of all subcontinua containing S); moreover, the above uniqueness characterizes hereditarily unicoherent continua, see [18, Theorem 1.1, p. 179]. Let a continuum X be hereditarily unicoherent. A retraction $r : 2^X \to X$ is said to be *internal* (compare [4, (3.8), p. 11]) provided that

(6.1) $$r(A) \in I(A) \quad \text{for each} \quad A \in 2^X.$$

It can be observed that if a continuum X is hereditarily unicoherent, then a retraction $r : 2^X \to X$ is internal (in the sense defined here at the beginning of Chapter 5) if and only if condition (6.1) is satisfied, i.e., that the two definitions coincide.

Given a topological Hausdorff space X, a *mean* μ on X is defined as a mapping $\mu : X \times X \to X$ such that for all $x, y \in X$ we have

(6.2) $$\mu(x, x) = x;$$

(6.3) $$\mu(x, y) = \mu(y, x).$$

A mean $\mu : X \times X \to X$ on a space X is said to be *associative* provided that

(6.4) $$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) \quad \text{for all} \quad x, y, z \in X;$$

if X is a dendroid, then a mean $\mu : X \times X \to X$ is said to be *internal* provided that

$$\mu(x, y) \in xy \quad \text{for all} \quad x, y \in X. \tag{6.5}$$

The reader is referred to Chapter 5 of [4, p. 18] for more information on means defined on continua. In particular, the following result is known (see [4, Proposition 5.11, p. 19]).

Proposition 6.2. *Let a continuum X be given. If there exists a retraction $r : 2^X \to X$, then there exists a mean $\mu : X \times X \to X$. Furthermore, if the retraction r is associative (internal), then the mean μ is associative (internal, respectively).*

An example is known (see [4, Example 5.52, p. 25]) of a smooth dendroid X that admits no mean, and consequently, by Proposition 6.2, that admits no retraction from 2^X onto X. This dendroid has been constructed in the 3-space, and it is nonplanable (it cannot be embedded in the plane). It follows from our Corollary 5.4 that such a phenomenon is not possible on the plane. Namely, Theorem 5.2 and Corollary 5.4 imply the following two results.

Theorem 6.3. *If a smooth dendroid X can be embedded in the plane $\mathbb{R}^2$ so that an initial point of X is accessible from $\mathbb{R}^2 \setminus X$, then there exists an associative and internal retraction $r : 2^X \to X$ which can be extended to a star-internal retraction from $Z(\mathbb{R}^2)$ onto $\mathbb{R}^2$.*

Theorem 6.4. *For every plane smooth dendroid X there exists a star-internal retraction $r : 2^X \to X$ which can be extended to a star-internal retraction from $Z(\mathbb{R}^2)$ onto $\mathbb{R}^2$.*

As a particular case of Question 5.1 we have the following one.

Question 6.1. Is it true that every plane smooth dendroid X admits an associative or internal retraction $r : 2^X \to X$?

Proposition 6.2 together with Theorems 6.3 and 6.4 imply the following corollaries.

Theorem 6.5. *If a smooth dendroid X can be embedded in the plane $\mathbb{R}^2$ so that an initial point of X is accessible from $\mathbb{R}^2 \setminus X$, then it admits an associative and internal mean.*

Theorem 6.6. *Every plane smooth dendroid admits a mean.*

Remark 6.2. The above mentioned example of a (nonplanable) smooth dendroid admitting no mean shows that planability of the considered smooth dendroid is an indispensable assumption in Theorems 6.5 and 6.6 (as well as in Theorems 6.3 and 6.4).

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