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GENERAL TOPOLOGY

Convex Structure on the Space of Order Arcs

by

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Summary. A new convex structure (definition due to D. W. Curtis) is constructed on the space of all maximal order arcs.

1. Introduction. The first construction of a convex structure on the space of order arcs was given by Curtis [1] where it was used to investigate contractability of hyperspaces. Then the convex structure was used in [2] to prove that the subset of a Whitney level composed of all those sets having a given point in common is an ANR (see [2, Theorem]).

In this paper we construct a convex structure on the space of all maximal order arcs which is simpler then the one in [1]. This construction has all properties needed in the proof of Curtis' Theorem on contractability of hyperspaces [1, (5.4)]. It has also some additional properties — for example it is symmetric (see 2.5 bellow).

2. Results. Let us start with recalling the definition of a convex structure.

DEFINITION 2.1 [1, (2.1)]. Let, for every natural number n, $P_n = \{(t_1, t_2, \dots, t_n) \in [0,1]^n : t_1 + t_2 + \dots + t_n = 1\}$ and let (Y, d) be any metric space. A convex structure in (Y, d) is a sequence of sets $M_n \subset Y^n$ and mappings $k_n : M_n \times P_n \to Y$ satisfying the following conditions:

(1) $k_n((y, \dots, y), (t_1, \dots, t_n)) = y$;

(2) $k_n((y_1, \dots, y_n), (t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)) = k_{n-1}((y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n), (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n))$;

(3) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall n \, \forall (t_1, \dots, t_n) \in P_n \forall (y_1, \dots, y_n) \in M_n \, \forall (z_1, \dots, z_n) \in M_n : d(y_i, z_i) < \delta \Rightarrow d(k_n((y_1, \dots, y_n), (t_1, \dots, t_n)), k_n((z_1, \dots, z_n), (t_1, \dots, t_n))) < \varepsilon$.

Moreover, a set $C \subset Y$ is said to be *convex* if for every natural n we have $C^n \subset M_n$ and $k_n(C^n \times P_n) \subset C$.

REMARK 2.2. Observe that if Y is a compact space and d and d' are equivalent metrics on Y, then every convex structure in (Y, d) is a convex structure in (Y, d'). This is a consequence of the uniform continuity of the identity function. Thus, in this case, we can consider a convex structure in Y without specyfying the metric.

We will use the following notations and definitions. The letter X denotes a metric continuum, i.e. a compact, connected metric space. The hyperspace of all subcontinua of X is denoted by $C(X)$, and $C(C(X))$ is denoted by $C^2(X)$. By a Whitney map we understand a mapping $\omega : C(X) \to [0, \infty)$ satisfying the following two conditions:

(i) $\omega(\{x\}) = 0$ for every $x \in X$ and

(ii) if A is a proper subcontinuum of B, then $\omega(A) < \omega(B)$.

An order arc in $C(X)$ is an arc in $C(X)$ which is ordered by the inclusion. The symbol $\Lambda(X)$ stands for the space of all maximal order arcs, i.e. $\Lambda(X) = \{\mathcal{A} \in C^2(X) : \mathcal{A}$ is an order arc, $\bigcup \mathcal{A} = X$ and $\bigcap \mathcal{A}$ is a one-point set$\}$. It is known that Whitney maps exist for all continua and that for every x in X there is an order arc between $\{x\}$ and X. The proof of these facts, as well as other information concerning hyperspace can be found in [3].

Let $\omega : C(X) \to [0, \infty)$ be any fixed Whitney map. For a given order arc $\mathcal{A}$ and a number $t \in [0, \omega(X)]$ we denote by $\mathcal{A}(t)$ the only continuum $A \in \mathcal{A}$ with $\omega(A) = t$. In particular, $\mathcal{A}(0)$ is a one-point set and $\mathcal{A}(\omega(X)) = X$.

Now recall Curtis construction of a convex structure on $\Lambda(X)$.

