

ATRIODIC ABSOLUTE RETRACTS FOR HEREDITARILY UNICOHERENT CONTINUA

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ABSTRACT. Let X be an absolute retract for the class of hereditarily unicoherent continua that contains no simple triod. In this paper we prove that (a) X is atriodic, (b) X is either an arc or an indecomposable continuum having only arcs as its proper subcontinua; (c) if X is either tree-like or circle-like, then it is arc-like.

1. INTRODUCTION

The class of hereditarily unicoherent continua and their various subclasses as e.g. of tree-like continua, λ -dendroids, dendroids and dendrites appear in a natural way in various regions of mathematical interest: the fixed point property, homogeneous spaces, continuous and upper semi-continuous decompositions, (hereditarily) indecomposable continua and many other areas of topology, and also out of topology. Therefore absolute retracts for hereditarily unicoherent continua and for some related classes seem to be worth of a separate study. This paper is a continuation of such a study initiated by T. Maćkowiak in [27] and developed later by the authors in [9] and in [8].

T. Maćkowiak proved that the simplest indecomposable continuum, so called bucket-handle, is a member of $\text{AR}(\mathcal{HU})$ (i.e., of the class of absolute retracts for hereditarily unicoherent continua, see [27, Corollary 4, p. 181]). Thus there are nondegenerate atriodic members of $\text{AR}(\mathcal{HU})$ different from an arc. Atriodic case

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turns out to be special in various areas of continuum theory, as for example in the study of homogeneous continua (see e.g. [16], [17], [18], [19], [25], [32], [33], [34]), of dynamical systems (see e.g. [1], [2] and references therein), and also in the fixed point property (see [31], for example). In this paper the authors present their research on atriodic absolute retracts for hereditarily unicoherent continua showing many strong properties of atriodic members of $\text{AR}(\mathcal{HU})$.

The paper consists of five sections. After the Introduction, some basic definitions, auxiliary concepts and results are collected in the second section. The new results of this paper are presented in Sections 3, 4 and 5. Let \mathcal{AAP} be the class of all continua having the arc approximation property (for the definition see the next section, the paragraph before Proposition 2.1), and \mathcal{APK} stand for the class of all continua having the arc property of Kelley (see the next section, the paragraph after Proposition 2.1). It is known (see [9, Corollary 3.7 and Proposition 3.4]) that

$$\text{AR}(\mathcal{HU}) \subset \mathcal{APK} \subset \mathcal{AAP}.$$

Investigating properties of those members of $\text{AR}(\mathcal{HU})$ and $\text{AR}(\mathcal{TL})$ that contain no simple triods (equivalently, that are atriodic, see Corollary 5.1), we have observed that a great part of the results holds for some larger classes of continua, namely for those with the arc approximation property and/or the arc property of Kelley. These results are gathered in Sections 3 and 4, respectively.

In Section 3 we prove that in the class \mathcal{AAP} atriodicity is equivalent to the property that a continuum contains no simple triod, and that atriodic members of \mathcal{AAP} are precisely arcs, simple closed curves and indecomposable continua having only arcs for proper subcontinua. As we prove in Section 4, atriodic members of \mathcal{APK} turn out to have a local product structure about their non-end points, of an arc times some 0-dimensional compact set. We also study three classical examples of atriodic members of \mathcal{APK} . The main result of Section 5 says that all atriodic members of $\text{AR}(\mathcal{HU})$ that are either tree-like or circle-like, are arc-like.

This paper is devoted to study general properties of continua in $\text{AR}(\mathcal{HU})$ containing no simple triods. In [10, Theorem 3.6] the authors show that the inverse limit of arcs with confluent bonding mappings is in $\text{AR}(\mathcal{HU})$ (this extends the above mentioned result of Maćkowiak). According to this result, there are many continua in the class studied in this paper.

2. PRELIMINARIES AND AUXILIARY RESULTS.

By a *space* we mean a topological space, and a *mapping* means a continuous function. Given a space X and its subspace $Y \subset X$, a mapping $r : X \rightarrow Y$

is called a *retraction* if the restriction $r|_Y$ is the identity. Then Y is called a *retract* of X . The reader is referred to [4] and [20] for needed information on these concepts.

Let \mathcal{C} be a class of *compacta*, i.e., of compact metric spaces. Following [20, p. 80], we say that a space $Y \in \mathcal{C}$ is an *absolute retract for the class \mathcal{C}* (abbreviated $\text{AR}(\mathcal{C})$) if for any space $Z \in \mathcal{C}$ such that Y is a subspace of Z , Y is a retract of Z . The concept of an AR space originally had been studied by K. Borsuk, see [4].

Let X be a metric space with a metric d . For a mapping $f : A \rightarrow B$, where A and B are subspaces of X , we define

$$d(f) = \sup\{d(x, f(x)) : x \in A\}.$$

Further, we denote by $B(p, \varepsilon)$ the (open) ball in X centered at a point $p \in X$ and having the radius ε . For a subset $A \subset X$ we put $N(A, \varepsilon) = \bigcup\{B(a, \varepsilon) : a \in A\}$. The symbol \mathbb{N} stands for the set of all positive integers, and \mathbb{R} denotes the space of real numbers.

By a *continuum* we mean a connected compactum. Given a continuum X , we denote by $C(X)$ the hyperspace of all nonempty subcontinua of X equipped with the Hausdorff metric H (see e.g. [29, (0.1), p. 1 and (0.12), p. 10]). We write $A = \text{Lim } A_n$, if $\lim H(A, A_n) = 0$.

A *curve* means a 1-dimensional continuum. A continuum X is said to be *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected. X is said to be *hereditarily unicoherent* if all of its subcontinua are unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A locally connected dendroid is called a *dendrite*. A *tree* means a graph containing no simple closed curve.

