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GENERAL TOPOLOGY

Some Counterexamples Concerning Whitney Levels

by

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Summary. We prove, by showing a counterexample, that the covering property and λ-connectedness are not Whitney properties. Moreover, we construct an example of a continuum with the covering property and having a Whitney level which is not unicoherent.

In this paper we construct an example showing that the covering property and λ-connectedness of continua are not Whitney properties. Thus this example gives a negative answer to two questions: a question raised by Nadler ([5], (14.76.9), p. 510), and a question by Krasinkiewicz and Nadler ([4], Problem 1, p. 179; cf. [5], (14.36), p. 432).

Furthermore another example is constructed which gives a negative answer to a question by B. Hughes raised in [5], (14.73.3), p. 500, whether the covering property of a continuum X implies that each Whitney level $\mu^{-1}(t)$ of $C(X)$ is unicoherent.

We consider metric continua and continuous mappings only. Let a continuum X be given. By a Whitney map for X we understand a mapping μ from the hyperspace $C(X)$ of all subcontinua of X into non-negative reals such that $\mu(\{x\})=0$ for each $x \in X$ and, for each two continua A and B contained in X such that A is a proper subcontinuum of B, we have $\mu(A)<\mu(B)$.

For a given Whitney map μ for X and a number $t \in [0, \mu(X)]$ the set $\mu^{-1}(t)$ is a continuum ([5], (14.2), p. 400) and it is called a Whitney level of $C(X)$.

A topological property P is called a Whitney property if from the fact that a continuum X has P it follows that each Whitney level has P. A continuum X is said to have the covering property provided for each Whitney map μ and for $t \in [0, \mu(X)]$ no proper subcontinuum of $\mu^{-1}(t)$ covers X.

We will use the symbol d for the distance in a continuum X, dist—for the Hausdorff distance in $C(X)$, and Dist—for the Hausdorff distance in $C(C(X))$.

Recall (see [5], (14.73.5), p. 485) a useful

1. Proposition. *For a continuum X the following are equivalent:*
— *X has the covering property,*
— *for each Whitney map μ for X and for each $t \in [0, \mu(X)]$ the continuum $\mu^{-1}(t)$ is irreducible,*
— *for some Whitney map μ for X and for each $t \in [0, \mu(X)]$ the continuum $\mu^{-1}(t)$ is irreducible.*

We start with

2. Lemma. *Let a half-ray H approximate a continuum Y. Put $X = Y \cup H$. Let $\mu: C(X) \to [0, \infty)$ be a Whitney map. If $t \in [0, \mu(Y))$, then $\mu^{-1}(t)$ is the union of a continuum $\mu^{-1}(t) \cap C(Y)$ and a half-ray H' which approximates some subcontinuum of $\mu^{-1}(t) \cap C(Y)$. If $t \in [\mu(Y), \mu(X))$, then $\mu^{-1}(t)$ is an arc.*

Proof. Assume $t \in [0, \mu(Y))$. First, note that $\mu^{-1}(t) \cap C(Y)$ is a continuum. Really, $\mu | C(Y)$ is a Whitney map for Y and $\mu^{-1}(t) \cap C(Y)$ is a Whitney level of $C(Y)$, so it is a continuum.

Second, note that if a subcontinuum $A \in \mu^{-1}(t)$ intersects Y, then it is contained in Y, hence $\mu^{-1}(t)$ is the union of $\mu^{-1}(t) \cap C(Y)$ and $\mu^{-1}(t) \cap C(H)$. So it is enough to show that $\mu^{-1}(t) \cap C(H)$ is a half-ray. Really, each element of $\mu^{-1}(t) \cap C(H)$ is an arc contained in H and it is determined by its end-point lying near to the end-point of H. So the mapping which assigns to an arc its end-point lying near to the end-point of H is a homeomorphism of $\mu^{-1}(t) \cap C(H)$ onto H, which means $H' = \mu^{-1}(t) \cap C(H)$ is a half-ray.

Assume $t = \mu(Y)$. Then $\mu^{-1}(t) \cap C(Y)$ is degenerate, and therefore $\mu^{-1}(t)$ is the union of the point and a half-ray H' having the point as the remainder under the one-point compactification, hence $\mu^{-1}(t)$ is an arc.

Finally, assume $t \in (\mu(Y), \mu(X))$. Then a continuum $A \in \mu^{-1}(t)$ is an arc contained in H or a continuum containing Y and intersecting H. As previously, the mapping which assigns to a continuum $A \in \mu^{-1}(t)$ its end-point lying near to the end-point of H is a homeomorphism of $\mu^{-1}(t)$ onto a subcontinuum of H, i.e., onto an arc.

