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An open Whitney map for the hyperspace of a circle

W. J. Charatonik

All spaces considered in this paper are assumed to be metric. For a given continuum X we call its hyperspace the space 2^X of all its nonempty compact subsets equipped with the Hausdorff metric. A Whitney map is a map $w: 2^X \to [0,w(X)]$ satisfying 1) $w(\{x\}) = 0$ for all $x \in X$, and 2) if $A, B \in 2^X$ and A is a proper subset of B, then $w(A) < w(B)$.

Nadler has asked in [4], (14.69), p. 472 whether there is an open Whitney map for the hyperspace of a circle. In this paper we construct such a Whitney map. As far as I know, it is the first example of a noncontractible continuum with an open Whitney map. Goodykoontz and Nadler have proved in [3], (2.12), p. 677 that admissible Whitney maps (see [3], (2.1), p. 674 for the definition) are open, but all known examples of continua having hyperspaces with admissible Whitney maps are contractible. Here we construct an open Whitney map which is not admissible (since the circle does not have an admissible Whitney map, see [3], (2.5), p. 675).

THEOREM 1. Let S be a circle. There is an open Whitney map for 2^S.

Proof. We can assume, for convenience, that S is the unit circle in the plane, and denote by d an arc-length metric on S. Let $\{x_0, x_1, \ldots\}$ be a dense subset of S and let $A \in 2^S$. We put $f_n(A) = \operatorname{diam}\{d(x_n,x): x \in A\}$. Define

$$w(A) = \sum_{n=0}^{\infty} 3^{-n} f_n(A) . \qquad (*)$$

This paper is in final form and will not be submitted for publication elsewhere.

One can easily verify that w is a Whitney map. We show w is open. To this end take an arbitrary $A \in 2^S$ and an open subset $\underline{U} \subset 2^S$ with $A \in \underline{U}$. We have to show that w(A) is in the interior of $w(\underline{U})$. If A is one-point or infinite, the conclusion follows from [4], (14.65), p. 469. So assume A is finite but not one-point. We shall construct continua $\underline{K}$ and $\underline{L}$ with $A \in \underline{K} \cap \underline{L}$ and $\underline{K} \cup \underline{L} \subset \underline{U}$ and such that there are numbers u_1 and u_2 with $w(\underline{L}) = [u_1, w(A)]$ and $w(\underline{K}) = [w(A), u_2]$. Note that the existence of such continua ends the proof. Denote by B(A,t) the closed ball in S of radius t about the set A, and put $\underline{K} = \{B(A,t): t \in [0,\varepsilon]\}$, where $\varepsilon > 0$ is such that $\underline{K} \subset \underline{U}$. Obviously $w(\underline{K})$ is of the form $[w(A), u_2]$. To construct the continuum $\underline{L}$ consider two cases.

Case 1: $f_0(A) \neq 0$. There are at most two points of A — denote them by a_1 and a_2 — such that $d(x_0, a_1) = d(x_0, a_2)$ is the maximum of all distances $d(x_0, a)$ for all $a \in A$ (if there is only one such a point we put $a_1 = a_2$). By the assumption $f_0(A) \neq 0$ there are other points in A. Define, for t small enough, $a_i(t)$ as that point of S lying near a_i and satisfying $d(x_0, a_i(t)) = d(x_0, a_i) - t$, and let $A(t) = A \cup \{a_1(t), a_2(t)\} \setminus \{a_1, a_2\}$. Define $\underline{L} = \{A(t): t \in [0,\varepsilon]\}$, where ε is such a small number that $\underline{L} \subset \underline{U}$ and $d(x_0, a_i(\varepsilon))$ is greater than $\max\{d(x_0,a): a \in A \setminus \{a_1,a_2\}\}$. Observe that $f_0(A(t)) = f_0(A) - t$ and $f_n(A(t)) \leq f_n(A) + t$ for $n \in \{1,2,\ldots\}$, and hence

$$w(A(t)) = f_0(A(t)) + \sum_{n=1}^{\infty} 3^{-n} f_n(A(t))$$

$$\leq f_0(A) - t + \sum_{n=1}^{\infty} 3^{-n} f_n(A) + t \cdot \sum_{n=1}^{\infty} 3^{-n}$$

$$= w(A) - t + t/2.$$

So, for $t > 0$, we get $w(A(t)) < w(A)$ and therefore $w(\underline{L})$ is really of the form $[u_1, w(A)]$.

Case 2: $f_0(A) = 0$. Now, A consists of two points, $A = \{a,b\}$, with $d(x_0,a) = d(x_0,b)$. Let J be the shorter of two arcs joining a and b (if a and b are antipodal, we choose the arc arbitrarily). Define, for small t, a(t) and b(t) as such points of J which satisfy $d(a,a(t)) = t$ and $d(b,b(t)) = t$, and put $A(t) = \{a(t), b(t)\}$.

