PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

A CONTINUUM X WHICH HAS NO CONFLUENT WHITNEY MAP FOR 2^X

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ABSTRACT. An example is shown of a continuum X which has no confluent Whitney map for 2^X. This answers two problems asked by Nadler [N].

Nadler has asked in [N, (14.63) and (14.64), pp. 468 and 469] whether for every continuum X there exists a monotone (or an open) Whitney map for 2^X. We answer both these questions in the negative by showing an example of a continuum X which has no confluent Whitney map for 2^X.

All considered spaces are assumed to be metric and all mappings are continuous. We denote by R the space of reals and by H the half-line $[1, +\infty)$. The symbol $\operatorname{diam}(A)$ stands for the diameter of a set A. Given a continuum X we use small letters for points of X, capitals to designate subsets of X, and script letters for subsets of 2^X. A mapping $f\colon X \to Y$ of a continuum X onto Y is said to be confluent provided that for each subcontinuum K of Y and each component A of $f^{-1}(K)$ we have $f(A) = K$. It is well known that monotone mappings and open mappings between continua are confluent (cf. [N, (0.45.3), p. 21], where further information is given).

THEOREM. *There exists a rational continuum X in the plane which admits no confluent Whitney map for* 2^X.

PROOF. Let S denote the unit circle in R^2. Define functions f and g mapping H into R^2 by

$$f(t) = (1 + 1/t)\exp(it) \quad \text{and} \quad g(t) = (1 - 1/t)\exp(-it),$$

and put $M = f(H)$ and $L = g(H)$. The space $X = M \cup S \cup L$ is a rational continuum in R^2 (cf. [N, (16.35), p. 558]). We show there is no confluent Whitney map from 2^X into $[0, +\infty)$.

Let μ be an arbitrary Whitney map for 2^X. Put $\mathscr{B} = \{B \in 2^X\colon \operatorname{diam}(B) \geqslant 1\}$. Thus $\mathscr{B}$ is a compact subset of 2^X. Let $t_0 = \inf\{\mu(B)\colon B \in \mathscr{B}\}$ and note $t_0 > 0$. Consider the segment $[0, t_0/2] \subset [0, \mu(X))$. We show that there exists a component of $\mu^{-1}([0, t_0/2])$ whose image under μ is not the whole segment. To this end put $p_k = f(2\pi k) \in M$ and $q_k = g(2\pi k) \in L$ for $k \in \{1, 2, \dots\}$. Observe that the sets $\{p_k, q_k\}$ tend to the one-point set $\{1\} \subset S$, so $\mu(\{p_k, q_k\})$ tends to zero as k tends

Received by the editors November 21, 1983.
1980 *Mathematics Subject Classification*. Primary 54B20; Secondary 54C10, 54F20.
Key words and phrases. Confluent, continuum, hyperspace, Whitney map.

to infinity. Hence there exists a number s such that $\mu(\{p_s, q_s\}) \in [0, t_0/2]$. Let $\mathscr{C}$ be the component of $\mu^{-1}([0, t_0/2])$ which contains the point $\{p_s, q_s\}$ of 2^X, and assume on the contrary that $\mu(\mathscr{C}) = [0, t_0/2]$. Put $\mathscr{T} = \{A \in 2^X\colon A \cap M \neq \varnothing \neq A \cap L\}$. Note $\{p_s, q_s\} \in \mathscr{C} \cap \mathscr{T}$ and $\mathscr{C} \setminus \mathscr{T}$ is nonempty (because it contains a one-point set according to the assumption). Denote by $\mathscr{L}$ the component of $\mathscr{C} \cap \mathscr{T}$ containing the point $\{p_s, q_s\}$. Define mappings $m\colon \mathscr{T} \to H$ and $l\colon \mathscr{T} \to H$ putting $m(A) = \min f^{-1}(A)$ and $l(A) = \min g^{-1}(A)$, and note $m(\{p_s, q_s\}) = l(\{p_s, q_s\}) = 2\pi s$. Observe that $\mathscr{L}$ has a limit point in $\operatorname{bd}(\mathscr{C} \cap \mathscr{T})$, i.e., there exists a sequence of points A_n of $\mathscr{L}$ such that $m(A_n) \to +\infty$ or $l(A_n) \to +\infty$ as $n \to +\infty$. Consider a function $\arg\colon \mathscr{L} \to R$ defined by $\arg(A) = m(A) + l(A) - 4\pi s$. Thus arg is a continuous function, and $\arg(\{p_s, q_s\}) = 0$. Note that $\arg(A_n) \to +\infty$ as $n \to +\infty$, so the image of $\mathscr{L}$ under the function arg is an unbounded from the right and connected set in R containing the point 0. Thus there exists a set A_0 in $\mathscr{L}$ such that $\arg(A_0) = \pi$. Denote $a = m(A_0)$ and $b = l(A_0)$. Putting $x = f(a)$ and $y = g(b)$ we have $x, y \in A_0$. Calculate

$$\rho(x, y) = \left[(1 + 1/a)^2 + (1 - 1/b)^2 - 2(1 + 1/a)(1 - 1/b) \cdot \cos(a + b)\right]^{1/2},$$

where ρ is the euclidean metric in the plane. Now $\arg(A_0) = \pi$ implies $a + b = 4\pi s + \pi$, whence $\cos(a + b) = -1$, so all terms in the square brackets are positive, and therefore $\rho(x, y) > 1$. Hence $\operatorname{diam}(A_0) > 1$, thus $A_0 \in \mathscr{B}$, and therefore $\mu(A_0) \geqslant t_0$, a contradiction to the fact $A_0 \in \mathscr{L} \subset \mathscr{C} \subset \mu^{-1}([0, t_0/2])$. The proof is complete.

COROLLARY. *There is neither a monotone nor an open Whitney map for 2^X, where X is the continuum described above.*

REMARKS. Note the following. (1) Each Whitney map for the hyperspace 2^X of an arbitrary continuum X is weakly confluent (see [N, (0.45.4), p. 22] for the definition) as a mapping onto a segment. (2) Each Whitney map for the hyperspace $C(X)$ of all subcontinua of an arbitrary continuum X is monotone and open (see [N, (14.44), p. 453]).

REFERENCES

[N] Sam B. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York and Basel, 1978.

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