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On self-homeomorphic spaces *

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Abstract

We investigate spaces in which a homeomorphic image of the whole space can be found in every open set. Such spaces are called *self-homeomorphic*. Some modifications of this notion are also studied.

Key words: Continuum; Dendrite; Self-homeomorphic

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1. Introduction

In this paper we investigate spaces in which small parts “look like” the whole space. In the literature such spaces were usually called *self-similar* but “look like” was meant in the metric sense. Because we are interested in topological properties, we discuss some formal topological definitions of “look like” and “small parts”, and we call spaces under consideration *self-homeomorphic*. We introduce four different types of self-homeomorphic spaces and investigate relationships between

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them, establishing implications between the definitions. We also give examples which show that none of the proven implications can be reversed.

Next we describe a method of constructing self-homeomorphic spaces as limits of convergent sequences $F(A), F^2(A), F^3(A), \dots$, where F is a mapping between hyperspaces of compact subsets, and A is any compact subset of a space X . The idea comes from [2, 3.7, pp. 80–85], and here we extend it in two ways. First, we consider construction of F as the union $\bigcup_{n=1}^{\infty} f_n(A)$, where $f_n : X \rightarrow X$ are any mappings, while in [2] only finite unions were considered. Second, we prove that under some conditions on the f_n the limit space is self-homeomorphic.

We next consider the structure of the set of local cut points of self-homeomorphic spaces. We show an example that in general this structure does not determine a self-homeomorphic space. Afterwards we study the set of points at which a self-homeomorphic space is pointwise self-homeomorphic. We prove that for continua this set is dense and uncountable, while in general it can be empty.

Special attention is given to self-homeomorphic dendrites. We introduce a special class of self-homeomorphic dendrites, namely those having points of all orders of ramification on every arc. These dendrites are generalizations of universal dendrites of a given order, so the results obtained generalize those for universal dendrites. In particular, their topological characterization is established.

We end the paper posing some problems.

Some basic definitions, which will be used throughout the paper, are necessary. By a continuum we mean a compact, connected metric space. A neighborhood of a point x in a topological space X is a set N satisfying $x \in \text{int } N$. A point $x \in X$ is called a cut point of X [9, III, p. 41] if $X \setminus \{x\}$ is not connected. A point $x \in X$ is called a local cut point of X provided there is a connected neighborhood N of x , such that x is a cut point of N . Note that a local cut point is a point of local connectedness of the space X . We will, however, mainly use the notion of a local cut point in locally connected spaces anyway. For connected sets, this definition of a cut point is equivalent to the definition of a separating point in [9, III, 8, p. 58], and the definition of a local cut point is equivalent to that of local separating point in [9, III, 9, p. 61].

Given a point x in a space X , we define the order of x in X as the number of arcs having x as their common endpoint and pairwise disjoint out of x . Points of order one are called endpoints and points of order three or more are called ramification points. This definition of order of a point does agree with the definition of order of a point in the sense of Menger–Urysohn (see e.g. [7, §51, I, p. 274]) for locally connected continua ([7, §51, I, 8. p. 277 and the remark below it]).

For two real numbers x and y the symbol $\langle x, y \rangle$ denotes the point in \mathbb{R}^2 with coordinates x and y , and (x, y) denotes an open interval of reals with endpoints x and y .

If p and q are points of an arcwise connected space, the symbol pq stands for the arc with endpoints p and q . An arc pq is said to be free in a topological space if $pq \setminus \{p, q\}$ is an open set. For continua, this is equivalent to the condition that pq does not contain any ramification points or limit points of its complement.

2. Definitions and basic properties of self-homeomorphic spaces

In this section we introduce four types of self-homeomorphic spaces and we discuss interrelations between them and some of their basic properties.

Definition 2.1. A topological space X is called *self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X .

Definition 2.2. A topological space X is called *strongly self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ with nonempty interior such that V is homeomorphic to X .

Definition 2.3. A topological space X is called *pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a set V such that $x \in V \subseteq U$ and V is homeomorphic to X . The space X is called *pointwise self-homeomorphic* if it is pointwise self-homeomorphic at each of its points.

