

CONTINUA HOMEOMORPHIC TO ALL THEIR CONFLUENT IMAGES

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Abstract. Continua homeomorphic to all their confluent images are investigated. It is known that if such a continuum is locally connected then it is the arc (and conversely) and that if such continuum is hereditarily indecomposable then it is the pseudo-arc (the converse is not known). Some questions are asked.

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A continuum means a compact connected metric space. A map is a continuous transformation. A map $f : X \rightarrow Y$ from a continuum X onto Y is called confluent (see [3], p. 213) if for every subcontinuum K of Y and every component C of $f^{-1}(K)$ we have $f(C) = K$. All open and all monotone maps are confluent (see [3], V and VI, p. 214).

In this note we are interested in continua such that all their confluent images are homeomorphic. We write $X \in (*)$ if X is such. It is known that an arc ([4], Corollary 20, p. 32) and the Lelek fan ([6], Proposition 2) have property $(*)$. Other examples of continua with $(*)$ are not known till now, however there is a conjecture that the pseudo-arc is such.

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Recall that a surjective map $f : X \rightarrow Y$ between continua X and Y is called weakly confluent if for every subcontinuum K of Y there is a component C of $f^{-1}(K)$ with $f(C) = K$ (see [8], p. 98). By definition each confluent map is weakly confluent.

The proposition below justifies using confluent maps in (*) by showing that extending the class of maps to weakly confluent ones leads to the empty family of continua with the considered property.

1. Proposition. There is no nondegenerate continuum homeomorphic to all its weakly confluent images.

Proof. Each nondegenerate continuum X may be mapped onto an arc, and since each map onto an arc is weakly confluent (see [11], Lemma, p. 236), an arc is a weakly confluent image of any continuum. Since the double winding map $f : [0, 1] \rightarrow S^1$ from the interval onto the unit circle defined by $f(x) = e^{4\pi i x}$ is weakly confluent, we can see the circle is a weakly confluent image of any continuum, too, and the conclusion follows.

Now we discuss locally connected continua with property (*). We show that every such a continuum must be an arc. To this end we need the following proposition.

2. Proposition. Every locally connected continuum can be mapped onto an arc under a confluent map.

Proof. Let X be a locally connected continuum. Then by Bing-Moise theorem (see [2], Theorem 8, p. 1109 and [9], Theorem 4, p. 1119) X has a convex metric d . Moreover we can assume $\text{diam } X = 1$. Let $a, b \in X$ be two points such that $d(a, b) = \text{diam } X = 1$. Define a map $f : X \rightarrow [-1, 1]$ by $f(x) = d(x, a) - d(x, b)$. We show the map is confluent. Let $[y_1, y_2]$ be an arbitrary subinterval of the range $[-1, 1]$ and let $x \in X$ be such that $f(x) \in [y_1, y_2]$. We have to show that there is a continuum C with $x \in C$ and $f(C) = [y_1, y_2]$. Let I_1 be a convex arc in X with end-points a and x .

We show the restriction $f|_{I_1}$ is a monotone map of I_1 onto $[-1, f(x)]$. Really, let \leq be a natural order on I_1 from a to x . Take $z_1, z_2 \in I_1$ with $z_1 \leq z_2$. Then, by the convexity of the metric d , we have $f(z_2) = d(z_2, a) - d(z_2, b) \geq d(z_1, a) + d(z_2, z_1) - [d(z_1, b) + d(z_2, z_1)] = f(z_1)$.

Hence, we can take a point x_1 in the arc I_1 with $f(x_1) = y_1$ and such that

the image of the arc J_1 from x_1 to x is $[y_1, f(x)]$. Similarly, we can find a convex arc J_2 from x to x_2 with $f(x_2) = y_2$ and such that $f(J_2) = [f(x), y_2]$. The union $J_1 \cup J_2$ is the needed continuum and the proof is complete.

3. Theorem. The arc is the only locally connected continuum with property (*).

Proof. Really, the arc has property (*) as it was established in [4], Corollary 20, p. 32, and the uniqueness follows from Proposition 2.

4. Remark. In [5], p. 27 a result is shown due to T. Maćkowiak saying that the only locally connected continua homeomorphic to all their monotone images are the arc and the circle. Since the circle can be openly mapped onto the arc, Theorem 3 is a consequence of that Maćkowiak's result. However, the proof presented here seems to be much simpler than that in [5].

Now we shall go to the opposite extreme in the structure of continua. Recall that a continuum is said to be hereditarily indecomposable if for every two its subcontinua A and B with $A \cap B \neq \emptyset$ we have $A \subset B$ or $B \subset A$. By the pseudo-arc we mean a hereditarily indecomposable arc-like continuum.

5. Proposition. If there is a continuum X which is hereditarily indecomposable and has property (*), then X is the pseudo-arc.

Proof. Really, by Theorem 1 and 2 of [1], p. 10 and 11, every hereditarily indecomposable continuum can be mapped onto the pseudo-arc, and since every map onto any hereditarily indecomposable continuum (in particular onto the pseudo-arc) is confluent (see [7], Theorem 4, p. 243) the conclusion follows.

6. Remark. It is not known whether the pseudo-arc has property (*).

As the reader can see from Theorem 3 and Proposition 5, the rest of continua having property (*) is situated somewhere "between" locally connected and hereditarily indecomposable ones. We pass now to a discussion about such continua.

Recall some definitions. A continuum is said to be hereditarily unicoherent if the intersection of any two its subcontinua is connected. A dendroid is defined as a hereditarily unicoherent and arcwise connected continuum. A dendroid with only one ramification point (a common point of three arcs disjoint out of it) is called a fan. A dendroid X is smooth at a point $p \in X$ if for every $x \in X$ and every

sequence $\{x_n\}$ tending to x the arcs from x_n to p tend to the arc from x to p . The Lelek fan means a smooth fan with a dense set of end-points. Recall the following proposition.

7. Proposition. ([6], Corollary). The Lelek fan is the only smooth fan with property (*).

We end the paper asking some questions.

8. Question. Is the Lelek fan the only fan (dendroid, smooth dendroid, hereditarily decomposable continuum, respectively) having property (*)?

9. Question. Is the arc the only hereditarily decomposable continuum with the property of Kelley (see e.g. [10], (16.10), p. 538 for the definition) having the property (*)?

10. Question. What are other continua with property (*)?

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