We show that for $t > 0$ the inequality $w(A(t)) < w(A)$ holds. Really, if x_n is such a point that the straight line containing x_n and the point $\langle 0,0 \rangle$ separates a(t) and b(t), then we have $f_n(A(t)) = f_n(A)$; in the opposite case we have $f_n(A(t)) < f_n(A)$.

Since $\{x_0, x_1, \ldots\}$ is dense, there are points for which the inequality holds.

So choose $\varepsilon > 0$ in such a way that $A(t) \in \underline{U}$ for $t \leq \varepsilon$. Then $\underline{L} = \{A(t): t \in [0, \varepsilon]\}$ is the required continuum, and the proof is complete.

Recall that a map $f: X \to Y$ from a continuum X onto Y is

- monotone, if $f^{-1}(y)$ is connected for every $y \in Y$;
- confluent, if for every subcontinuum $K \subset Y$ and every component C of $f^{-1}(K)$ we have $f(C) = K$.

Each monotone and each open map is confluent.

One can ask whether the Whitney map w is also monotone. We answer this question in the affirmative by showing a more general result. Its proof in the present short form is due to Janusz R. Prajs.

THEOREM 2. Each confluent Whitney map is monotone.

Proof (Prajs). Let X be a continuum, let $w: 2^X \to [0, w(X)]$ be a confluent Whitney map and let $t \in [0, w(X)]$. We have to show that $w^{-1}(t)$ is connected. By the assumption each component of $w^{-1}([0,t])$ is mapped under w onto $[0,t]$, but since $w^{-1}(0)$ is connected (it is homeomorphic to X) we see that $w^{-1}([0,t])$ is connected. Now $w^{-1}(t)$ is the intersection of two continua $w^{-1}([0,t]) \cap w^{-1}([t,w(X)])$ and it is connected by the unicoherence of 2^X (see [4], (1.176), p. 178).

Remark. One may expect that (*) defines an open Whitney map for an arbitrary locally connected continuum, where the metric d is a convex one. This is not true, as the following example shows.

Let a continuum X on the plane be defined as

$$X = \{\langle x,y\rangle: x^2 + y^2 = 1\} \cup \{\langle x,y\rangle: x = y \text{ and } x^2 + y^2 \leq 1\}.$$

Put $x_0 = \langle 0,1\rangle$, $x_1 = \langle 0,0\rangle$, $a = \langle 1,0\rangle$ and $b = \langle -1,0\rangle$ and choose $x_2, x_3, \ldots$ as arbitrary points of X which form a dense subset. Let d be the arc-length metric on X. Define by (*) a Whitney map $w: 2^X \to [0, w(X)]$. Put $\underline{U} = \{A \in 2^X: \text{dist}(A, \{a,b\}) < \pi/4\}$ (here dist denotes the Hausdorff distance). We show $w(\underline{U})$ is not open. Really, we have $f_0(\{a,b\}) = f_1(\{a,b\}) = 0$ and $w(\{a,b\}) = \sum_{n=2}^{\infty} 3^{-n} f_n(\{a,b\})$. Take $A \in \underline{U}$ with $A \neq \{a,b\}$ and choose $a' \in A$ and $b' \in A$ with $d(a,a') < \pi/4$, $d(b,b') < \pi/4$ and $a \neq a'$ or $b \neq b'$. We get $f_0(\{a',b'\}) \neq 0$ or $f_1(\{a',b'\}) \neq 0$, and arguing as in the proof of Theorem 1 we see that $f_0(\{a',b'\}) + 3^{-1}f_1(\{a',b'\}) > \sum_{n=2}^{\infty} 3^{-n}f_n(\{a,b\}) - \sum_{n=2}^{\infty} 3^{-n}f_n(\{a',b'\})$, whence $w(A) \geq w(\{a',b'\}) > w(\{a,b\})$ and therefore $w(\{a,b\})$ is not an interior point of $w(\underline{U})$.

P r o b l e m. Has every locally connected continuum X an open Whitney map for 2^X ?

Examples are known of continua without confluent Whitney maps even (see [1] and Theorem 24 of [2]) but all these examples are not locally connected.

R e f e r e n c e s

[1] W. J. Charatonik, A continuum X which has no confluent Whitney map for 2^X, Proc. Amer. Math. Soc. 92 (1984), 313-314.

[2] W. J. Charatonik, R^i-continua and hyperspaces, Topology Appl. (to appear).

[3] J. T. Goodykoontz, Jr. and S. B. Nadler, Jr., Whitney levels in hyperspaces of certain Peano continua, Trans. Amer. Math. Soc. 274 (1982), 671-694.

[4] S. B. Nadler, Jr., Hyperspaces of sets, Dekker, New York and Basel, 1978.

Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Added in proof. A. Illanes (Proc. Amer. Math. Soc. 98(1986), 516-518) has recently solved the above problem in the affirmative.