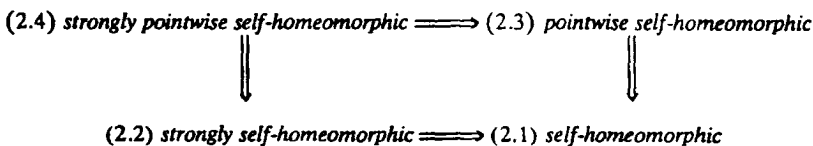
Definition 2.4. A topological space X is called *strongly pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a neighborhood V of x such that $x \in V \subseteq U$ and V is homeomorphic to X . The space X is called *strongly pointwise self-homeomorphic* if it is strongly pointwise self-homeomorphic at each of its points.

Recall that metric spaces X and Y are called *similar* if there is a surjection $f: X \rightarrow Y$ such that there is a constant c satisfying $d(f(x), f(y)) = cd(x, y)$. Such a map f is called a *similarity*. If we replace the condition that V is homeomorphic to X in each of the above definitions by the condition that V is similar to X , we get definitions of self-similar spaces. Thus every (strongly, pointwise or strongly pointwise) self-similar space is (strongly, pointwise or strongly pointwise) self-homeomorphic, but not conversely. Of course we could consider any other class of mappings besides homeomorphisms or similarities. For example, if we use affine transformations we get analogous definitions of self-affine spaces. However, homeomorphisms and similarities seem to be the most interesting ones.

An arc, an n -dimensional cube and a one-point union of two n -dimensional cubes with the identification point on the boundary of both of the n -dimensional cubes are examples of strongly pointwise self-homeomorphic continua.

We note some basic properties of self-homeomorphic spaces. Easy proofs are omitted.

Theorem 2.5. *The following diagram of implications applies to the above definitions.*



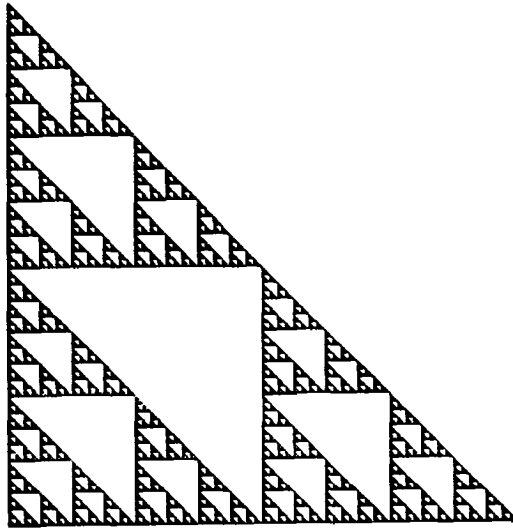


Fig. 1.

The examples below show that none of the implications in the above diagram can be reversed and that there is no relationship between strongly self-homeomorphic and pointwise self-homeomorphic spaces.

Example 2.6. The pseudo-arc can be defined as a chainable, hereditarily indecomposable continuum (see [3, Theorem 1, p. 44] for the proof that all pseudo-arcs are homeomorphic). Thus, every subcontinuum of the pseudo-arc is again a pseudo-arc. This property was first proven by Moise in [8, Theorem 13, p. 594]. Therefore, the pseudo-arc is an example of a continuum that is self-homeomorphic and pointwise self-homeomorphic but not strongly self-homeomorphic and not strongly pointwise self-homeomorphic. Hence neither of the implications $(2.4) \Rightarrow (2.3)$ and $(2.2) \Rightarrow (2.1)$ can be reversed. Also (2.3) does not imply (2.2) .

Example 2.7. The Sierpiński triangle (see Fig. 1 and the description in Example 3.9) is a strongly self-homeomorphic and pointwise self-homeomorphic, locally connected continuum. It does not have cut points, but it has local cut points. Hence it is not strongly pointwise self-homeomorphic and the implications $(2.4) \Rightarrow (2.3)$ and $(2.4) \Rightarrow (2.2)$ cannot be reversed. Moreover even (2.2) and (2.3) together do not imply (2.4) .

Example 2.8. There is a continuum which is strongly self-homeomorphic, but not pointwise self-homeomorphic. To construct the continuum let $a_n = \langle 1/n, 0 \rangle$ for $n \in \{1, 2, \dots\}$ and let $a_0 = \langle 0, 0 \rangle$. Denote by I_n the straight line segment with endpoints a_n and t , where $t = \langle 0, 1 \rangle$ and $n \in \{0, 1, \dots\}$. Then $X_1 = \bigcup_{i=0}^{\infty} I_n$ is the well-known harmonic fan (i.e., the cone over a harmonic sequence).