A continuum is said to be *decomposable* provided that it can be represented as the union of two of its proper subcontinua. Otherwise it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable (hereditarily indecomposable)* provided that each of its subcontinua is decomposable (indecomposable, respectively). A hereditarily unicoherent and hereditarily decomposable continuum is called a λ -*dendroid*. A continuum is said to be *tree-like (arc-like, circle-like)* provided that it is the inverse limit of an inverse sequence of trees (arcs, circles, respectively).

A continuum X is called a *triod* provided that there is a subcontinuum Z of X (the *center* of the triod) such that $X \setminus Z$ is the union of three mutually disjoint nonempty open sets in X . A continuum is said to be *atriodic* provided that it does not contain any triod. It is known that each arc-like continuum is atriodic (see e.g. [30, Corollary 12.7, p. 233]).

Recall that, for any $k \in \mathbb{N}$, a simple k -od with the center p means the union of k arcs every two of which have exactly one point p in common and such that p is an end point of each of the two.

For any space X define $T(X)$ as the set of all centers of simple triods in X , and let $C_T(X)$ be the collection of all nonempty subcontinua K of X such that for each $\varepsilon > 0$ there is a continuum K_ε in X satisfying

$$K \subset K_\varepsilon \subset N(K, \varepsilon) \quad \text{and} \quad K_\varepsilon \cap T(X) \neq \emptyset.$$

In this paper we will use the following concept of relative arcwise connectedness of a set. A subset Y of a space X is said to be *relatively arcwise connected in X* provided that for every two points $p, q \in Y$ and for each neighborhood U of Y in X there is an arc in U joining p and q . This concept captures a very common phenomenon. For instance, in the Euclidean spaces, manifolds, the Hilbert cube, and more generally, in each locally connected, (locally) compact metric space, each continuum is relatively arcwise connected.

Let \mathcal{D}_0 denote the class of dendrites, \mathcal{D} — the class of dendroids, $\lambda\mathcal{D}$ — of λ -dendroids, \mathcal{TL} — of tree-like continua, and \mathcal{HU} — the class of hereditarily unicoherent ones. Then

$$(2.0) \quad \mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{TL} \subset \mathcal{HU}.$$

According to a classical result of K. Borsuk [4, (13.5), p. 138] each dendrite is an absolute retract for the class of all compacta. Consequently any dendrite D is an absolute retract for each class \mathcal{C} of compacta (abbreviated $\text{AR}(\mathcal{C})$) such that $D \in \mathcal{C}$. More generally, if $\mathcal{C}_1 \subset \mathcal{C}_2$ for some classes \mathcal{C}_1 and \mathcal{C}_2 of spaces, then

$$\mathcal{C}_1 \cap \text{AR}(\mathcal{C}_2) \subset \text{AR}(\mathcal{C}_1).$$

In particular, we have

$$\text{AR}(\mathcal{D}_0) = \mathcal{D}_0 \subset \text{AR}(\mathcal{D}) \cap \text{AR}(\lambda\mathcal{D}) \cap \text{AR}(\mathcal{TL}) \cap \text{AR}(\mathcal{HU}).$$

Note that the class absolute retracts of all unicoherent continua coincides with the class of retracts of the Hilbert cube, thus it also coincides with the class of absolute retracts of all compacta. This class is relatively well studied, and we do not investigate it here.

In the rest of this section we collect concepts and results used in the body of the paper, mostly introduced and studied in our very recent papers, and therefore perhaps not known to the reader. In this way we help the reader in understanding our arguments applied in proofs of results in the next sections.

We start with recalling the following two concepts. A continuum X is said to have the *property of Kelley* provided that for each point $p \in X$, for each

subcontinuum K of X containing p and for each sequence of points p_n converging to p there exists a sequence of subcontinua K_n of X containing p_n and converging to the continuum K (see e.g. [21, p. 167] or [29, Definition 16.10, p. 538]).

A continuum X is said to have the *arc approximation property* provided that for each point $x \in X$, for each subcontinuum K of X containing x there exists a sequence of arcwise connected subcontinua K_n of X containing x and converging to the continuum K (see [12, Section 3, p. 113]). The next proposition was noticed in [12, Proposition 3.10, p. 116].

Proposition 2.1. *If a continuum has the arc approximation property, then each arc component of the continuum is dense.*

A continuum X is said to have the *arc property of Kelley* (see [9, Definition 3.3]) provided that for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points p_n converging to p there exists a sequence of arcwise connected subcontinua K_n of X containing p_n and converging to the continuum K . One can observe the following (compare [9, Proposition 3.4]).

Proposition 2.2. *A continuum has the arc property of Kelley if and only if it has the arc approximation property and the property of Kelley.*

The next theorem is one of the main results of [9] (see [9, Theorem 3.5]).

Theorem 2.3. *Let \mathcal{K} be any class of continua listed in (2.0). Then any member of $\text{AR}(\mathcal{K})$ has the arc property of Kelley.*

Let K, K_1, K_2, \dots be compact sets in a metric space with a metric d . We say that the sequence $\{K_n\}$ converges to K *homeomorphically* provided there exists a sequence of homeomorphisms $h_n : K \rightarrow K_n$ such that $\lim d(h_n) = 0$.

The following theorem is shown in [9, Theorem 3.13 and Corollary 3.15].

Theorem 2.4. *Let a continuum X have the arc property of Kelley, Y be either an arc or a simple triod in X and p be an end point of Y . Then for each sequence of points p_n converging in X to p there is a sequence of sets $Y_n \subset X$, each homeomorphic to Y , such that $p_n \in Y_n$ and the sequence Y_n converges to Y homeomorphically.*

3. CONTINUA WITH THE ARC APPROXIMATION PROPERTY THAT CONTAIN NO SIMPLE TRIODS

Let us begin with the following observation.

Observation 3.1. *Let X be a continuum such that each of its proper subcontinua is relatively arcwise connected in X . Then exactly one of the two following conditions holds.*

(3.1.1) X contains a simple closed curve;

(3.1.2) X is hereditarily unicoherent.

