

Dendrites with a closed set of end points

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Abstract

Dendrites with a closed set of end points are investigated. It is shown that such dendrites are precisely those that do not contain two particular ones. A universal dendrite with a closed set of end points is constructed (also with ramification points of a given order). Open and monotone images of the considered dendrites are studied. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this article we investigate dendrites with a closed set of end points. After the introduction and preliminaries, some general properties of these dendrites are proven in Section 3. The main result of this section is a characterization of dendrites with a closed set of end points as those that do not contain two particular ones.

Classes of dendrites with a closed set of end points and, for each integer $n \geq 3$, with orders of ramification points less than or equal to n are considered in Section 4. For each of them a universal element in the class is constructed.

In the last two sections we characterize open as well as monotone images of dendrites with a closed set of end points, and of universal dendrites.

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2. Preliminaries

A *dendrite* is a locally connected continuum containing no simple closed curve. It is known that every subcontinuum of a dendrite is a dendrite, see [6, §51, VI, Theorem 4, p. 301]; in particular every dendrite is hereditarily locally connected.

By the *order* of a point x in a dendrite X we mean the Menger–Urysohn order, see [6, §51, I, p. 274], or equivalently (see, for example, [6, §51, I, p. 274]) the order in the classical sense, i.e., the number of arcs emanating from x and disjoint out of x (see [1, p. 229] and [7, p. 301]). For dendrites this number is equal to the number of components of $X \setminus \{x\}$, (see [11, (1.1), (iv), p. 88]) and we will denote it by $\text{ord}(x, X)$. If this order is infinite, then it is countable, and the diameters of components of $X \setminus \{x\}$ tend to zero [11, (2.6), p. 92]. In this case we write $\text{ord}(x, X) = \omega$. Points of order 1 are called *end points*, and the set of all end points of a dendrite X is denoted by $E(X)$. Points of order 3 or more are called *ramification points* and the set of all ramification points is denoted by $R(X)$. For any $n \in \{1, 2, \dots, \omega\}$, the symbol $R_n(X)$ denotes the set of all points of X of order n . Thus $E(X) = R_1(X)$, and $R(X) = \bigcup \{R_n(X) : n \in \{3, 4, \dots, \omega\}\}$ and $X = E(X) \cup R_2(X) \cup R(X)$. It is known that $R_2(X)$ is a dense subset of X [6, §51, VI, Theorem 8, p. 302], and that $R(X)$ is at most countable [6, §51, VI, Theorem 7, p. 302].

By a *tree* we mean a dendrite with finitely many end points, or equivalently a dendrite containing no points of order ω with $R(X)$ finite. Given two points x and y in a dendrite X , the symbol xy denotes the unique arc joining x and y . The arc is called *free* if $xy \setminus \{x, y\}$ is open in X or, equivalently, if $xy \setminus \{x, y\}$ contains no ramification points of X .

Fix a point $p \in X$. We will write $x \leq_p y$ if $x \in py$ and $x <_p y$ if $x \leq_p y$ and $x \neq y$. Then \leq_p is a closed partial order in X , and for every $A \subset X$ there is a unique element $a \in X$ such that for any $x \in A$ we have $a \leq_p x$ and a is the greatest element with this property. Then we write $a = \inf_{\leq_p} A$.

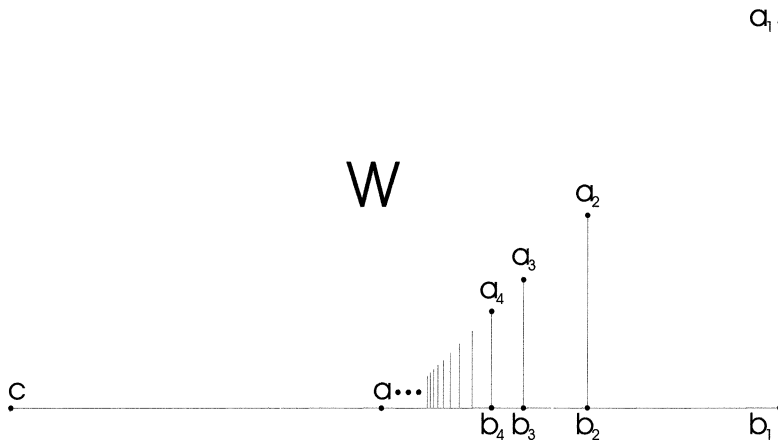


Fig. 1.

In the paper we will need two special dendrites. One of them is the dendrite with only one ramification point, whose order is ω . We will denote this dendrite by F_ω . Now we will construct the other one (see Fig. 1).

Let \overline{pq} stand for the straight line segment joining p and q in the plane. In the plane, let $a = (0, 0)$, $a_n = (1/n, 1/n)$, $b_n = (1/n, 0)$ and $c = (-1, 0)$. Define $W_R = \overline{ab_1} \cup \bigcup \{\overline{a_n b_n} : n \in \mathbb{N}\}$ and $W = \overline{ca} \cup W_R$. Note that neither F_ω nor W has the set of its end points closed. We will show in Section 3 that every dendrite whose set of end points is not closed contains either F_ω or W .

For a given set $A \subset X$, the symbols $\text{cl } A$, $\text{int } A$ and $\text{bd } A$ denote the closure, the interior and the boundary of the set A , respectively.

3. General properties

In this section we will provide a characterization of dendrites with a closed set of end points and we will show some other general properties of those dendrites. We start with the following characterization of trees.

Theorem 3.1. *A dendrite X is a tree if and only if X contains neither a copy of F_ω nor of W_R .*

Proof. Suppose X is a tree. Then each of its subcontinua is a tree, so X contains no copy of F_ω nor of W_R .

Now assume that X contains no copy of F_ω or of W_R . Then $\text{ord}(x, X)$ is finite for every $x \in X$. Suppose that $E(X)$ is infinite and take a convergent non constant sequence $\{x_n\}_{n=1}^\infty$ of end points of X . Put $x = \lim_{n \rightarrow \infty} x_n$. Because $\text{ord}(x, X)$ is finite, we may assume that all points of the sequence $\{x_n\}_{n=1}^\infty$ are in one component of $X \setminus \{x\}$. Define $p_n = \inf_{\leq_x} \{x_1, \dots, x_n\}$. Thus $\{p_n\}_{n=1}^\infty$ is a decreasing (with respect to the order \leq_x) sequence of points. Because $p_n \in xx_n$ and the arcs xx_n converge to $\{x\}$ we have $\lim_{n \rightarrow \infty} p_n = x$. Choose a subsequence $\{p_{n_k}\}_{k=1}^\infty$ of the sequence $\{p_n\}_{n=1}^\infty$ satisfying $p_{n_i} \neq p_{n_j}$ for $i \neq j$. Let Y be the minimal continuum containing the points x_{n_k} for $k \in \{1, 2, \dots\}$. Then there is a homeomorphism $h : Y \rightarrow W_R$ such that $h(x_{n_k}) = a_k$ and $h(p_{n_k}) = b_k$, so X contains a copy of W_R . This finishes the proof. \square

Theorem 3.2. *Every subcontinuum of a dendrite with a closed set of end points is a dendrite with a closed set of end points.*

Proof. Assume a dendrite Y is contained in a dendrite X with a closed set of end points. Consider a sequence $\{y_n\}_{n=1}^\infty$ of points of $E(Y)$ with $\lim_{n \rightarrow \infty} y_n = y$. We may assume that $y_n \notin E(X)$. Then, for every component C_n of $X \setminus Y$ such that $y_n \in \text{cl } C_n$, choose $x_n \in C_n \cap E(X)$. Because $\{\text{cl } C_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint continua in a hereditarily locally connected continuum X , they form a null-sequence (see [11, Chapter 5, (2.6), p. 92]). Thus $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = y$ and, by closedness of $E(X)$, we have $y \in E(X)$, so $y \in E(Y)$. \square

Theorem 3.3. *A dendrite has a closed set of end points if and only if it contains no copy of F_ω or of W .*

Proof. First observe that F_ω and W do not have closed sets of end points, so neither does any dendrite that contains one of them by Theorem 3.2.