Let a continuum X be the union of a continuum Y and of a half-ray approximating Y. We say that H approximates each subcontinuum of Y provided that for each subcontinuum A of Y there is a sequence of arcs $A_1, A_2, \ldots$ contained in H and tending to A.

3. Corollary. *Let a continuum X be the union of a continuum Y and a half-ray H which approximates each subcontinuum of Y. Then, for each $t \in [0, \mu(Y))$, the Whitney level $\mu^{-1}(t)$ is the union of the continuum $\mu^{-1}(t) \cap C(Y)$ and the half-ray H' which approximates the whole continuum $\mu^{-1}(t) \cap C(Y)$.*

4. PROPOSITION. *Let a continuum X be the union of a continuum Y and a half-ray H which approximates Y. Then X has the covering property if and only if H approximates each subcontinuum of Y.*

Proof. Assume X approximates each subcontinuum of Y. Then by 2 and 3 the Whitney levels of $C(X)$ are irreducible continua, and therefore by 1 the continuum X has the covering property.

Inversely, assume X has the covering property, and suppose on the contrary that there is a subcontinuum A of Y such that H does not approximate A. Put $t = \mu(A)$. Then by 2 the Whitney level $\mu^{-1}(t)$ is the union of the continuum $\mu^{-1}(t) \cap C(Y)$ and of the half-ray H' approximating a subcontinuum of $\mu^{-1}(t) \cap C(Y)$. Since A is not approximated by H, we have $A \in \mu^{-1}(t) \setminus \mathrm{cl}\, H'$. So $\mathrm{cl}\, H'$ is a proper subcontinuum of $\mu^{-1}(t)$. We shall show $\mathrm{cl}\, H'$ covers X. Really, H' covers H and, by compactness, $\mathrm{cl}\, H'$ covers X. This contradicts the covering property of X.

Now we shall show the covering property is not a Whitney property. This answers a question from [5], (14.76.9), p. 510.

5. EXAMPLE. There exists a continuum X, a Whitney map $\mu: C(X) \to [0, \infty)$ and a number $t \in [0, \mu(X))$ such that X has the covering property, but $\mu^{-1}(t)$ has not.

5.1. Construction. We shall construct a continuum X satisfying the assumptions of 3. Let Y be the one-point union of unit arcs I_1, I_2 and I_3, respectively. Now we construct a half-ray H contained in $Y \times [0, 1]$ and approximating each subcontinuum of $Y \times \{0\}$ (we will also write Y instead of $Y \times \{0\}$). Let $\{z_{i,m}: i \in \{1, 2, \ldots\}$ and $m \in \{1, \ldots, 6\}\}$ be a set of points of $[0, 1]$ satisfying three conditions:

$$z_{i,m} < z_{j,n} \quad \text{for} \quad j < i \quad \text{and each} \quad m, n \in \{1, \ldots, 6\},$$

$$z_{i,m} < z_{i,n} \quad \text{for} \quad n < m \quad \text{and each} \quad i \in \{1, 2, \ldots\},$$

$$z_{i,m} < 1/i \quad \text{for each} \quad i \in \{1, 2, \ldots\} \quad \text{and} \quad m \in \{1, \ldots, 6\}.$$

Let $\{(x_i, y_i)\}_{i=1}^{\infty}$ be a sequence of all pairs of rationals from the interval $[0, 1]$ such that each pair appears infinitely many times in the sequence. Denote by b_i a point in I_2 such that $d(e, b_i) = x_i$ and by c_i a point in I_3 is such that $d(e, c_i) = y_i$.

We will use the symbol $\overline{p_1 \ldots p_n}$ for the broken line being the union of $n-1$ straight line segments joining consecutively points p_1 and p_2, p_2 and $p_3, \ldots, p_{n-1}$ and p_n. In particular, $\overline{p_1 p_2}$ is a segment with end-points p_1 and p_2.

Put

$$H_i = \overline{(a, z_{i,1})(e, z_{i,2})(b_i, z_{i,3})(e, z_{i,4})(c_i, z_{i,5})(e, z_{i,6})(a, z_{i+1,1})}.$$

Note that H_i is an arc and $H_i \cap H_{i+1} = \{(a, z_{i+1})\}$. Put $H = \bigcup_{i=1}^{\infty} H_i$ and $X = Y \cup H$. We can see that H is a half-ray approximating Y. We shall show

5.2. The half-ray H approximates each subcontinuum of Y. Let P be a subcontinuum of Y. Thus P is a segment contained in one of the segments I_1, I_2 or I_3, or it is the union of three segments $A_1 \cup A_2 \cup A_3$ (maybe degenerate) contained in I_1, I_2 and I_3 respectively and having the point e in common.