PROOF. Assume that X is not hereditarily unicoherent. Then there are continua A and B in X such that the intersection $A \cap B$ has at least two different components K and L . Choose points $p \in K$ and $q \in L$. Then for sufficiently small $\varepsilon > 0$ there is no continuum in $N(A, \varepsilon) \cap N(B, \varepsilon)$ intersecting both K and L . By the assumption, however, there are arcs $A_1 \subset N(A, \varepsilon)$ and $B_1 \subset N(B, \varepsilon)$ each containing both p and q . Then the union $A_1 \cup B_1 \subset X$ must contain a simple closed curve. Clearly (3.1.1) and (3.1.2) are mutually exclusive conditions. \square

Lemma 3.2. *Let a continuum X have the arc approximation property and let a subcontinuum $K \subset X$ contain a triod. Then $K \in C_T(X)$.*

PROOF. Let d be a metric on X . Take $\varepsilon > 0$, and let A, B, C be subcontinua of K such that the sets $A \setminus (B \cup C)$, $B \setminus (A \cup C)$ and $C \setminus (A \cup B)$ are nonempty, and the union $A \cup B \cup C$ is a triod with a continuum $Z = A \cap B = B \cap C = C \cap A$. Fix a point $z \in Z$ and some points $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. Let $\delta > 0$ be so small that $\delta < \varepsilon$ and

$$B(a, \delta) \cap (B \cup C) = B(b, \delta) \cap (C \cup A) = B(c, \delta) \cap (A \cup B) = \emptyset.$$

Applying the arc approximation property we see that there are arcs $A_1 \subset N(A, \frac{\delta}{2})$, $B_1 \subset N(B, \frac{\delta}{2})$, $C_1 \subset N(C, \frac{\delta}{2})$ having the point z as their common end point, and some points a_1, b_1, c_1 as the other end points, respectively, such that

$$\max\{d(a, a_1), d(b, b_1), d(c, c_1)\} < \frac{\delta}{2}.$$

Then there is a simple triod in the union $A_1 \cup B_1 \cup C_1$ that contains the points a_1, b_1 and c_1 . We let $K_\varepsilon = K \cup A_1 \cup B_1 \cup C_1 \subset N(K, \varepsilon)$. Therefore K belongs to $C_T(X)$. \square

Lemma 3.3. *Let a continuum X have the arc approximation property and let a proper subcontinuum K of X be such that $K \notin C_T(X)$. Then K is relatively arcwise connected in X .*

PROOF. Suppose on the contrary that there are two different points $a, b \in K$ that belong to two distinct arc components of a neighborhood U of K in X . Choose a $\delta > 0$ such that $N(K, \delta) \subset U$, the complement $X \setminus N(K, \delta)$ has nonempty interior in X , and $N(K, \delta) \cap T(X) = \emptyset$. Since all arc components of X are dense in X

according to Proposition 2.1, there are arcs A and B in $N(K, \delta)$ each having a point in $\text{bd } N(K, \frac{\delta}{2})$ as one of its end points, with the points a and b as the other end points of A and of B , respectively. Evidently, A and B are contained in two different arc components of $N(K, \delta)$. Since X has the arc approximation property, there are arcwise connected continua A_n in $N(K, \delta)$ containing the point a and converging to $K \cup B$. Then, for sufficiently large n , the set $(K \cup A \cup B \cup A_n) \setminus K$ has at least three components. Thus the continuum $K \cup A \cup B \cup A_n$ is a triod with K as its center. According to Lemma 3.2 the triod is in $C_T(X)$, so there is a continuum K_1 in $N(K, \delta)$ such that $K \subset K \cup A \cup B \cup A_n \subset K_1$ and $K_1 \cap T(X) \neq \emptyset$. Hence, K belongs to $C_T(X)$, a contradiction. \square

The following theorem is the main result of this section.

Theorem 3.4. *For each continuum X the following conditions are equivalent.*

- (3.4.1) X has the arc approximation property and contains no simple triod;
- (3.4.2) X has the arc approximation property and is atriodic;
- (3.4.3) each proper nondegenerate subcontinuum of X is an arc.

PROOF. The implication from (3.4.3) to (3.4.1) is trivial, and the one from (3.4.1) to (3.4.2) follows by Lemma 3.3. It remains to show that (3.4.2) implies (3.4.3).

First, notice that by the assumption, the sets $T(X)$ and, consequently, $C_T(X)$, are empty. Therefore by Lemma 3.3 each proper subcontinuum of X is relatively arcwise connected in X . Next, observe that X contains no simple closed curve as a proper subcontinuum. In fact, suppose that S is a simple closed curve in X such that $S \neq X$. Since the arc component of X containing S is dense in X by Proposition 2.1, there is an arc A in X having exactly one point in common with S . The union $S \cup A$ contains a (simple) triod, a contradiction.

Consequently, in view of Observation 3.1, either X is a simple closed curve, or X is hereditarily unicoherent. In the former case condition (3.4.3) is obviously satisfied. So, assume that X is hereditarily unicoherent, and let K be a proper subcontinuum of X . Since K is relatively arcwise connected in X , for every two points $p, q \in K$ there is an arc $pq \subset X$. Since X is hereditarily unicoherent, the arc pq is contained in K . Thus we have proved that each proper subcontinuum of X is arcwise connected. Thus, again by the hereditary unicoherence of X , each proper subcontinuum of X is a dendroid. Since arcs are the only nondegenerate dendroids containing no (simple) triod, condition (3.4.3) follows. The proof is complete. \square

Corollary 3.5. *For each arcwise connected, nondegenerate continuum X the following conditions are equivalent.*

(3.4.1) X has the arc approximation property and contains no simple triod;

(3.5.1) X is either an arc or a simple closed curve.