Now assume that X contains neither a copy of F_ω nor of W . We will show that X has a closed set of end points. Assume on the contrary that x is not an end point, but $x = \lim_{n \rightarrow \infty} x_n$ where $x_n \in E(X)$. Then, proceeding as in the proof of Theorem 3.1, we conclude that X contains a copy Y of W_R with x being the image of a , and since x is not an end point of X we can choose a point z in a different component of $X \setminus \{x\}$ from the one in which all the points x_n are. Then $Y \cup xz$ is homeomorphic to W , a contradiction. This finishes the proof. \square

Proposition 3.4. *If X is a dendrite with a closed set of end points and $\{r_n\}_{n=1}^\infty$ is a convergent sequence of distinct ramification points, then $\lim_{n \rightarrow \infty} r_n$ is an end point.*

Proof. Let $r = \lim_{n \rightarrow \infty} r_n$. By Theorem 3.3 X contains no copy of F_ω , hence it contains no point of order ω , so we may assume that all the points r_n are in one component of $X \setminus \{r\}$. Define $p_n = \inf_{\leq r} \{r_1, \dots, r_n\}$. Then p_n is a ramification point and all points p_n lie in the arc rp_1 . For each n choose an end point e_n of X in a component of $X \setminus \{p_n\}$ that is disjoint with the arc rp_1 . Then $\lim_{n \rightarrow \infty} e_n = r$, so $r \in E(X)$, since $E(X)$ is closed. \square

Corollary 3.5. *If X is a dendrite with a closed set of end points, then $\text{cl } R(X) \subset R(X) \cup E(X)$.*

Corollary 3.6. *If X is a dendrite with a closed set of end points, then $R(X)$ is discrete.*

We close this section with the following observation.

Observation 3.7. *A dendrite X does not contain a copy of W if and only if for every arc $ab \subset X$ the set $(ab \setminus \{a, b\}) \cap \text{cl } R(X)$ is discrete.*

4. Universal dendrites

In this section we construct, for any number $n \in \{3, 4, \dots\}$, a dendrite G^n and we provide topological characterizations of them. Next we show that, for any $n \in \{3, 4, \dots\}$, the dendrite G^n is universal in the class of all dendrites with a closed set of end points and orders of ramification points not more than n . We also construct, for any increasing sequence of natural numbers $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$ a dendrite $G(\mathbf{r})$ with orders of ramification points in the set $\{r_1, r_2, \dots\}$, and we prove that $G(\mathbf{r})$ is universal in the class of all dendrites with a closed set of end points. We will start with the construction of the dendrites G^n (see Fig. 2).

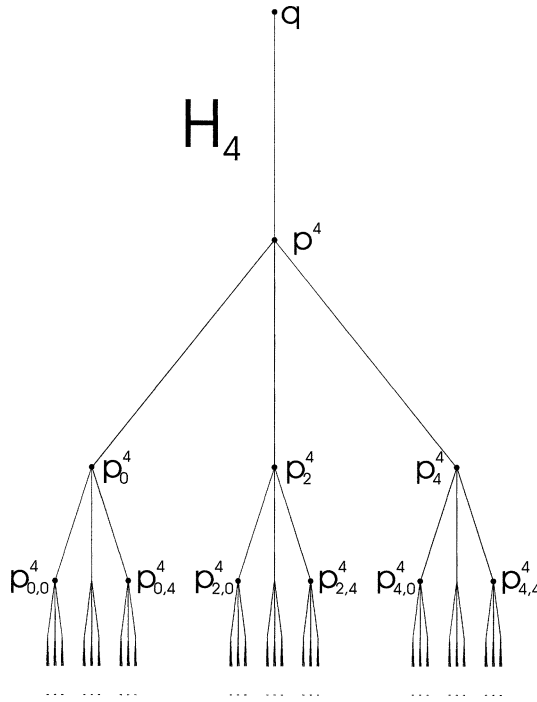


Fig. 2.

Let $n \in \{3, 4, \dots\}$ be fixed. For given numbers $\alpha_1, \dots, \alpha_i \in \{0, 1, \dots, 2n - 4\}$ we define $E_{\alpha_1, \dots, \alpha_i}^n$ as the set of numbers x in $[0, 1]$ such that if x is written in enumeration system of base $2n - 3$, then the first i digits are exactly $\alpha_1, \dots, \alpha_i$, i.e.,

$$E_{\alpha_1, \dots, \alpha_i}^n = \left[\frac{\alpha_1}{2n - 3} + \dots + \frac{\alpha_i}{(2n - 3)^i}, \frac{\alpha_1}{2n - 3} + \dots + \frac{\alpha_{i-1}}{(2n - 3)^{i-1}} + \frac{\alpha_i + 1}{(2n - 3)^i} \right].$$

If all of the digits $\alpha_1, \dots, \alpha_i$ are even, then let $p_{\alpha_1, \dots, \alpha_i}^n$ be the point in the plane whose first coordinate is the middle of the interval $E_{\alpha_1, \dots, \alpha_i}^n$ and whose second coordinate is $1/2^i$. Note that if $i = 0$, then we have $E^n = [0, 1]$ and $p^n = (1/2, 1)$. Put $q = (1/2, 2)$ and define

$$H^n = \overline{p^n q} \cup \text{cl} \left[\bigcup \left\{ \overline{p_{\alpha_1, \dots, \alpha_{i-1}}^n p_{\alpha_1, \dots, \alpha_{i-1}, \alpha_i}^n} : i \in \mathbb{N} \text{ and } \alpha_1, \dots, \alpha_i \in \{0, 2, \dots, 2n - 4\} \right\} \right].$$

Note that for a given infinite sequence $\alpha_1, \alpha_2, \dots$ the intersection $\bigcap \{E_{\alpha_1, \dots, \alpha_i}^n : i \in \mathbb{N}\}$ is a one point set. Denote the point by $e_{\alpha_1, \alpha_2, \dots}$, put $p_{\alpha_1, \alpha_2, \dots} = (e_{\alpha_1, \alpha_2, \dots}, 0) \in H^n$, and note that

$$E(H^n) = \{q\} \cup \{p_{\alpha_1, \alpha_2, \dots} : \alpha_i \in \{0, 2, \dots, 2n - 4\}\}.$$

Finally define G^n as the union of two homeomorphic copies H^n and $(H^n)'$ of H^n with the corresponding points q and q' identified. Then G^n is a dendrite such that every ramification point of G^n has order n and the set of end points of G^n is homeomorphic to the Cantor set. We will show that these two properties characterize G^n .