Now divide every arc ta_n , $n \in \{1, 2, \dots\}$ into two parts. Let t_n be the midpoint. Attach a harmonic fan with top t_n having arc $t_n a_n$ as the limit arc. Similarly, attach a harmonic fan with the top t and the arc tt_n as the limit arc. The obtained continuum is X_2 which is the second approximation of the continuum X . For the next approximation, divide each maximal free arc of X_2 into two equal parts and attach harmonic fans to each of them in the same manner. Then the closure $\text{cl}(\bigcup_{n=1}^{\infty} X_n)$ is the continuum X .

Observe that for any ramification point $r \in X$ there is a homeomorphic copy of X with the point r as the image of the point t under the homeomorphism, and that this copy has nonempty interior. This shows that X is a strongly self-homeomorphic continuum. On the other hand, small continua containing the point a_0 are arcs, so X is not pointwise self-homeomorphic. Hence (2.2) does not imply (2.3) and, together with Example 2.6, this shows that definitions (2.2) and (2.3) are not related.

Proposition 2.9. *A connected strongly pointwise self-homeomorphic space is locally connected.*

Note that none of the other types of self-homeomorphic continua are necessarily locally connected, as one can see in Examples 2.6 and 2.8. The next example is a continuum which is both pointwise self-homeomorphic and strongly self-homeomorphic, but not locally connected.

Example 2.10. Let X_1 be as in Example 2.8. Divide each arc ta_n , $n \in \{0, 1, \dots\}$ into two parts. Let t_n be the midpoint. Attach a harmonic fan with top t_n having arc $t_n a_n$ as the limit arc. Similarly, attach a harmonic fan with the top t and the arc tt_n as the limit arc. Note that this example is different from Example 2.8 in that harmonic fans are attached on the limit arcs as well as the free arcs. The obtained continuum is X_2 which is the second approximation of the continuum X . For the next approximation, divide each maximal arc without ramification points of each harmonic fan in X_2 into two equal parts and attach harmonic fans in the same manner. Then the closure $\text{cl}(\bigcup_{n=1}^{\infty} X_n)$ is the continuum X .

We close this section with an easy proposition on Cartesian products.

Proposition 2.11. *All four types of self-homeomorphic spaces are preserved under Cartesian products (of arbitrarily many factors).*

The converse of Proposition 2.11 is not true. If a Cartesian product is (strongly, pointwise or strongly pointwise) self-homeomorphic, then the factor space need not be self-homeomorphic. As an example, let X be a simple triod and let Y be a Hilbert cube. Then $X \times Y$ is the Hilbert cube again, so it is strongly pointwise self-homeomorphic, while X is not self-homeomorphic.

3. Methods of constructing self-homeomorphic spaces

In this section we describe a method of constructing self-homeomorphic spaces. The method comes from [2, 3.7, pp. 80–85] and here we discuss it in details. Namely, for any given compact metric space X and for a set of contractive self-mappings $\{f_i; i \in I\}$ we define a mapping $F: 2^X \rightarrow 2^X$ by $F(A) = \text{cl}(\cup\{f_i(A); i \in I\})$. The mapping F has the only fixed set. Conditions are presented under which this fixed set is self-homeomorphic, or strongly self-homeomorphic or pointwise self-homeomorphic. Also some examples are described.

Let us recall some common notations. For a given compact metric set X and a positive number r we put $N(A, r) = \{x \in X: \text{there exists } a \in A \text{ with } d(a, x) < r\}$.

For two nonempty compact subsets A and B of X we define the Hausdorff distance $\text{dist}(A, B) = \inf\{r > 0: A \subseteq N(B, r) \text{ and } B \subseteq N(A, r)\}$, and we denote by 2^X the hyperspace of all nonempty compact subsets of X equipped with the Hausdorff distance.

If $f: X \rightarrow X$ is a mapping, then $\hat{f}: 2^X \rightarrow 2^X$ denotes the induced mapping defined by $\hat{f}(A) = f(A)$.