PROOF. The implication from (3.5.1) to (3.4.1) is trivial. To see that (3.4.1) implies (3.5.1) observe that X must be decomposable, because it is arcwise connected. Thus, by Theorem 3.4, the continuum X is the union of two arcs. Therefore X is locally connected. Since it contains no simple triod, X must be either an arc or a simple closed curve. \square

Corollary 3.6. *Let a non-arcwise connected continuum X have the arc approximation property and contain no simple triod. Then X is indecomposable, 1-dimensional, and it has only arcs as its proper nondegenerate subcontinua. In particular, the composants of X are precisely its arc components.*

PROOF. Indeed, since all arc components of X are dense in X according to Proposition 2.1, each of them has empty interior. By Theorem 3.4 each proper subcontinuum K of X is an arc, and thus it is contained in one arc component of X . Therefore K has empty interior in X , so X is indecomposable (see [26, §48, V, Theorem 2, p. 207]). Since each proper subcontinuum of X is 1-dimensional, it follows that $\dim X = 1$ (see e.g. [15, Theorem 1.9.8, p. 93]). \square

4. CONTINUA WITH THE ARC PROPERTY OF KELLEY THAT CONTAIN NO SIMPLE TRIODS

In this section we first exhibit the global structure of hereditarily unicoherent continua having the arc property of Kelley and being embeddable into a surface (Theorem 4.3). Second, we present a local structure of continua with the arc property of Kelley and containing no simple triods. Namely, we prove that at most points of such continua their open neighborhoods are homeomorphic to the Cartesian products of a compact 0-dimensional set and an open interval (Theorem 4.4) and give some consequences of this result. Next, we analyze three classical examples (the lakes of Wada, the Case-Chamberlin curve and the Ingram continuum) that illustrate results and properties previously considered.

To show the first of the main results of this section we need two lemmas.

Lemma 4.1. *Let a continuum X having the arc property of Kelley be embedded in a surface, and let $a \in T(X)$. Then X is locally arcwise connected at a .*

PROOF. Suppose that X is not locally arcwise connected at a . Then there are a neighborhood U of a in X and a sequence of points a_n in X converging to a such that for each $n \in \mathbb{N}$ the points a and a_n belong to two different arc components

of U . Take a simple triod T in U whose center is a , let p be an end point of T , and ap be the arc from a to p in T . According to Theorem 2.4 there are arcs A_n in U such that $a_n \in A_n$ and A_n converge to the arc ap homeomorphically. Thus, for each $n \in \mathbb{N}$, the arcs ap and A_n are contained in two different arc components of U . Note that there are points $p_n \in A_n$ such that $\lim p_n = p$. Again applying Theorem 2.4 we see that there are simple triods $T_n \subset U$ such that $p_n \in T_n$ and T_n converge to T homeomorphically. Moreover, the triods T and T_n are disjoint as they lie in two different arc components of U . This is impossible in a surface by the local separation property of an arc in the plane. The proof is complete. \square

Lemma 4.2. *Let a continuum X having the arc property of Kelley be embedded in a surface. Then each proper subcontinuum of X is relatively arcwise connected in X .*

PROOF. Let K be a proper subcontinuum of X . If $K \notin C_T(X)$, then the conclusion follows from Lemma 3.3. So, assume that $K \in C_T(X)$. Let p and q be points in K and U be a neighborhood of K in X . Since $K \in C_T(X)$, there are a continuum $K_1 \subset X$ and a point $z \in K_1 \cap T(X)$ such that $K \subset K_1 \subset U$. Let P and Q be subcontinua of K irreducible between p and z and between q and z , respectively. Since X has the arc property of Kelley, there are two sequences $\{P_n\}$ and $\{Q_n\}$ of arcwise connected continua in X , each converging to K and such that $p \in P_n$ and $q \in Q_n$ for each $n \in \mathbb{N}$. Observe that $P_n \cup Q_n \subset U$ for almost all n . Then there are points $a_n \in P_n$ and $b_n \in Q_n$ such that $\lim a_n = \lim b_n = z$. Since X is locally arcwise connected at z by Lemma 4.1, there are arcs A_n and B_n in X such that $a_n, z \in A_n$, $b_n, z \in B_n$, $\text{Lim } A_n = \text{Lim } B_n = \{z\}$ and $A_n \cup B_n \subset U$ for almost all n . Then the union $P_n \cup A_n \cup B_n \cup Q_n$, for sufficiently large n , contains an arc between p and q , and is contained in U . \square

The following theorem is our main result on continua with the arc property of Kelley embedded in surfaces. Note that if such a continuum is a dendroid, then it has the property of Kelley, and thus is it smooth in the sense of [21, p. 194] (see [14, Corollary 5, p. 730]).

Theorem 4.3. *Let a continuum X having the arc property of Kelley and containing no simple closed curve be embedded in a surface. Then X is either a dendroid or an indecomposable continuum having only arcs for its proper nondegenerate subcontinua.*

PROOF. Since by Lemma 4.2 each proper subcontinuum of X is relatively arcwise connected in X , and since X has the arc property of Kelley, and consequently the

arc approximation property by Proposition 2.2, X is hereditarily unicoherent by Observation 3.1. Therefore X is a dendroid whenever it is arcwise connected.

Assume that X is not arcwise connected. Observe that, by Lemma 4.2, every two points of X that belong to a proper subcontinuum of X , also belong to an arc in X . Therefore each proper subcontinuum of X is arcwise connected by the hereditary unicoherence of X .

Since each arc component of X is dense in X according to Propositions 2.1 and 2.2, and since there are at least two of them, each proper subcontinuum of X has empty interior in X . Therefore X is indecomposable (see [26, §48, V, Theorem 2, p. 207]), and thus X is locally arcwise connected at no of its points. It follows from Lemma 4.1 that X contains no simple triod. The rest of the conclusion follows from Corollary 3.6. The proof is complete. \square

By an *end point of a continuum* X we mean a point of X which is an end point of each arc containing it and contained in X .