Consider first the case when P is a segment contained in I_1, or in I_2, or in I_3. Since the pair (1,1) occurs infinitely many times in the sequence, there is a subsequence $i_1, i_2, \ldots$ of indices such that $(x_{i_k}, y_{i_k}) = (1,1)$ for $k \in \{1, 2, \ldots\}$. Then we can find a sequence of arcs $\{P_k\}_{k=1}^{\infty}$ such that $P_k \subset H_{i_k}$ and P_k tend to P.

Secondly, consider P as the union of three segments. Denote the other end-points of A_1, A_2 and A_3 by p, q and r respectively and put $p' = d(e, p)$, $q' = d(e, q)$ and $r' = d(e, r)$. Let $\{q_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ be two sequences of rationals tending to the numbers q' and r', respectively. Since each pair of rationals appears in the sequence $\{(x_i, y_i)\}_{i=1}^{\infty}$ infinitely many times, we can construct a subsequence $\{(x_{i_k}, y_{i_k})\}_{k=1}^{\infty}$ such that $x_{i_k} = q_k$ and $y_{i_k} = r_k$ for all $k \in \{1, 2, \ldots\}$. So by the definition, the points b_{i_k} tend to q and c_{i_k} tend to r. Now let for $i \in \{1, 2, \ldots\}$ $z'_{i,1}$ be a point from the interval $[z_{i,2}, z_{i,1}]$ such that $(p, z'_{i,1}) \in \overline{(a, z_{i,1})\,(e, z_{i,2})}$ and similarly let $z'_{i,6}$ be a point from $[z_{i+1,1}, z_{i,6}]$ such that $(p, z'_{i,6}) \in \overline{(e, z_{i,6})\,(a, z_{i+1,1})}$. Put

$$P_k = \overline{(p, z'_{i_k,1})\,(e, z_{i_k,2})\,(b_{i_k}, z_{i_k,3})\,(e, z_{i_k,4})\,(c_{i_k}, z_{i_k,5})\,(e, z_{i_k,6})\,(p, z'_{i_k,6})}.$$

We can see that P_k is a subarc of H_{i_k} and the sequence $\{P_k\}_{k=1}^{\infty}$ tends to the continuum P. Therefore, by 4, the continuum X has the covering property.

5.3. The Whitney map for X. We shall construct*) a Whitney map $\mu: C(X) \to [0, \infty)$. First, define a Whitney map $\mu_1: C(Y) \to [0, 3]$. If a subcontinuum $A \subset Y$ is a segment contained in one of the segments I_1, I_2 or I_3, then let $\mu_1(A)$ be its length. If A is the union of three segments $\overline{ep}, \overline{eq}, \overline{er}$ contained in I_1, I_2 and I_3 respectively, then we put $\mu_1(A) = d(e, p) + d(e, q) + d(e, r)$.

Let f denote the projection of X onto Y and g — the projection of X into $[0, 1]$. Let $\mu_2: C([0, 1]) \to [0, 1]$ be defined as the length of the appropriate interval. For each $A \in C(X)$ put $\mu(A) = \mu_1(f(A)) + \mu_2(g(A))$. One can easily verify that μ is really a Whitney map for X.

Observe that for $A \subset Y$ we have $\mu(A) = \mu_1(A)$, so $\mu(I_1) = \mu(I_2) = \mu(I_3) = 1$.

Now we shall show

5.4. $\mu^{-1}(1)$ does not have the covering property. Note that by 3

*) The author thanks K. Omiljanowski for a simplification of the construction.

the set $\mu^{-1}(1)$ is the union of $\mu^{-1}(1) \cap C(Y)$ and a half-ray H' which approximates the whole $\mu^{-1}(1) \cap C(Y)$. Arguing like in [6], Example 7a), p. 154 we can see $\mathscr{D} = \mu^{-1}(1) \cap C(Y)$ is a disk. In fact, it is homeomorphic to a subset of the 3-space composed of points (x, y, z) satisfying $x, y, z \in [0, 1]$ and $x+y+z = 1$.