THEOREM 2.3 [1, (4.1)]. *Put* $M_n = \{(\mathcal{A}_1, \ldots, \mathcal{A}_n) \in (\Lambda(X))^n : \mathcal{A}_1(0) = \cdots = \mathcal{A}_n(0)\}$, *and, for* $(t_i, \ldots, t_n) \in P_n$ *and* $i \in \{1, \ldots, n\}$, *let* $\tau_i = t_i/(t_i + \ldots + t_n)$. *If all* $t_i > 0$ *for* $i \in \{1, \ldots, n\}$, *then define* $k_n((\mathcal{A}_1, \ldots, \mathcal{A}_n), (t_1, \ldots, t_n)) = \{\mathcal{A}_1(\tau_1) \cup \ldots \cup \mathcal{A}_{i-1}(\tau_{i-1}) \cup \mathcal{A}_i(t) : i \in \{1, \ldots, n\}$ *and* $t \in [0, \tau_i]\}$. *If* $t_i = 0$ *for some* i, *then we define* $k_n((\mathcal{A}_1, \ldots, \mathcal{A}_n), (t_i, \ldots, t_n))$ *according to* (2) *of* 2.1. *Then the sets* M_n *and the mappings* k_n *form a convex structure on* $\Lambda(X)$ *such that the sets* $\mathcal{L}_x = \{\mathcal{A} \in \Lambda(X) : \mathcal{A}(0) = \{x\}\}$ *are convex.*

Now we show how we can define convex structure on $\Lambda(X)$ in another way.

THEOREM 2.4. *Let* $M_n = \{(\mathcal{A}_1, \ldots, \mathcal{A}_n) \in (\Lambda(X))^n : \mathcal{A}_1(0) = \cdots = \mathcal{A}_n(0)\}$. *For* $(\mathcal{A}_1, \ldots, \mathcal{A}_n) \in M_n$ *and, let* $t_0 = \max\{t_1, \ldots, t_n\}$ *and let* $k_n((\mathcal{A}_1, \ldots, \mathcal{A}_n), (t_1, \ldots, t_n)) = \{\mathcal{A}_1(st_1) \cup \ldots \cup \mathcal{A}_n(st_n) : s \in [0, \omega(X)/t_0]\}$. *Then the sets* M_n *and mappings* k_n *form a convex structure on* $\Lambda(X)$ *such that the sets* $\mathcal{L}_x = \{\mathcal{A} \in \Lambda(X) : \mathcal{A}(0) = \{x\}\}$ *are convex.*

P r o o f. It is not very hard to see that conditions (1) and (2) of 2.1 are satisfied. We verify (3).

Denote by "dist" the Hausdorff distance for $C(X)$ and define a metric D on $\Lambda(X)$ by formula $D(\mathcal{A},\mathcal{B}) = \sup\{\text{dist}(\mathcal{A}(t),\mathcal{B}(t)) : t \in [0,\omega(X)]\}$. The metric D is equivalent to the metric obtained by the restriction of the Hausdorff distance on $C^2(X)$ to $\Lambda(X)$. Now we show that (3) of 2.1 is satisfied for the metric D with $\delta = \varepsilon$ and, therefore, by 2.2, that it is satisfied for any other metric on $\Lambda(X)$.

Let $(\mathcal{A}_1,\dots,\mathcal{A}_n), (\mathcal{B}_1,\dots,\mathcal{B}_n) \in M_n$ be sequences such that $D(\mathcal{A}_i,\mathcal{B}_i) < \varepsilon$ for $i \in \{1,\dots,n\}$, and let $(t_1,\dots,t_n) \in P_n$. Then, for every $s \in [0,\omega(X)/t_0]$, we have $\text{dist}(\mathcal{A}_1(st_1)\cup\dots\cup\mathcal{A}_n(st_n),\mathcal{B}_1(st_1)\cup\dots\cup\mathcal{B}_n(st_n)) < \varepsilon$. Hence we have $D(k_n((\mathcal{A}_1,\dots,\mathcal{A}_n),(t_1,\dots,t_n)),k_n((\mathcal{B}_1,\dots,\mathcal{B}_n),(t_1,\dots,t_n))) < \varepsilon$. Thus we have done it. Convexity of $\mathcal{L}_x$ for $x \in X$ is obvious.

REMARK 2.5. The convex structure defined in 2.4 is symmetric, i.e., for every sequence $(\mathcal{A}_1,\dots,\mathcal{A}_n) \in M_n$ and for every permutation $p : \{1,\dots,n\} \to \{1,\dots,n\}$ we have $k_n((\mathcal{A}_1,\dots,\mathcal{A}_n),(t_1,\dots,t_n)) = k_n((\mathcal{A}_{p(1)},\dots,\mathcal{A}_{p(n)}),(t_{p(1)},\dots,t_{p(n)}))$.

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REFERENCES

[1] D. W. Curtis, *Application of a selection theorem to hyperspace contractibility*, Can. J. Math., **37** (1985) 747–759.

[2] M. Lynch, *Whitney levels in $C_p(X)$ are ARs*, Proc. Amer. Math. Soc., **97** (1986) 748–750.

[3] S. B. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York 1978.