Theorem 4.4. *If a continuum X has the arc property of Kelley and contains no simple triod, then each non-end point of X has a neighborhood homeomorphic to the Cartesian product of a compact 0-dimensional set and an arc.*

PROOF. Assume $p \in X$ is not an end point of X . Then by Theorem 2.4, applied to the two arcs going out of p , there is an open neighborhood U of p containing no end point of X . Again by Theorem 2.4 there is in U a regular family of curves in the sense of [2, Definition 5.3, p. 460]. Thus by [2, Theorem 5.7, p. 460] U is a flowbox manifold, i.e., every point x of U has an open neighborhood V_x homeomorphic to the Cartesian product of a set S_x and the open interval $(0, 1)$. By Corollaries 3.5 and 3.6 we have $\dim X = 1$, whence $\dim S_x = 0$. By compactness of X we can have closed neighborhoods of x homeomorphic to the Cartesian product of a compact 0-dimensional set and an arc. \square

Corollary 4.5. *If a continuum X has the arc property of Kelley and contains no simple triod, then X is an arc, or X is a circle, or each non-end point of X has a neighborhood homeomorphic to the Cartesian product of the Cantor set and an arc.*

PROOF. If X is arcwise connected and contains no simple triod, then it is an arc or a circle (see Corollary 3.5). So, assume that X is not arcwise connected. Then by Proposition 2.1 each arc component of X is dense, so X contains no free arc. Consequently, the 0-dimensional factor of the Cartesian product considered in Theorem 4.4 has no isolated point, so it is the Cantor set. Therefore the conclusion follows from Theorem 4.4. \square

Corollary 4.6. *Let a continuum X have the arc property of Kelley and contain no simple triod. Then the set of end points of X is closed and has empty interior.*

Definition 4.7. A continuum X is said to have *the local property of Kelley at a point $p \in X$* provided that there exists a neighborhood $U(p)$ of p such that for each continuum $K \subset U(p)$ with $p \in K$ and for each sequence of points $\{p_n\}$ converging to p there is a sequence of continua $\{K_n\}$ with $p_n \in K_n$ converging to K .

Note that the property of Kelley at a point as defined in [36, II, p. 292] implies the local property of Kelley at the point in the sense of Definition 4.7.

Theorem 4.8. *If each nondegenerate proper subcontinuum of a continuum X is an arc, and if X has the local property of Kelley at each of its points, then X has the property of Kelley.*

PROOF. Let K be a subcontinuum of X , let $p \in K$, and $\{p_n\}$ be a sequence of points of X converging to p . We have to find a sequence of continua $\{K_n\}$ in X with $p_n \in K_n$ for each $n \in \mathbb{N}$ and converging to K . This is obvious either if K is the singleton $\{p\}$ (we take $K_n = \{p_n\}$) or if $K = X$ (we take $K_n = X$). So assume that K is a nondegenerate proper subcontinuum of X . Hence K is an arc. Consider two cases.

CASE 1. p is an end point of K .

Let $\mu : C(X) \rightarrow [0, \infty)$ be a Whitney map for the hyperspace $C(X)$ of subcontinua of X (the reader is referred to [29, (0.50), p. 24] or to [21, Chapter 13, p. 105] for the definition, basic properties and existence theorems of this concept). Given $\varepsilon > 0$, denote by x_ε the other end point of a subarc L_ε of K with p as one end point and such that $\mu(L_\varepsilon) = \varepsilon$. Define

$$E = \{\varepsilon \geq 0 : \text{for each } n \in \mathbb{N} \text{ there exists } C_n \in C(X) \text{ such that } p_n \in C_n \text{ and } \text{Lim } C_n = L_\varepsilon\}.$$

Using the standard diagonal procedure one can observe that $\sup E \in E$. Suppose that $\varepsilon_0 = \sup E < \mu(K)$. Let $U(x_{\varepsilon_0})$ be a neighborhood of x_{ε_0} as in Definition 4.7 of the local property of Kelley, and let M be an arc in $K \cap U(x_{\varepsilon_0})$ such that $L_{\varepsilon_0} \cap M = \{x_{\varepsilon_0}\}$. Since $\varepsilon_0 \in E$, there is a sequence $\{C_n\}$ of subcontinua of X such that $p_n \in C_n$ for each $n \in \mathbb{N}$ and $\text{Lim } C_n = L_{\varepsilon_0}$. Take a sequence of points $q_n \in C_n$ such that $\lim q_n = x_{\varepsilon_0}$. Since X has the local property of Kelley at x_{ε_0} , there is a sequence $\{C'_n\}$ of subcontinua of X such that $q_n \in C'_n$ for each $n \in \mathbb{N}$ and $\text{Lim } C'_n = M$. Observe that $\text{Lim}(C_n \cup C'_n) = L_{\varepsilon_0} \cup M \supsetneq L_{\varepsilon_0}$, whence $\mu(L_{\varepsilon_0} \cup M) > \varepsilon_0$. Since $L_{\varepsilon_0} \cup M \subset K$ and $p_n \in C_n \cup C'_n$, we have a contradiction

with the definition of ε_0 . Therefore $\varepsilon_0 = \mu(K)$, i.e., $L_{\varepsilon_0} = K$. This implies the conclusion in Case 1.

CASE 2. p is not an end point of K .

Then K is the union of two arcs K_1 and K_2 such that $K_1 \cap K_2 = \{p\}$. We apply the previous case for K_1 and K_2 separately, finding sequences of continua $C_n^{(1)}$ and $C_n^{(2)}$ such that

$$p_n \in C_n^{(1)} \cap C_n^{(2)} \text{ for each } n \in \mathbb{N}, \quad \text{Lim } C_n^{(1)} = K_1 \quad \text{and} \quad \text{Lim } C_n^{(2)} = K_2.$$

Then the sequence of continua $C_n = C_n^{(1)} \cup C_n^{(2)}$ satisfies the conclusion. \square

One can ask if a hereditarily unicoherent continuum having the arc property of Kelley and containing no simple triods must be either arc-like or circle-like. The answer is negative. This can be shown by any of the three well known examples, that are recalled below. The authors thank Professor Paweł Krupski for calling the first two of them to their attention.