Theorem 4.1. *Let an integer $n \geq 3$ be fixed. A dendrite X is homeomorphic to G^n if and only if $E(X)$ is homeomorphic to the Cantor set and $R(X) = R_n(X)$.*

Proof. Since G^n has the required properties, only one implication needs to be proved. So, let X be a dendrite with $E(X)$ homeomorphic to the Cantor set and with $R(X) = R_n(X)$. We will construct a homeomorphism $h : G^n \rightarrow X$. By Corollary 3.6 we can choose a point a in a free arc, and we put $h(q) = a$. Let b be an end point of the maximal free arc that contains a . Then b is neither an end point of X nor a point of order 2 in X . Put $h(p^n) = b$ and let $h|_{qp^n}$ be a homeomorphism of the arc qp^n onto ab . Since $\text{ord}(b, X) = n$ and since $R(X)$ is discrete (see Corollary 3.6) there are $n - 1$ maximal free arcs not containing a whose common end point is b . Denote by $b_0, b_2, \dots, b_{2n-4}$ the other end points of those arcs and put $h(p_k^n) = b_k$ for $k \in \{0, 2, \dots, 2n - 4\}$. Moreover, let $h|_{p^n p_k^n}$ be a homeomorphism onto the arc bb_k .

We will proceed by induction on the number of indices. Define $H_m^n \subset H^n$ as the continuum irreducible with respect to containing all points of the form $p_{\alpha_1, \dots, \alpha_i}^n$ for $i \leq m$, and q . Then $H_m^n \subset H_{m+1}^n$ and $H^n = \text{cl} \bigcup \{H_m^n : m \in \mathbb{N}\}$. Assume h is defined on H_m^n . We will define h on H_{m+1}^n . Let $\alpha_1, \dots, \alpha_m$ be given. Since $\text{ord}(h(p_{\alpha_1, \dots, \alpha_m}^n), X) = n$ we have $n - 1$ maximal free arcs not containing $h(p_{\alpha_1, \dots, \alpha_{m-1}}^n)$. Denote by $b_{\alpha_1, \dots, \alpha_m, 0}, b_{\alpha_1, \dots, \alpha_m, 2}, \dots, b_{\alpha_1, \dots, \alpha_m, 2n-4}$ the other end points of those arcs, put $h(p_{\alpha_1, \dots, \alpha_{m+1}}^n) = b_{\alpha_1, \dots, \alpha_{m+1}}$ for any $\alpha_{m+1} \in \{0, 2, \dots, 2n - 4\}$ and let $h|_{p_{\alpha_1, \dots, \alpha_m}^n p_{\alpha_1, \dots, \alpha_{m+1}}^n}$ be a homeomorphism of the arc $p_{\alpha_1, \dots, \alpha_m}^n p_{\alpha_1, \dots, \alpha_{m+1}}^n$ onto $b_{\alpha_1, \dots, \alpha_m} b_{\alpha_1, \dots, \alpha_{m+1}}$.

In this way we have defined $h|_{\bigcup \{H_m^n : m \in \mathbb{N}\}}$. One can verify that $h|_{\bigcup \{H_m^n : m \in \mathbb{N}\}}$ is continuous and one-to-one. We will define h on the set $E(H^n) = \text{cl} \bigcup \{H_m^n : m \in \mathbb{N}\} \setminus \bigcup \{H_m^n : m \in \mathbb{N}\}$. Put $h(p_{\alpha_1, \alpha_2, \dots}^n) = \lim_{k \rightarrow \infty} b_{\alpha_1, \dots, \alpha_k}$. To show h is well defined we prove that the sequence $\{b_{\alpha_1, \dots, \alpha_k}\}_{k=1}^{\infty}$ is convergent. Note that $b_{\alpha_1, \dots, \alpha_k} \leq_a b_{\alpha_1, \dots, \alpha_{k+1}}$, so the sequence is increasing with respect to the order \leq_a . Since the closure of an increasing sequence of arcs in a dendrite is an arc, all points $b_{\alpha_1, \dots, \alpha_k}$ lie on a (closed) arc and form an increasing sequence. Therefore the sequence is convergent. The limit $h(p_{\alpha_1, \alpha_2, \dots}^n)$ is an end point of X by Proposition 3.4. Finally, since X contains no simple closed curve, the mapping h is one-to-one.

It remains to be shown that h is continuous. Since $h|_{\bigcup \{H_m^n : m \in \mathbb{N}\}}$ is continuous, we have to verify continuity of h at the end points of H^n . To this aim define for any $x \in X$ the set $U(x) = \{y \in X : x <_a y\}$ and similarly for $x \in H^n$ the set $V(x) = \{y \in H^n : x <_q y\}$ and observe that they are open in X and in H^n , respectively. Note that the family $\{U(b_{\alpha_1, \dots, \alpha_k}) : k \in \mathbb{N}\}$ is a local base of neighborhoods of a point $\lim_{k \rightarrow \infty} b_{\alpha_1, \dots, \alpha_k}$ and $h^{-1}(U(b_{\alpha_1, \dots, \alpha_k})) = V(p_{\alpha_1, \dots, \alpha_k}^n)$. This finishes the proof of continuity of $h|_{H^n}$. Analogously we can define h on $(H^n)'$ and observe that since $\text{ord}(a, X) = 2$ the function h is surjective, so the proof is complete. \square

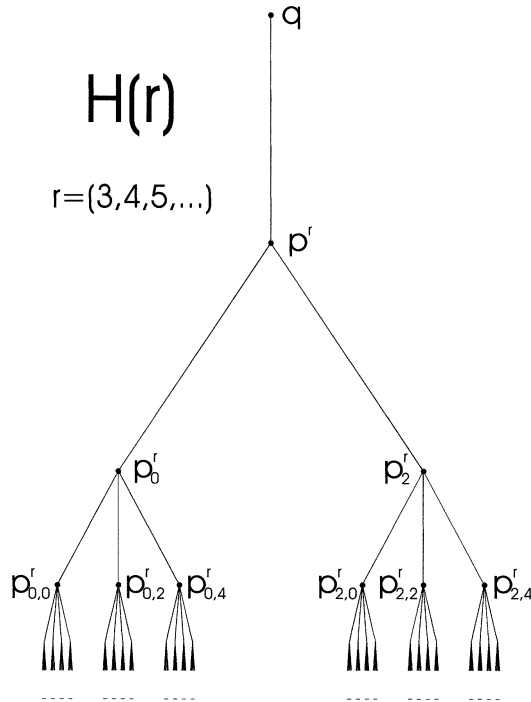


Fig. 3.

Recall that a space X is *universal in a class of spaces* if it belongs to the class and it contains a homeomorphic copy of every element of that class. Now we will construct, for any increasing sequence of natural numbers $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$ a dendrite $G(\mathbf{r})$ with orders of ramification points in the set $\{r_1, r_2, \dots\}$, and then we will establish the universality properties of the dendrites $G(\mathbf{r})$ and G^n . We start with the construction (see Fig. 3).