We shall show H' does not approximate each subcontinuum of $\mathscr{D}$. To this end, for $x \in [0, 1]$, define $A_x = \overline{ep} \cup \overline{eq} \cup \overline{er}$, where $p \in I_1$, $q \in I_2$ and $r \in I_3$ satisfy $d(e, p) = x$ and $d(e, q) = d(e, r) = (1-x)/2$. Observe that $A_x \in \mu^{-1}(1)$ and put $\mathscr{A} = \{A_x : x \in [0, 1/2]\}$. Then $\mathscr{A}$ is an arc belonging to $C(\mathscr{D})$.

We prove that $\mathscr{A}$ is not approximated by H', which implies by 4 that $\mu^{-1}(1)$ has not the covering property. Suppose on the contrary that there is a sequence of arcs $\{\mathscr{A}_k\}_{k=1}^{\infty}$ tending to $\mathscr{A}$ such that each $\mathscr{A}_k$ is contained in H'. Take in index k such that $\mathrm{Dist}(\mathscr{A}_k, \mathscr{A}) < 1/8$. Observe that no element of $\mathscr{A}_k$ contains a point, the first coordinate of which is equal to a. Really, it follows from the fact that the distance beetwen a point having its first coordinate equal to a and any point of any element of $\mathscr{A}$ is greater than or equal to 1/2. So we conclude that there is an index i_k such that $\mathscr{A}_k \subset C(H_{i_k})$.

Since there is a continuum $P \in \mathscr{A}_k$ such that $\mathrm{dist}(P, A_0) < 1/8$ (recall that $A_0 = A_x$ for $x = 0$), the numbers x_{i_k} and y_{i_k} from the definition of H_{i_k} satisfy $x_{i_k} > 1/2 - 1/8 = 3/8$ and $y_{i_k} > 1/2 - 1/8 = 3/8$ and therefore $d(e, b_{i_k}) > 3/8$ and $d(e, c_{i_k}) > 3/8$.

Denote by Q a continuum belonging to $\mathscr{A}_k$ and satisfying $\mathrm{dist}(Q, A_{1/2}) < 1/8$. Thus the points $(b_{i_k}, z_{i_k,3})$ and $(c_{i_k}, z_{i_k,5})$ do not belong to Q. Then either $Q \subset \overline{(a, z_{i_k,1})(e, z_{i_k,2})(b_{i_k}, z_{i_k,3})}$ or $Q \subset \overline{(b_{i_k}, z_{i_k,3})(e, z_{i_k,4})(c_{i_k}, z_{i_k,5})}$, or else $Q \subset \overline{(c_{i_k}, z_{i_k,5})(e, z_{i_k,6})(a, z_{i_k+1,1})}$, and therefore $\mathrm{dist}(Q, A_{1/2}) > 1/4$. The condition completes the proof that $\mathscr{A}$ is not approximated by H', and therefore by 4 we see $\mu^{-1}(1)$ has not the covering property. This finishes the proof that the covering property is not a Whitney property.

6. Remark. A continuum X is said to be λ-connected (in the sense of Hagopian, see [1], p. 371; cf. [5], (0.30), p. 16) if for each two points $a, b \in X$ there is a hereditarily decomposable continuum containing a and b. The example described in 5.1 shows that λ-connectedness is not a Whitney property. Really, X is hereditarily decomposable, while $\mu^{-1}(1)$ is an irreducible continuum containing a disk. This answers a question of Krasinkiewicz and Nadler ([4], Problem 1, p. 179; cf. [5], (14.36), p. 432).

Recall that there is another, earlier definition of the term "λ-connected". Namely Knaster and Mazurkiewicz in [3], p. 85 have defined a continuum X to be λ-connected provided that for every two points a and b of X there exists an irreducible continuum of type λ from a to b, i.e., such an irreducible continuum from a to b whose indecomposable subcontinua have empty interiors. Note that each continuum which is λ-connected in the sense of Hagopian is also λ-connected in the sense of Knaster–Mazurkiewicz. These two concepts coincide for plane continua (see [2], Theorem 2, p. 119).

Observe that the continuum X constructed in 5.1 gives a negative answer to Krasinkiewicz–Nadler question quoted above if λ-connectedness is meant in the sense of Hagopian, but it is not good enough to answer this question if λ-connectedness is understood in the sense of Knaster–Mazurkiewicz. So the following question remains open.

7. QUESTION. Is λ-connectedness of continua in the sense of Knaster–Mazurkiewicz a Whitney property?

The next example shows a negative answer to a question of B. Hughes ([5], (14.73.22), p. 500).

8. EXAMPLE. There exist a continuum X, a Whitney map μ for X and a number $t \in (0, \mu(X))$ such that X has the covering property, but $\mu^{-1}(t)$ is not unicoherent.