Example 4.9. *Lakes of Wada.*

The continuum Y which is the common boundary of three plane domains, known under the name of “lakes of Wada” (see [37, p. 60]) described precisely by Bing in [3, Section 8, Figure 1, p. 222] is hereditarily unicoherent, each of its proper subcontinua is an arc (so it contains no simple triod), each of its points lies in an open subset homeomorphic with the product of the Cantor set and an open interval, so it has the local property of Kelley at each of its points, and therefore by Theorem 4.8 the continuum Y has the arc property of Kelley and, being the common boundary of three plane domains, it is neither arc-like nor circle-like.

Example 4.10. *Case-Chamberlin curve.*

Let M be the Case-Chamberlin curve (see [5, Section 5, An example, p. 76]), i.e., $M = \varprojlim \{M_n, f_n\}$, where each M_n is homeomorphic to the union B of two tangent circles having a point $p \in B$ as the point of tangency, and where each f_n is a fixed mapping $f : B \rightarrow B$ defined in [5, p. 78].

Using definition of f one can show the following claim.

CLAIM 1. Each nondegenerate proper subcontinuum of M is an arc.

Since $f(p) = p$, the point $e = (p, p, p, \dots)$ is in M .

CLAIM 2. The point e is an end point of M .

Indeed, let K be a proper subcontinuum of M such that $e \in K$. Choose $n \in \mathbb{N}$ so that $\pi_n(K)$ is a proper subcontinuum of $M_n = B$ (here $\pi_n : M \rightarrow M_n = B$ denotes the projection mapping in the inverse sequence). Let A_i for $i \in \{1, 2, 3, 4\}$

be the four arms of a 4-od in B defined as follows (we use notation from [5, Section 5, p. 78]):

$$A_1 = \{(u, 1) : \arg(u) \in [0, \frac{\pi}{2}]\}, \quad A_2 = \{(u, 1) : \arg(u) \in [\frac{3\pi}{2}, 2\pi]\},$$

$$A_3 = \{(1, u) : \arg(u) \in [0, \frac{\pi}{2}]\}, \quad A_4 = \{(1, u) : \arg(u) \in [\frac{3\pi}{2}, 2\pi]\}.$$

Denote $S = A_1 \cup A_2 \cup A_3 \cup A_4 \subset B$ and observe that $f^2(A_i) = B$ for each $i \in \{1, 2, 3, 4\}$. Thus if $m > n + 1$, then $\pi_m(K)$ is a subset of S (since it is a continuum containing e , the image of which under f^2 is not the whole B). Consider the set π_{m+1} . It has to be contained in the component of $f^{-1}(S)$ that contains the point p . The component is the union of four arcs

$$A'_1 = \{(u, 1) : \arg(u) \in [0, \frac{\pi}{8}]\}, \quad A'_2 = \{(u, 1) : \arg(u) \in [(2 - \frac{1}{8})\pi, 2\pi]\},$$

$$A'_3 = \{(1, u) : \arg(u) \in [0, \frac{\pi}{16}]\}, \quad A'_4 = \{(1, u) : \arg(u) \in [(2 - \frac{1}{16})\pi, 2\pi]\}.$$

By the definition of f we have

$$f(A'_1) = f(A'_3) = A_1 \quad \text{and} \quad f(A'_2) = f(A'_4) = A_3.$$

This means that $\pi_m(K)$ is contained in $A_1 \cup A_3$. Since this is true for all $m > n + 1$, hence $\pi_{m+1}(K) \subset (A_1 \cup A_3) \cap (A'_1 \cup A'_2 \cup A'_3 \cup A'_4) = A'_1 \cup A'_3$. But $f(A'_1 \cup A'_3) = A_1$, thus $\pi_m(K) \subset A_1$ for each $m > n + 1$. Therefore K is the inverse limit of arcs $\pi_m(K) \subset A_1$ with homeomorphisms as the bonding mappings. Since the point p is an end point of each of these arcs, it follows that $e = (p, p, p, \dots)$ is an end point of K . Thus e is an end point of M as needed, and Claim 2 is proved.

It is stated in [24, Theorem 3, p. 380] that each continuum has the property of Kelley at any of its end points. Thus Claim 2 implies the next one.

CLAIM 3. The continuum M has the property of Kelley at e .

CLAIM 4. The continuum M has the local property of Kelley at each point distinct from e .

Really, let $x = (x_1, x_2, x_3, \dots) \in M$ with $x \neq e$. Then there is $n \in \mathbb{N}$ such that $x_n \neq p$. Choose an arc L_n in $X_n = B$ with $x_n \in L_n$. Then $\pi_n^{-1}(L_n)$ is a neighborhood of x in M and this neighborhood is homeomorphic to the product of the Cantor set and an arc. Thus M has the local property of Kelley at x , and Claim 4 is shown.

It follows from Claims 3 and 4 that M has the local property of Kelley at each of its points. Thus Theorem 4.8 implies that M has the property of Kelley, and thus it has the arc property of Kelley.

Finally, M is neither circle-like nor tree-like (hence nor arc-like), see [5, Section 5, Q6 and Q9, p. 81 and 82 respectively]. The argument is complete.