For given numbers

$$\alpha_1 \in \{0, 1, \dots, 2r_1 - 4\}, \quad \alpha_2 \in \{0, 1, \dots, 2r_2 - 4\}, \quad \dots, \quad \alpha_i \in \{0, 1, \dots, 2r_i - 4\}$$

we define an interval $E_{\alpha_1, \dots, \alpha_i}^r$ as follows. Put $E^r = [0, 1]$, and

$$E_0^r = \left[0, \frac{1}{2r_1-3}\right], \quad \dots, \quad E_{\alpha_1}^r = \left[\frac{\alpha_1}{2r_1-3}, \frac{\alpha_1+1}{2r_1-3}\right], \quad \dots, \quad E_{2r_1-4}^r = \left[\frac{2r_1-4}{2r_1-3}, 1\right].$$

Assume we have defined $E_{\alpha_1, \dots, \alpha_i}^r$ for some $i \geq 1$. Divide the interval $E_{\alpha_1, \dots, \alpha_i}^r$ into $2r_i - 3$ congruent consecutive intervals $E_{\alpha_1, \dots, \alpha_i, 0}^r, E_{\alpha_1, \dots, \alpha_i, 1}^r, \dots, E_{\alpha_1, \dots, \alpha_i, 2r_i-4}^r$. One can verify that

$$E_{\alpha_1, \dots, \alpha_i}^r = \left[\frac{\alpha_1}{2r_1-3} + \frac{\alpha_2}{(2r_1-3)(2r_2-3)} + \dots + \frac{\alpha_{i-1}}{(2r_1-3)\dots(2r_{i-1}-3)} + \frac{\alpha_i}{(2r_1-3)\dots(2r_i-3)}, \right. \\ \left. \frac{\alpha_1}{2r_1-3} + \frac{\alpha_2}{(2r_1-3)(2r_2-3)} + \dots + \frac{\alpha_{i-1}}{(2r_1-3)\dots(2r_{i-1}-3)} + \frac{\alpha_i+1}{(2r_1-3)\dots(2r_i-3)} \right].$$

If all numbers $\alpha_1, \dots, \alpha_i$ are even, then let $p_{\alpha_1, \dots, \alpha_i}^r$ be the point in the plane whose first coordinate is the middle of the interval $E_{\alpha_1, \dots, \alpha_i}^r$ and the second coordinate is $1/2^i$. Note that if $i = 0$, then we have $E^r = [0, 1]$ and $p^r = (1/2, 1)$. Put $q^r = (1/2, 2)$ and define

$$H(\mathbf{r}) = \overline{p^r q^r} \cup \text{cl} \left[\bigcup \{ \overline{p_{\alpha_1, \dots, \alpha_{i-1}}^r p_{\alpha_1, \dots, \alpha_{i-1} \alpha_i}^r} : i \in \mathbb{N} \text{ and } \alpha_i \in \{0, 2, \dots, 2r_i - 4\} \} \right].$$

Note that $\text{ord}(p_{\alpha_1, \dots, \alpha_i}^r, H(\mathbf{r})) = r_i$, so $H(\mathbf{r})$ contains points of arbitrarily large order. As previously, let $p_{\alpha_1, \alpha_2, \dots}^r = \lim_{i \rightarrow \infty} p_{\alpha_1, \dots, \alpha_i}^r$. Then $E(H(\mathbf{r})) = \{q^r\} \cup \{p_{\alpha_1, \alpha_2, \dots}^r : \alpha_i \in \{0, 2, \dots, 2r_i - 4\}\}$. Finally define $G(\mathbf{r})$ as the union of two homeomorphic copies $H(\mathbf{r})$ and $H'(\mathbf{r})$ of $H(\mathbf{r})$ with the corresponding points q^r and $(q^r)'$ identified. Then $G(\mathbf{r})$ is a dendrite whose set of end points is homeomorphic to the Cantor set.

Theorem 4.2. *For any sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$ the dendrite $G(\mathbf{r})$ (the dendrite G^n for $n \in \{3, 4, \dots\}$) is universal in the class of all dendrites having a closed set of end points (and orders of ramification points less than or equal to n).*

Proof. We will show the proof for the dendrite $G(\mathbf{r})$; the proof in the case of G^n is analogous, and even simpler.

Let X be a dendrite with a closed set of end points. We will construct an embedding $f : X \rightarrow H(\mathbf{r}) \subset G(\mathbf{r})$. Denote by $V(X)$ the union of $R(X)$ and the set of all isolated points of $E(X)$. Thus $V(X)$ is countable, so we can write $V(X) = \{t_1, t_2, \dots\}$. First we put $s_1 = t_1$ and let $f(s_1)$ be a point of $H(\mathbf{r})$ satisfying $\text{ord}(s_1, X) < \text{ord}(f(s_1), H(\mathbf{r}))$.

Assume we have defined $s_1, \dots, s_n, f(s_1), \dots, f(s_n)$, and if $s_i s_j$ is a free arc in X , then $f|_{s_i s_j}$ is defined and it is a homeomorphism. We will say that two points of $V(X)$ are *adjacent* if the arc joining them is free. Let s_{n+1} be the first point in the sequence t_1, t_2, \dots such that $s_{n+1} \notin \{s_1, \dots, s_n\}$ and s_{n+1} has an adjacent s_k in the set $\{s_1, \dots, s_n\}$. Then we choose $f(s_{n+1})$ as a point in $H(\mathbf{r})$ satisfying $f(s_k) <_q f(s_{n+1})$ and such that $f(s_{n+1})$ is an end point of $H(\mathbf{r})$ if s_{n+1} was an end point of X or $\text{ord}(s_{n+1}, X) \leq \text{ord}(f(s_{n+1}), H(\mathbf{r}))$ if s_{n+1} was a ramification point. Finally, let $f|_{s_k s_{n+1}}$ be a homeomorphism. In this way we have defined the embedding f for every $x \in X$ except non isolated end points of X . Proceeding as in the proof of Theorem 4.1, we can extend f to an embedding of X into $H(\mathbf{r})$ and prove its continuity. \square

Remark 4.3. As the reader can verify in the proof of the above theorem, the considered embedding embeds the set of end points of X into the set of end points of $G(\mathbf{r})$ (of G^n , respectively). Moreover, since X contains a free arc, we may construct an embedding f in such a way that $q^r \in f(X) \subset G(\mathbf{r})$.

In connection with Theorem 3.3 the classes of those dendrites that do not contain a copy of F_ω , and those that do not contain a copy of W are of some interest. The first one is just the class of dendrites with finite orders of ramification points, and it is known that this class has a universal element (see [3, Theorem 6.8, p. 230]). We will show that the other class, which appears in a natural way in Section 6, also has a universal element P . Now we will describe the dendrite P , and then we will show its universality and give a topological characterization.

Let p be any point in the plane \mathbb{R}^2 . Let p_1, p_2, \dots be a sequence of points in \mathbb{R}^2 tending to p and such that no three of the points p, p_1, p_2, \dots are collinear. Define $P_1 = \bigcup\{\overline{pp_n} : n \in \mathbb{N}\}$. Then P_1 is homeomorphic to F_ω . For each $n, m \in \mathbb{N}$, let p_{nm} be a point in $\mathbb{R}^2 \setminus P_1$ such that the sequence p_{n1}, p_{n2}, \dots tends to p_n , no three of the points $p_n, p_{n1}, p_{n2}, \dots$ are collinear, and that the continuum $P_2 = P_1 \cup \bigcup\{\overline{p_n p_{nm}} : n, m \in \mathbb{N}\}$ is a dendrite. We will continue constructing dendrites P_3, P_4, \dots in the same manner, such that we get an increasing sequence of dendrites $P_1 \subset P_2 \subset P_3 \subset \dots \subset P_n \subset P_{n+1} \subset \dots$ in the plane having the property that the closure of their union is also a dendrite. Finally define

$$(4.4) \quad P = \text{cl}\left(\bigcup\{P_n : n \in \mathbb{N}\}\right).$$

Theorem 4.5. *A dendrite X is homeomorphic to the dendrite P defined by (4.4) if and only if each of the following three conditions is satisfied:*

- (4.5.1) X contains no copy of W ;
- (4.5.2) $E(X) \subset \text{cl } R(X)$;
- (4.5.3) $R(X) = R_\omega(X)$.