8.1. Construction. Let S be the unit circle $|z| = 1$ in the complex plane. We will construct the continuum X contained in $S \times [-1, 1]$. Given a natural number n, define

$$H_{2n-1} = \left\{(x, y) \in S \times [0, 1]: x = \exp(2\pi i\theta) \text{ and } y = \frac{2}{3}\left(\frac{1}{2n}\theta + \frac{1}{2n-1}\left(\frac{3}{2} - \theta\right)\right) \text{ and } \theta \in \left[0, \frac{3}{2}\right]\right\}$$

and

$$H_{2n} = \left\{(x, y) \in S \times [0, 1]: x = \exp(2\pi i\theta) \text{ and } y = \frac{2}{3}\left(\frac{1}{2n}\theta + \frac{1}{2n+1}\left(\frac{3}{2} - \theta\right)\right) \text{ and } \theta \in \left[0, \frac{3}{2}\right]\right\}.$$

Thus H_{2n-1} is an arc in $S \times [0, 1]$ with end-points $(1, 1/(2n-1))$ and $(-1, 1/2n)$, and H_{2n} is an arc in $S \times [0, 1]$ with end-points $(1, 1/(n+1))$ and $(-1, 1/2n)$ for all $n \in \{1, 2, \ldots\}$. Observe that $H_{2n-1} \cap H_{2n} = \{(-1, 1/2n)\}$ and $H_{2n} \cap H_{2n+1} = \{(1, 1/(2n+1))\}$, and $H_n \cap H_m \neq \emptyset$ if and only if $|n-m| \leq 1$. Hence $H = \bigcup_{i=1}^{\infty} H_i$ is a half-ray approximating the circle S. Put $Y = S \cup H$. Let s be the central symmetry map of the cylinder $S \times [-1, 1]$ onto itself with respect to its center, and put $X = Y \cup s(Y)$. Since $Y \subset S \times [0, 1]$, we have $Y \cap s(Y) = S$.

8.2. The covering property for X. Let $\mu: C(X) \to [0, \infty)$ be the Whitney map constructed in [5], (0.50.1), p. 25 having the same value for isometric subcontinua of X. We shall prove X has the covering property by showing $\mu^{-1}(t)$ is an irreducible continuum for each $t \in [0, \mu(X)]$ (see 1).

Observe that an arc $A \subset S$ is not the limit of a sequence of arcs contained in H if and only if A properly contains the arc $A_0 = \{z \in S: \text{im } z \leqslant 0\}$ and none of the points -1 and 1 is an end-point of A. Similarly we see that an arc A is not the limit of a sequence of arcs contained in $s(H)$ if and only if it properly contains the arc $s(A_0)$ and none of the points -1 and 1 is an end-point of A. Since there is no arc satisfying both these conditions, each arc contained in S is the limit of a sequence of arcs contained in H or $s(H)$.

Let $t \in [0, \mu(S))$. Then $\mu^{-1}(t)$ is, by 2, the union of two half-rays F and F' approximating a subcontinuum of $\mu^{-1}(t) \cap C(S)$, which is a circle by [5], (14.7), p. 403. Since each subcontinuum of S is approximated by H or $s(H)$, the whole circle $\mu^{-1}(t) \cap C(S)$ is approximated by $F \cup F'$. So $\mu^{-1}(t)$ is irreducible.

If $t \in [\mu(S), \mu(X))$, then arguing like in 2 one can show $\mu^{-1}(t)$ is an arc. So $\mu^{-1}(t)$ is irreducible for all $t \in [0, \mu(X)]$.

Now let $t \in (\mu(A_0), \mu(S))$. We shall show

8.3. $\mu^{-1}(t)$ is not unicoherent. Really, in this case we have $\mu^{-1}(t) = \text{cl } F \cup \text{cl } F'$. We prove $\text{cl } F \cap \text{cl } F'$ is not connected. First, note $\text{cl } F \cap \text{cl } F'$ is contained in $\mu^{-1}(t) \cap C(S)$. Secondly $\text{cl } F \cap C(S)$ and $\text{cl } F' \cap C(S)$ are proper subcontinua of the circle, and the union of these subcontinua is the whole circle. So $\text{cl } F \cap \text{cl } F' = (\text{cl } F \cap C(S)) \cap (\text{cl } F' \cap C(S))$ is not connected.

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В. Я. Харотоник, **Некоторые контрпримеры связанные с уровнями Уитни**

Доказано одним контрпримером, что свойство покрытия и λ-связь не являются свойством Уитни. Кроме этого сконструирован пример континуума со свойством покрытия, котрый имеет не-уникогерентный уровень Уитни.