Example 4.11. *Ingram continuum.*

The Ingram continuum Y defined in [22], is tree-like by its definition, it is atriodic (see [22, Theorem 1, p. 100]), it has the property of Kelley as it is shown in [23, Section 4, p. 355], and it has the arc approximation property (thus it has the arc property of Kelley according to Proposition 2.2) because each nondegenerate proper subcontinuum of it is an arc. This last fact, that is stated in [23, the third paragraph on p. 353], can be explained as follows. Let X be the continuum constructed in [35, Example 4.2, p. 1034] (compare also [23, p. 353]), and let $\Phi : X \rightarrow Y$ be the confluent surjective mapping (see [23, Section 2, p. 352, and Theorem 4.1, p. 355]). For any nondegenerate proper subcontinuum Q of Y let L be a component of $\Phi^{-1}(Q)$. Thus L is a nondegenerate proper subcontinuum of X , so it is an arc by [35, Theorem 4.3, p. 1037]. Since the partial mapping $\Phi|_L : L \rightarrow Q$ is a confluent surjection (see [6, I, p. 213]) and since a nondegenerate confluent image of an arc is an arc by [7, Corollary 20, p. 32], it follows that Q is an arc. Finally, Y is not arc-like by [22, Theorem 3, p. 106]; since each tree-like and circle-like continuum is arc-like, it follows that Y is not circle-like.

5. ATRIODIC ABSOLUTE RETRACTS FOR HEREDITARILY UNICOHERENT CONTINUA

In this section we investigate properties of those members of $\text{AR}(\mathcal{HU})$ (and of their subsets: $\text{AR}(\lambda\mathcal{D})$ and $\text{AR}(\mathcal{TL})$) that contain no simple triods (equivalently, that are atriodic, see Corollary 5.1). The obtained results concern global structure of such continua. In particular we show that any circle-like member of $\text{AR}(\mathcal{HU})$ has to be arc-like.

As a consequence of Theorems 3.4 and 2.3 and Proposition 2.2 we get the following corollary.

Corollary 5.1. *A member of $\text{AR}(\mathcal{HU})$ is atriodic if and only if it contains no simple triod.*

The next result is related to Example 4.11.

Theorem 5.2. *Let a continuum X in $\text{AR}(\mathcal{TL})$ contain no simple triod. Then X is arc-like.*

PROOF. By [11, Corollary 5.5] for each $\varepsilon > 0$ the continuum X admits an ε -mapping to a tree contained in X . Such a tree must be an arc by the assumption, which ends the proof. \square

Remark 5.3. The Ingram curve (see Example 4.11) is a tree-like continuum with the arc property of Kelley that is not in $\text{AR}(\mathcal{HU})$, according to Theorem 5.2. It is the only example (known to the authors) with these properties.

Any continuum in $\text{AR}(\lambda\mathcal{D})$ has the arc approximation property by Theorem 2.3 and Proposition 2.2. Applying Corollaries 3.5 and 3.6 we have the following.

Corollary 5.4. *Let a nondegenerate continuum X in $\text{AR}(\lambda\mathcal{D})$ contain no simple triod. Then X is an arc.*

The following question is related to the above results.

Question 5.5. Let a continuum X in $\text{AR}(\mathcal{HU})$ contain no simple triod. Is then X an arc-like continuum?

The rest of this section is devoted to a structure of those members of $\text{AR}(\mathcal{HU})$ which are circle-like. All known examples of members of $\text{AR}(\mathcal{HU})$ are tree-like continua. In [10, Theorem 3.6] the authors show that the inverse limit of trees with confluent bonding mappings is in $\text{AR}(\mathcal{HU})$. We do not know if tree-likeness is necessary for being in $\text{AR}(\mathcal{HU})$. In the next theorem, however, we will prove this property for circle-like continua in $\text{AR}(\mathcal{HU})$. To show the result we will need the following construction.

Construction 5.6. Let X be a circle-like, not arc-like continuum, and let S be an indecomposable solenoid. Represent X as the inverse limit of an inverse sequence of circles C_n with essential bonding mappings $f_n : C_{n+1} \rightarrow C_n$. Thus for each $n \in \mathbb{N}$ the projections $\pi_n : X \rightarrow C_n$ are essential, too. Embed X and the circles C_n in \mathbb{R}^3 in such a way that the projections π_n from X onto C_n are $\frac{1}{n}$ -translations in \mathbb{R}^3 , i.e., if $x = \{x_1, x_2, \dots\} \in X$, then $d(x, x_n) < \frac{1}{n}$. We may moreover assume that $C_m \cap C_n = \emptyset$ for $m \neq n$. For each $n \in \mathbb{N}$ let T_n be a solid torus which is a small closed neighborhood of C_n , such that the following conditions are satisfied.

(5.6.1) $T_m \cap T_n = \emptyset$ for $m \neq n$;

(5.6.2) $\text{Lim } T_n = \text{Lim } C_n = X$;

(5.6.3) for each $n \in \mathbb{N}$ there are retractions $g_n : T_n \rightarrow C_n$ satisfying $d(x, g_n(x)) < \frac{1}{n}$ for $x \in T_n$.

Let S_n be a copy of S in T_n such that $g_n|_{S_n} : S_n \rightarrow C_n$ is topologically equivalent to the projection of the solenoid S_n onto its first factor in the inverse limit representation. For each $n \in \mathbb{N}$ we join the pair S_n, S_{n+1} by a spiral A_n , i.e., A_n is a homeomorphic copy of the real line satisfying the following conditions.

$$(5.6.4) \quad \text{cl } A_n \setminus A_n = S_n \cup S_{n+1};$$

$$(5.6.5) \quad \text{Lim } A_n = \text{Lim } C_n = \text{Lim } S_n = X;$$

$$(5.6.6) \quad A_n \cap (X \cup S_m \cup A_k) = \emptyset \quad \text{for any } m \text{ and for any } k \neq n.$$

Put

$$(5.6.7) \quad Y(X, S) = X \cup \bigcup \{S_n \cup A_n : n \in \mathbb{N}\},$$

and observe that $Y(X, S)$ is a hereditarily unicoherent continuum.