Proof. The proof is similar to that of Theorem 4.1. Since the dendrite P has the required properties, only one implication needs to be proved. So, let a dendrite X satisfy (4.5.1)–(4.5.3). We will construct a homeomorphism $h : P \rightarrow X$.

Choose a point $a \in R(X)$ and put $h(p) = a$. Let a_1, a_2, \dots be ramification points of X lying in different components of $X \setminus \{a\}$ such that the arcs aa_i are free, and every component of $X \setminus \{a\}$ contains one of those points. Such points do exist by Observation 3.7. For each $i \in \mathbb{N}$ put $h(p_i) = a_i$ and let $h|_{pp_i} : pp_i \rightarrow aa_i$ be a homeomorphism. In this way $h|_{P_1}$ is defined.

We proceed by induction on the number of indices. Assuming that $h|_{P_n}$ is defined for some $n \in \mathbb{N}$, we will define h on $P_{n+1} \setminus P_n$. Consider $p_{\alpha_1 \dots \alpha_n} \in P_n$. Then $h(p_{\alpha_1 \dots \alpha_n}) = a_{\alpha_1 \dots \alpha_n}$ is a point of order ω in X , so there are ramification points $a_{\alpha_1 \dots \alpha_n 1}, a_{\alpha_1 \dots \alpha_n 2}, \dots$ of X belonging to distinct components of $X \setminus \{a_{\alpha_1 \dots \alpha_n}\}$ and such that each component of $X \setminus \{a_{\alpha_1 \dots \alpha_n}\}$ that does not contain the point a contains some of the points $a_{\alpha_1 \dots \alpha_n 1}, a_{\alpha_1 \dots \alpha_n 2}, \dots$. We also require that for each $i \in \mathbb{N}$ the arcs $a_{\alpha_1 \dots \alpha_n} a_{\alpha_1 \dots \alpha_n i}, \dots$ are maximal free arcs in X . Then we put $h(p_{\alpha_1 \dots \alpha_n i}) = a_{\alpha_1 \dots \alpha_n i}$ and define

$$h|_{p_{\alpha_1 \dots \alpha_n} p_{\alpha_1 \dots \alpha_n i}} : p_{\alpha_1 \dots \alpha_n} p_{\alpha_1 \dots \alpha_n i} \rightarrow a_{\alpha_1 \dots \alpha_n} a_{\alpha_1 \dots \alpha_n i}$$

as a homeomorphism. In this way we have defined $h|_{P_{n+1}}$, and thus, by the inductive procedure, $h|_{\bigcup\{P_n : n \in \mathbb{N}\}}$ is defined.

To extend h to the whole of P observe that for any sequence $\alpha_1, \alpha_2, \dots$ of positive integers the points $a_{\alpha_1} a_{\alpha_2}, \dots$ lie in one arc and they form an increasing sequence with respect to the order \leq_a . Thus the limit of the sequence exists, and it is an end point of X by (4.5.2). Hence putting $h(\lim_{i \rightarrow \infty} p_{\alpha_1 \dots \alpha_i}) = \lim_{i \rightarrow \infty} a_{\alpha_1 \dots \alpha_i}$ the function h is well defined and continuous on $E(P) = P \setminus \bigcup\{P_n : n \in \mathbb{N}\}$, and therefore it is well defined and continuous on P . It is one-to-one by its construction, and surjective by condition (4.5.2). The proof is complete. \square

Following the proof of Theorem 4.2 the reader can prove the next theorem.

Theorem 4.6. *The dendrite P defined by (4.4) is universal in the class of all dendrites containing no copy of W .*

5. Open images

In this section we will investigate open images of dendrites with a closed set of end points, in particular those of $G(\mathbf{r})$ and of G^n . Let us recall that a mapping $f : X \rightarrow Y$ is *open* if the image of every open set in X is open in Y . The mapping is called *interior* at a point $x \in X$ if for any set U in X , the condition $x \in \text{int } U$ implies $f(x) \in \text{int } f(U)$. It is known that f is open if and only if it is interior at every point of X .

We start by recalling a result by Whyburn (see [11, Chapter 8, (7.31), p. 137]).

Proposition 5.1 (G.T. Whyburn). *The order of a point is never increased under an open mapping.*

Proposition 5.2. *The open image of a dendrite with a closed set of end points is a dendrite with a closed set of end points.*

Proof. Let $f : X \rightarrow Y$ be an open mapping of a dendrite X with a closed set of end points. Then Y is a dendrite by [11, (7.36), p. 68]. By Proposition 5.1, $f(E(X)) \subset E(Y)$ and since $f^{-1}(E(Y)) \setminus E(X)$ is finite (see [2, (6.18), p. 25]) the set $E(Y)$ is equal to $f(E(X))$ plus possibly a finite set, so it is closed. \square

Theorem 5.3. *The following conditions are equivalent for a dendrite X :*

(5.3.1) *for any increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^{\infty}$ with $r_1 \geq 3$, the dendrite X is an open image of $G(\mathbf{r})$;*

(5.3.2) *for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^{\infty}$ with $r_1 \geq 3$, the dendrite X is an open image of $G(\mathbf{r})$;*

(5.3.3) *$E(X)$ is closed in X .*

Proof. The implication (5.3.1) \Rightarrow (5.3.2) is obvious. Condition (5.3.2) implies (5.3.3) by Proposition 5.2. It remains to show that (5.3.3) implies (5.3.1). So assume X is a dendrite with a closed set of end points. By Remark 4.3 we may assume that X is contained in $G(\mathbf{r})$ in such a way that $E(X) \subset E(G(\mathbf{r}))$, $E(X) \cap E(H(\mathbf{r})) \neq \emptyset$, and $q^{\mathbf{r}} \in X$. Define $h : G(\mathbf{r}) \rightarrow [0, 2]$ as the second coordinate function, i.e., the projection of $G(\mathbf{r})$ onto $[0, 2]$ that satisfies $h(q^{\mathbf{r}}) = 2$ and $h(E(G(\mathbf{r}))) = \{0\}$. Note that if $x <_{q^{\mathbf{r}}} y$ then $h(x) > h(y)$, and if $h(x) = h(y)$, then $\text{ord}(x, G(\mathbf{r})) = \text{ord}(y, G(\mathbf{r}))$.

We will construct an open retraction $f : G(\mathbf{r}) \rightarrow X$ that satisfies

(5.3.4) $h(f(x)) = h(x)$ for $x \in G(\mathbf{r})$.