Definition 3.1. Let (X, d) be a metric space. A map $f: X \rightarrow X$ is called *contractive* if there is a constant s satisfying $0 \leq s < 1$ such that $d(f(x), f(y)) \leq sd(x, y)$ for every $x, y \in X$. Any such number s is called a *contractivity factor* for f .

Theorem 3.2. Let X be a compact metric space and let I denote any set. For any $i \in I$ let $w_i: 2^X \rightarrow 2^X$ be a contractive mapping and assume all w_i have a common contractivity factor $s < 1$. Define a mapping $F: 2^X \rightarrow 2^X$ by $F(A) = \text{cl}(\cup\{w_i(A); i \in I\})$. Then F is contractive with contractivity factor s .

Proof. Take $A, B \in 2^X$. We have to show that $\text{dist}(F(A), F(B)) \leq s \text{dist}(A, B)$. By assumption it is enough to show that $F(B) \subseteq N(F(A), s \text{dist}(A, B) + \epsilon)$ for any $\epsilon > 0$.

Take a point $x \in F(B)$. By the definition there is a sequence x_k in $\cup\{w_i(B); i \in I\}$ with $x = \lim x_k$. Denote by $i(k)$ the element of I such that $x_k \in w_{i(k)}(B)$ and observe that since $w_{i(k)}$ is contractive we have $x_k \in N(w_{i(k)}(A), s \text{dist}(A, B) + \epsilon/2) \subseteq N(F(A), s \text{dist}(A, B) + \epsilon/2)$. Taking the limit for $k \rightarrow \infty$ we get $x \in N(F(A), s \text{dist}(A, B) + \epsilon)$. \square

Theorem 3.3. Let X be a compact metric space, and let I denote any set. For any $i \in I$ let $f_i: X \rightarrow X$ be a contractive mapping and assume all f_i have a common contractivity factor $s < 1$. Define a mapping $F: 2^X \rightarrow 2^X$ by

$$F(A) = \text{cl}\left(\cup\{\hat{f}_i(A); i \in I\}\right). \tag{3.3.1}$$

Then F is contractive with contractivity factor s .

Proof. Note that for each $i \in I$ the mappings \hat{f}_i are contractive with contractivity factor s by [2, Lemma 3, p. 80]. Putting $w_i = \hat{f}_i$ in the previous theorem we get the conclusion. \square

For any given compact metric space X and a set of contractive mappings $\{f_i: i \in I\}$ having common contractivity factor $s < 1$, let $F: 2^X \rightarrow 2^X$ be defined by (3.3.1). According to the Banach Contraction Mapping Theorem (see e.g. [2, Theorem 1, p. 76]) there is exactly one fixed set under F , i.e., a set $A \in 2^X$ such that $F(A) = A$. Such a set A can be obtained as the limit $\text{Lim } F^n(B)$ for any $B \in 2^X$. Then we will write $A = \mathcal{F}(X, \{f_i: i \in I\})$.

Now we present some conditions on mappings f_i under which the set $\mathcal{F}(X, \{f_i: i \in I\})$ is self-homeomorphic.

Theorem 3.4. *If the mappings $f_i: X \rightarrow X$ for $i \in I$ are embeddings, then $\mathcal{F}(X, \{f_i: i \in I\})$ is self-homeomorphic.*

Proof. Let $A = \mathcal{F}(X, \{f_i: i \in I\})$ and fix an open in A set $U \subseteq A$. Note that $f_i(A)$ is homeomorphic to A for any $i \in I$, and that $\cup\{f_i(A): i \in I\}$ is dense in A . By induction the set $f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots))$ is homeomorphic to A and has its diameter not greater than $s^n \text{diam}(A)$. Moreover, their union, i.e., $\cup\{f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots)): i_1, i_2, \dots, i_n \in I\}$ is dense in A . If n is sufficiently large one of the sets is contained in U . This finishes the proof. \square

Theorem 3.5. *If the mappings $f_i: X \rightarrow X$ for $i \in I$ are embeddings and satisfy $\text{int } f_i(X) \neq \emptyset$ and $\text{int } f_i(X) \cap \text{int } f_j(X) = \emptyset$ for $i, j \in I$ with $i \neq j$, then $\mathcal{F}(X, \{f_i: i \in I\})$ is strongly self-homeomorphic.*

Proof. To prove the theorem it is enough to observe that the sets $f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots))$ considered in the proof of Theorem 3.4 have nonempty interiors. \square