Lemma 5.7. *Let X be a circle-like, not arc-like continuum, S be an indecomposable solenoid, and $Y(X, S)$ be defined by (5.6.7). If $r : Y(X, S) \rightarrow X$ is a retraction, then $r(S_n) = X$ for all but finitely many indices $n \in \mathbb{N}$.*

PROOF. Keeping all the notation of Construction 5.6 and letting $f_1^n = f_1 \circ \dots \circ f_{n-1} : C_n \rightarrow C_1$ for each $n > 1$, we see that for each $n \in \mathbb{N}$ and each $x \in S_n$ we have $d(g_n(x), \pi_n(r(x))) < \frac{3}{n}$. Therefore for each $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}$ such that for each $n > n_0$ the mappings

$$\varphi_n = f_1^n \circ \pi_n \circ (r|_{S_n}) \quad \text{and} \quad \psi_n = f_1^n \circ (g_n|_{S_n})$$

that map S_n onto C_1 , are ε -close to each other. By [20, Chapter IV, Theorem 1.1, p. 111] (see also [28, Chapter 5, Theorem 5.1.1, p. 191]) for sufficiently large n the mappings φ_n and ψ_n are homotopic, and since ψ_n is essential as the composition of two essential mappings, φ_n is essential, too.

If $r(S_n)$ is a proper subcontinuum of X , then it is arc-like, and thus $\pi_n|_{r(S_n)}$ is inessential, so the whole φ_n also is inessential. This contradiction finishes the proof. \square

Corollary 5.8. *No solenoid is in $\text{AR}(\mathcal{HU})$.*

PROOF. Suppose X is a solenoid, and $X \in \text{AR}(\mathcal{HU})$. By [13, Theorem 8, p. 238] there is a solenoid S which admits no surjection onto X . Since $X \in \text{AR}(\mathcal{HU})$, there is a retraction $r : Y(X, S) \rightarrow X$, where $Y(X, S)$ is as in Construction 5.6, contrary to Lemma 5.7. \square

Theorem 5.9. *If a circle-like continuum X is in $\text{AR}(\mathcal{HU})$, then X is arc-like.*

PROOF. Let X be a circle-like continuum and $X \in \text{AR}(\mathcal{HU})$. By Theorem 2.3 the continuum X has the property of Kelley; in particular, every point of X belongs

to a nondegenerate arc. A theorem of P. Krupski [24, Theorem 1, p. 379] says that every circle-like continuum with the property of Kelley and such that each of its points lies on an arc as not an end point of the arc is a solenoid. Since X is not a solenoid by Corollary 5.8, there is a point $e \in X$ such that e is an end point of every arc containing e . Let S be the dyadic solenoid. Assuming that the continuum X is not arc-like, consider again the continuum $Y(X, S)$ defined by (5.6.7) and a retraction $r : Y(X, S) \rightarrow X$. By Lemma 5.7 there is an index $n_0 \in \mathbb{N}$ such that for each $n > n_0$ the image $r(S_n)$ is the whole continuum X . Since X contains no triod (compare e.g. [21, 39.7, p. 260]), the arc component of X containing the point e is a one-to-one image of the half line $[0, \infty)$. Let $h_1 : [0, \infty) \rightarrow X$ be the one-to-one mapping such that $h_1(0) = e$. Observe that

(5.9.1) for each arc $A \subset h_1([0, \infty))$ the function $h_1^{-1}|_A : A \rightarrow [0, \infty)$ is continuous.

Fix $n > n_0$. Let K be the arc component of S_n such that $e \in r(K)$. Then there is one-to-one mapping $h_2 : \mathbb{R} \rightarrow K$ such that

(5.9.2) $r(h_2(0)) = e$;

(5.9.3) for each $x \in \mathbb{R}$ the composition $g_n \circ (h_2|[x, x+2\pi))$ is a one-to-one mapping of $[x, x+2\pi)$ onto C_n ;

(in particular, $g_n(h_2(x)) = g_n(h_2(x+2\pi))$). Put $\alpha = h_1^{-1} \circ r \circ h_2 : \mathbb{R} \rightarrow [0, \infty)$. Observe that, by (5.9.1), α is continuous, and by (5.9.2), $\alpha(0) = 0$. Define $\beta : \mathbb{R} \rightarrow \mathbb{R}$ by $\beta(x) = \alpha(x+2\pi) - \alpha(x)$ for each $x \in \mathbb{R}$. Since $\beta(-2\pi) \leq 0$ and $\beta(0) \geq 0$, there is $x_0 \in [-2\pi, 0]$ such that $\beta(x_0) = 0$, i.e., $\alpha(x_0) = \alpha(x_0+2\pi)$. Since h_1 is one-to-one, it follows that

(5.9.4) $r(h_2(x_0)) = r(h_2(x_0+2\pi))$

and, by (5.9.3), $g_n|_{h_2([x_0, x_0+2\pi))}$ is one-to-one. Let $P = h_2([x_0, x_0+2\pi])$. Notice that P is an arc. Again by (5.9.3), the mapping $g_n|_P : P \rightarrow C_n$ identifies the two end points of P only. Define $s_n : C_n \rightarrow X$ by $s_n(c) = r((g_n|_P)^{-1}(c))$ for each $c \in C_n$, and observe that, by (5.9.1) and (5.9.4), s_n is a well defined mapping.

Note that, if n tends to infinity, the sequence of mappings $s_n \circ \pi_n : X \rightarrow X$ converges to the identity on X (both π_n and g_n are $\frac{1}{n}$ -translations), so $\pi_1 \circ s_n \circ \pi_n$ converges to the essential mapping $\pi_1 : X \rightarrow C_1$. Thus, for large n , the mappings $\pi_1 \circ s_n \circ \pi_n : X \rightarrow C_1$ are essential, too. On the other hand, for each such $n \in \mathbb{N}$, the set $s_n(C_n)$ is a subcontinuum of $h_1([0, \infty))$, so it is an arc, and thus the mappings $\pi_1 \circ s_n \circ \pi_n$ are inessential. This contradiction completes the proof. \square

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