We will proceed by induction. For any $x \in G(\mathbf{r})$ define

$$U(x) = \{y \in G(\mathbf{r}) : x <_{q^{\mathbf{r}}} y\}$$

and note that $U(x)$ is open in $G(\mathbf{r})$. We will show the construction of f on $H(\mathbf{r})$, the construction on $H'(\mathbf{r})$ is analogous. First, let $f|_{q^r p^r}$ be the identity on $q^r p^r$. Second, assume we have defined f on $h^{-1}([1/2^k, 2])$, for some $k \in \{0\} \cup \mathbb{N}$, in particular for any point of the form $p_{\alpha_1 \dots \alpha_k}^r$. Define $f(p_{\alpha_1 \dots \alpha_{k+1}}^r)$ as a point in $X \cap U(f(p_{\alpha_1 \dots \alpha_k}^r))$ that satisfies (5.3.4) and

(5.3.5) if $p_{\alpha_1 \dots \alpha_{k+1}}^r \in X$, then $f(p_{\alpha_1 \dots \alpha_{k+1}}^r) = p_{\alpha_1 \dots \alpha_{k+1}}^r$,

(5.3.6) if $\text{ord}(f(p_{\alpha_1 \dots \alpha_k}^r), X) = n + 1$, then there are n points among $p_{\alpha_1 \dots \alpha_k, 0}^r, p_{\alpha_1 \dots \alpha_k, 2}^r \dots p_{\alpha_1 \dots \alpha_k, 2k+2}^r$ with different images.

Then f is defined on the set $\{p_{\alpha_1 \dots \alpha_k, \alpha_{k+1}}^r : \alpha_{k+1} \in \{0, 2, \dots, 2k + 2\}\}$ and we can extend f on $h^{-1}([1/2^{k+1}, 1/2^k])$ in a unique way such that condition (5.3.4) is satisfied. This finishes the inductive procedure, so f is defined on $h^{-1}((0, 2])$ and, by taking limits, on the whole of $G(\mathbf{r})$. It is continuous by its construction. Further, f is interior at any point of $R(G(\mathbf{r}))$ by (5.3.6), and its interiority at points of order 2 is clear. Interiority of f at the end points of $G(\mathbf{r})$ is a consequence of openness of $f(U(p_{\alpha_1 \dots \alpha_k}^r))$ for any $p_{\alpha_1 \dots \alpha_k}^r \in R(G(\mathbf{r}))$, which again follows from (5.3.6). The proof is complete. \square

Corollary 5.4. *The following conditions are equivalent for a dendrite X :*

- (5.4.1) $E(X)$ is closed;
- (5.4.2) X contains neither a copy of F_ω nor of W ;
- (5.4.3) for any increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X is an open image of $G(\mathbf{r})$;
- (5.4.4) for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X is an open image of $G(\mathbf{r})$;
- (5.4.5) for any increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$ in such a way that X is an open retract of $G(\mathbf{r})$;
- (5.4.6) for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$ in such a way that X is an open retract of $G(\mathbf{r})$;
- (5.4.7) for any increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$ in such a way that $E(X) \subset E(G(\mathbf{r}))$;
- (5.4.8) for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$ in such a way that $E(X) \subset E(G(\mathbf{r}))$;
- (5.4.9) for any increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$;
- (5.4.10) for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$, the dendrite X can be embedded in $G(\mathbf{r})$;
- (5.4.11) X is an open image of some dendrite with a closed set of end points;
- (5.4.12) X can be embedded in some dendrite with a closed set of end points.

Proceeding as in the proof of Theorem 5.3, we can prove the following result.

Theorem 5.5. *A dendrite X is an open image of G^n if and only if $E(X)$ is closed and all orders of ramification points in X are less than or equal to n .*

Corollary 5.6. *The following conditions are equivalent for a dendrite X :*

- (5.6.1) $E(X)$ is closed and for any $x \in X$ we have $\text{ord}(x, X) \leq n$;
- (5.6.2) X contains neither a copy of F_ω nor of W and for any $x \in X$ we have $\text{ord}(x, X) \leq n$;
- (5.6.3) X is an open image of G^n ;
- (5.6.4) X can be embedded in G^n in such a way that X is an open retract of G^n ;
- (5.6.5) X can be embedded in G^n in such a way that $E(X) \subset E(G^n)$;
- (5.6.6) X can be embedded in G^n ;
- (5.6.7) X is an open image of some dendrite Y with a closed set of end points and satisfying the condition $\text{ord}(x, Y) \leq n$ for any $x \in Y$;
- (5.6.8) X can be embedded in some dendrite Y with a closed set of end points and satisfying the condition $\text{ord}(x, Y) \leq n$ for any $x \in Y$.

6. Monotone images

In this section we establish some characterizations of monotone images of dendrites with a closed set of end points. The main result is that such dendrites are exactly the ones that do not contain any copy of W . Let us recall that a mapping is *monotone* if preimages of points are connected. For compact spaces a mapping is monotone if and only if preimages of connected sets are connected (see [11, (2.2), p. 138]).

Theorem 6.1. *If a dendrite Y is a monotone image of a dendrite X , and Y contains a copy of W , then so does X .*

Proof. Let a mapping $f : X \rightarrow Y$ be monotone. We assume, with no loss of generality, that $W \subset Y$. Choose points $p_k \in f^{-1}(a_k)$. Taking subsequence if necessary we may assume that $\{p_k\}_{k=1}^\infty$ is convergent and put $p = \lim_{k \rightarrow \infty} p_k$.

We claim that all points p_k are in the same component of $X \setminus \{p\}$. Really, for any $i, j \in \mathbb{N}$ there is a continuum $K \subset W$ such that $a_i, a_j \in K$, but $a \notin K$. Then $f^{-1}(K)$ is a continuum that contains p_i and p_j , but does not contain p .

Put $q_k = \inf_{\leq p} \{p_1, \dots, p_k\}$. Then the sequence $\{q_k\}_{k=1}^\infty$ converges to p , so we can choose a subsequence $\{q_{k_i}\}_{i=1}^\infty$ that satisfies $q_{k_{i+1}} <_p q_{k_i}$ for $i \in \mathbb{N}$. Then the irreducible continuum M that contains the points p_{k_1}, p_{k_2}, \dots is homeomorphic to W_R . The homeomorphism maps the point p_{k_i} onto a_i for $i \in \{0\} \cup \mathbb{N}$ and maps q_{k_i} onto b_i for $i \in \mathbb{N}$.

Next we claim that $\text{ord}(p, X) \geq 2$. Indeed, let C_1 be the component of $Y \setminus \{a\}$ that contains $\{a_k : k \in \mathbb{N}\}$, and let C_2 be another component of $Y \setminus \{a\}$. Then $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are connected and disjoint. Since $p \in \text{bd } f^{-1}(C_1)$ the difference $X \setminus \{p\}$ has at least two components, the one that contains $f^{-1}(C_1)$ and the other one that contains $f^{-1}(C_2)$. This finishes the proof of the claim.

Finally, let r be a point in a different component of $X \setminus \{p\}$ from the one that contains $\{p_k : k \in \mathbb{N}\}$. Then the union $rp \cup M$ is homeomorphic to W . The proof is finished. \square

To provide a characterization of monotone images of dendrites with a closed set of end points we will use inverse limits. Some definitions and notation are in order first.

An *inverse sequence* $\{X_n, f_n^m\}$ is a sequence of metric spaces X_n and bonding mappings $f_n^m : X_m \rightarrow X_n$ for $n, m \in \mathbb{N}$ and $n \leq m$, such that f_n^n is the identity on X_n and $f_n^m \circ f_m^k = f_n^k$ for $n \leq m \leq k$. We denote by $X_\infty = \varprojlim \{X_n, f_n^m\}$ the *inverse limit* of the inverse sequence $\{X_n, f_n^m\}$, i.e., $X_\infty = \{(x_1, x_2, \dots) \in X_1 \times X_2 \times \dots : f_n^m(x_m) = x_n \text{ for every } n, m \in \mathbb{N} \text{ with } n \leq m\}$. An element of the inverse limit is called a *thread*. The mapping $f_n : X_\infty \rightarrow X_n$ defined by $f_n(x_1, x_2, \dots) = x_n$ is called the *nth projection*. The reader is referred to Engelking’s monograph [4] for more information about this subject.

Lemma 6.2. *Let $\{X_n, f_n^m\}$ be an inverse sequence of dendrites X_n and monotone bonding mappings. Let (x_1, x_2, \dots) be a thread in X_∞ . If there is an integer k such that for every $n \in \mathbb{N}$ we have $\text{ord}(x_n, X_n) \leq k$, then $\text{ord}(x, X_\infty) \leq k$.*

Proof. First note that X_∞ is a dendrite by [8, 10.36, p. 180]. Assume on the contrary, that $\text{ord}(x, X_\infty) > k$. Then X_∞ can be represented as the union $Q_1 \cup \dots \cup Q_{k+1}$ of $k + 1$ nondegenerate continua satisfying $Q_i \cap Q_j = \{x\}$ for $i \neq j$. By the assumption there are two indices i_0 and i_1 such that $f_n(Q_{i_0}) \cap f_n(Q_{i_1})$ is a nondegenerate continuum for infinitely many, and consequently for all but finitely many, indices n . Define $P_n = f_n(Q_{i_0}) \cap f_n(Q_{i_1})$. One can show that $f_n^m(P_m) \subset P_n$ for any $m, n \in \mathbb{N}$ with $n \leq m$. Then $Q_{i_0} \cap Q_{i_1} = \varprojlim \{P_n, f_n^m|P_m\}$, so $Q_{i_0} \cap Q_{i_1}$ is a nondegenerate continuum contrary to the assumption. \square

Proposition 6.3. *If X is a dendrite not containing any copy of W , then*

(6.3.1) *the set $E(X) \cup R_\omega(X)$ is closed;*

(6.3.2) *for any $x \in X \setminus (E(X) \cup R_\omega(X))$, there is a closed neighborhood of x which is a tree.*

Proof. To prove (6.3.1) it is enough to show that for any sequence e_1, e_2, \dots of end points the limit belongs to $E(X) \cup R_\omega(X)$, and that for any sequence r_1, r_2, \dots of elements of $R_\omega(X)$ the limit also belongs to $E(X) \cup R_\omega(X)$. First take a sequence of end points. If $e = \lim_{n \rightarrow \infty} e_n$ is not an end point or a point of order ω , then there is a subsequence e_{n_1}, e_{n_2}, \dots that is contained in one component of $X \setminus \{e\}$. Then, arguing as in Theorem 3.1, we can show that the irreducible continuum containing all the points e_{n_1}, e_{n_2}, \dots contains a copy of W_R and, since e is not an end point, X contains a copy of W , a contradiction.

Second, take a sequence r_1, r_2, \dots of points of $R_\omega(X)$ converging to a point $r \notin E(X) \cup R_\omega(X)$. Because each $r_n \in R_\omega(X)$ we can take an end point e_n close enough to r_n such that $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} r_n$ and by the previous case we are done. Thus (6.3.1) is proven.

To show (6.3.2) take a closed connected neighborhood U of the point x that satisfies $U \cap (E(X) \cup R_\omega(X)) = \emptyset$. We will show that U is a tree. If not, since $R_\omega(U) = \emptyset$, the dendrite U contains a copy of W_R by Theorem 3.1. Since $U \cap E(X) = \emptyset$ we can add an arc in X to the copy of W_R to construct a copy of W . This finishes the proof. \square

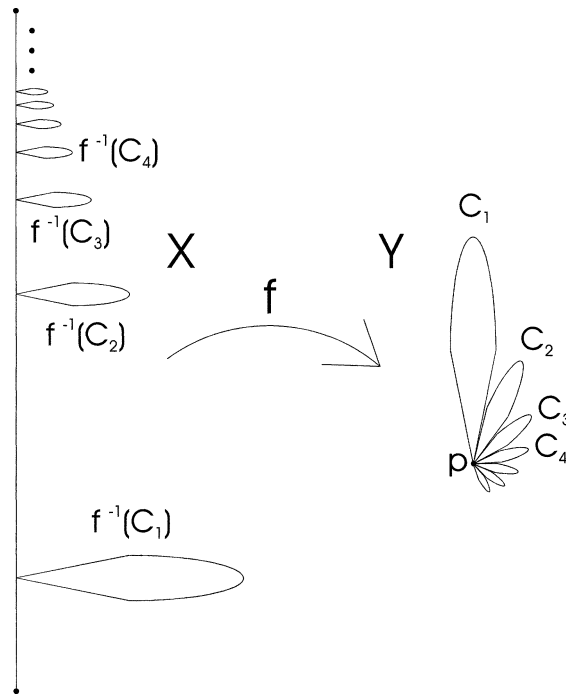


Fig. 4.

Now we will describe a monotone mapping between dendrites, called an *explosion*, that will be used in the proof of the next theorem.

Definition 6.4. Let a dendrite Y and a point $p \in R_\omega(Y)$ be given. By an *explosion of Y to a dendrite X at a point $p \in Y$* we mean a monotone mapping $f : X \rightarrow Y$ satisfying the following conditions (see Fig. 4):

- (6.4.1) $f^{-1}(p)$ is an arc whose end points are in $E(X)$;
- (6.4.2) for every $y \in Y \setminus \{p\}$ the preimage $f^{-1}(y)$ is a singleton;
- (6.4.3) if C_1 and C_2 are two different components of $Y \setminus \{p\}$, then $\text{cl } f^{-1}(C_1) \cap \text{cl } f^{-1}(C_2) = \emptyset$;
- (6.4.4) if C_1, C_2, \dots is a sequence of distinct components of $Y \setminus \{p\}$, then the sequence $f^{-1}(C_1), f^{-1}(C_2), \dots$ converges to an end point of the arc $f^{-1}(p)$.

One can observe that for any dendrite Y and any point $p \in R_\omega(Y)$ there is a dendrite X and an explosion of Y to X at p .

The following observation is a straightforward consequence of (6.4.2).

Observation 6.5. If $f : X \rightarrow Y$ is an explosion of Y to X at a point $p \in R_\omega(Y)$, then $f|(X \setminus f^{-1}(p))$ is a homeomorphism onto $Y \setminus \{p\}$.

Lemma 6.6. *If a dendrite Y does not contain any copy of W , and $f : X \rightarrow Y$ is an explosion of Y to X at a point $p \in R_\omega(Y)$, then X does not contain any copy of W as well.*

Proof. Suppose $W \subset X$. We will use the notation in the definition of W in Section 3. By Observation 6.5 we have $f(a) = p$. If $f(W_R)$ is in the closure of one component of $Y \setminus \{p\}$, then again by Observation 6.5 $f(W_R)$ is homeomorphic to W_R , and since $f(a) = p$ is not an end point of Y , we would have a copy of W in Y . Therefore $f^{-1}(p) \cap W_R$ is a nondegenerate arc, whence $f(a_n b_n)$ are in different components of $Y \setminus \{p\}$, for infinitely many $n \in \mathbb{N}$. Thus by (6.4.1) and (6.4.4), the point a is an end point of X , a contradiction. \square

Theorem 6.7. *If Y is a dendrite not containing a copy of W , then there is a dendrite X with a closed set of end points and a monotone surjective mapping $f : X \rightarrow Y$.*

Proof. We will construct the dendrite X as an inverse limit of a sequence of dendrites X_n with monotone bonding mappings $f_n^m : X_m \rightarrow X_n$.

Since $R_\omega(Y)$ is at most countable [6, §51, VI, Theorem 7, p. 302], we can write $R_\omega(Y) = \{p_1, p_2, \dots\}$ (the set can be finite). Put $X_1 = Y$ and let $f_1^2 : X_2 \rightarrow X_1$ be an explosion of X_1 to X_2 at p_1 . In general, let $f_n^{n+1} : X_{n+1} \rightarrow X_n$ be an explosion of X_n to X_{n+1} at the point $(f_1^n)^{-1}(p_n)$ if $R_\omega(Y)$ has at least n elements, or the identity of $X_{n+1} = X_n$ onto itself otherwise. Put $X = \varprojlim \{X_n, f_n^m\}$.

To show that $E(X)$ is closed it is enough, by Theorem 3.3, to prove that X does not contain a copy of F_ω nor of W .

Suppose on the contrary that there is a copy of F_ω in X , i.e., $R_\omega(X) \neq \emptyset$, and let $q \in R_\omega(X)$. Consider three cases.

Case 1. $f_1(q) \in E(X_1)$. Then, by the definition of the explosion, $f_n(q) \in E(X_n)$ for each $n \in \mathbb{N}$ and, by Lemma 6.2, we have $q \in E(X)$, a contradiction.

Case 2. $f_1(q) \notin E(X_1) \cup R_\omega(X_1)$. Then by (6.3.2) there is a closed neighborhood U of $f_1(q)$ in X_1 which is a tree. Thus, for any $n \in \mathbb{N}$, $(f_1^n)^{-1}(U)$ is homeomorphic to U , and consequently, $(f_1)^{-1}(U)$ is a neighborhood of q in X which is a tree, a contradiction again.

Case 3. $f_1(q) \in R_\omega(X_1)$. Then there is $n \in \mathbb{N}$ such that $f_1(q) = p_n$. Thus f_n^{n+1} is an explosion of X_n to X_{n+1} at $f_n(q)$, so $\text{ord}(f_{n+1}(q), X_{n+1}) \in \{1, 2, 3\}$ by condition (6.4.3). Hence, for every $m > n$ we have $\text{ord}(f_m(q), X_m) = \text{ord}(f_{n+1}(q), X_{n+1})$, and by Lemma 6.2, $\text{ord}(q, X) \leq 3$, which contradicts the assumption. This finishes the argument that $R_\omega(X) = \emptyset$.

Now we will show that X contains no copy of W . As before, assuming the contrary and using the notation as in the definition of W , we will consider three cases.

Case 1. $f_1(a) \in E(X_1)$. Then as before $a \in E(X)$, a contradiction.

Case 2. $f_1(a) \notin E(X_1) \cup R_\omega(X_1)$. Then arguing as before we can construct a neighborhood of a in X which is a tree, contradicting the assumption.

Case 3. $f_1(a) \in R_\omega(X_1)$. Then there is $n \in \mathbb{N}$ such that $f_1(a) = p_n$, and $\text{ord}(f_{n+1}(a), X_{n+1}) \in \{1, 2, 3\}$ by condition (6.4.3). Since X_{n+1} does not contain a copy of W by Lemma 6.6, there is a neighborhood U of $f_{n+1}(a)$ in X_{n+1} which is a tree. Thus $(f_{n+1})^{-1}(U)$ is a tree homeomorphic to U , which is a neighborhood of the point a in X , a contradiction. This finishes the proof of the theorem. \square

Proposition 6.8. *Let X be a dendrite such that $E(X)$ is uncountable. Then X contains a copy of G^3 .*

Proof. By [9, Theorem 1, p. 428] X contains an arc Z with uncountably many ramification points or it contains a Gehman dendroid (see [9, p. 423] for the definition). The first case is impossible since $R(X)$ is countable; in the second case X contains the Gehman dendrite, i.e., G^3 , by [10, Theorem 8, p. 212]. This ends the proof. \square

Proposition 6.9. *There is a monotone mapping from G^3 onto $G(\{3, 5, 9, 17, \dots, 2^n + 1, \dots\})$.*

Proof. Let $h: H^3 \rightarrow [0, 2]$ be the second coordinate function; in particular we have $h(p_{\alpha_1, \dots, \alpha_i}^3) = 1/2^i$. For $i \in \mathbb{N}$ define $B_i = h^{-1}([1/2^{i+1}, 1/2^i])$ and note that B_i has 2^{i-1} components.

Let m be a monotone mapping on H^3 that shrinks every component of $B_2, B_4 \cup B_5, B_7 \cup B_8 \cup B_9, B_{11} \cup B_{12} \cup B_{13} \cup B_{14}, \dots$ to a point. Then $m(p)$ has order 3, $m(p_0)$ and $m(p_1)$ have order 5, $m(p_{000}), m(p_{001}), m(p_{010}), \dots, m(p_{111})$ have order 9 and so on, thus $m(H^3)$ is homeomorphic to $H(\{3, 5, 9, 17, \dots\})$. Analogously we can define a monotone surjective mapping $m': (H^3)' \rightarrow H'(\{3, 5, 9, 17, \dots\})$. Then the union of the two mappings m and m' is a monotone mapping from G^3 onto $G(\{3, 5, 9, 17, \dots\})$. \square

The next corollary summarizes the results of this section.

Corollary 6.10. *The following conditions are equivalent for a dendrite X :*

- (6.10.1) X contains no copy of W ;
- (6.10.2) for every arc $ab \subset X$ the set $(ab \setminus \{a, b\}) \cap \text{cl } R(X)$ is discrete;
- (6.10.3) X is contained in P ;
- (6.10.4) X is a monotone image of a dendrite containing no copy of W ;
- (6.10.5) X is a monotone image of a dendrite with a closed set of end points;
- (6.10.6) X is a monotone image of $G(\mathbf{r})$ for every increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$;
- (6.10.7) X is a monotone image of $G(\mathbf{r})$ for some increasing sequence $\mathbf{r} = \{r_n\}_{n=1}^\infty$ with $r_1 \geq 3$;
- (6.10.8) X is a monotone image of G^3 ;
- (6.10.9) X is a monotone image of G^n for every $n \in \{3, 4, \dots\}$;
- (6.10.10) X is a monotone image of G^n for some $n \in \{3, 4, \dots\}$.

Proof. Condition (6.10.1) is equivalent to (6.10.2) by Observation 3.7. It is equivalent to (6.10.3) by Theorem 4.6 and to (6.10.4) by Theorems 6.1 and 6.7. Condition (6.10.1) implies (6.10.5) by Theorem 6.7. Condition (6.10.5) implies (6.10.6) by universality of $G(\mathbf{r})$ according to Theorem 4.2 and by the fact that each subdendrite of a dendrite is its monotone retract, see [5, Theorem, p. 157]. The implications from (6.10.6) to (6.10.7) and from (6.10.7) to (6.10.5) are trivial. Condition (6.10.6) implies (6.10.8) by Proposition 6.9. By the universality of G^n (Theorem 4.2) and again by [5, Theorem, p. 157] condition (6.10.8) implies (6.10.9). The implication from (6.10.9) to (6.10.10) is obvious, and (6.10.10) implies (6.10.1) by Theorem 6.1, because $G(\mathbf{r})$ has a closed set of end points, and so does not contain any copy of W by Theorem 3.3. The proof is complete. \square

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