Theorem 3.6. *If the mappings $f_i: X \rightarrow X$ for $i \in I$ are embeddings and the set I is finite, then $\mathcal{F}(X, \{f_i: i \in I\})$ is pointwise self-homeomorphic.*

Proof. Let $A = \mathcal{F}(X, \{f_i: i \in I\})$, and fix a point $x \in A$ and an open (in A) set $U \subseteq A$. Note that, since I is finite, we have $A = \cup\{f_i(A): i \in I\}$. Choose $i_1 \in I$ such that $x \in f_{i_1}(A)$. Similarly, we can choose a sequence i_1, i_2, \dots of elements of I such that $x \in f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots))$ and the considered sets are homomorphic to A and have their diameters tending to zero. So there is a number n such that $f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots))$ is contained in U . This concludes the proof. \square

The assumption of finiteness of the set I is essential in the above result as it will be seen by the following example. The constructed continuum is the same as the one described in Example 2.8.

Example 3.7. Let X of Theorem 3.5 be the triangle with vertices $a = \langle 0, 0 \rangle$, $c = \langle 1, 0 \rangle$ and $v = \langle 0, 1 \rangle$ in the plane. Denote by a_n the point $\langle 2^{-n}, 0 \rangle$ and by b_n the point $\langle 2^{-n-1}, \frac{1}{2} \rangle$. Thus the points a_n, b_n and v are colinear. Similarly define $c_n = \langle 3 \cdot 2^{-n-1}, 0 \rangle$ and $d_n = \langle 3 \cdot 2^{-n-2}, \frac{1}{2} \rangle$, so the points v, c_n, d_n are colinear and, for $n_1 \neq n_2$ the triangles $a_{n_1}, b_{n_1}, c_{n_1}$ and $a_{n_2}, b_{n_2}, c_{n_2}$ are disjoint, triangles $a_{n_1}, b_{n_1}, c_{n_1}$ and v, b_{n_2}, d_{n_2} are disjoint, and triangles v, b_{n_1}, d_{n_1} and v, b_{n_2}, d_{n_2} have only the point v in common. Denote by f_{2n} , for $n \in \{1, 2, \dots\}$, a linear homeomorphism of X onto the triangle a_n, b_n, c_n such that $f_{2n}(v) = b_n, f_{2n}(a) = a_n$ and $f_{2n}(c) = c_n$. Similarly define f_{2n-1} as a linear homeomorphism of X onto the triangle v, b_n, d_n satisfying $f_{2n-1}(v) = v, f_{2n-1}(a) = b_n$ and $f_{2n-1}(c) = d_n$. Then the continuum $A = \mathcal{F}(X, \{f_i; i \in I\})$ is the one described in Example 2.8. Now it follows from Theorem 3.5 that A is strongly self-homeomorphic. This also shows that the assumption of Theorem 3.6 that I is finite is essential there.

Theorem 3.8. *Under the assumptions of Theorem 3.6 the set $\mathcal{F}(X, \{f_i; i \in I\})$ either is a locally connected continuum, or is not connected.*

Proof. Assume $A = \mathcal{F}(X, \{f_i; i \in I\})$ is connected. Then, for a fixed $n \in \mathbb{N}$, $A = \cup \{f_{i_n}(f_{i_{n-1}}(\dots f_{i_1}(A)\dots)); i_1, \dots, i_n \in I\}$ is a finite union of continua of diameter not greater than $s^n \text{diam}(A)$. Because $s^n \text{diam}(A)$ can be arbitrarily small for sufficiently large n , the conclusion is a consequence of the Sierpiński characterization of locally connected continua, see e.g. [9, I, (15.7), p. 23]. \square

To illustrate how to apply the above theorems we show constructions of the Sierpiński triangle and the Sierpiński universal plane curve.

Example 3.9. To define $f_n : I^2 \rightarrow I^2$ for $n \in \{1, 2, 3\}$ divide I^2 into four isometric squares. Denote any three of these squares as A_1, A_2, A_3 . Let f_1, f_2 and f_3 be standard linear homeomorphisms of I^2 onto A_1, A_2 and A_3 respectively. Then the set $\mathcal{F}(I^2, \{f_1, f_2, f_3\})$ is homeomorphic to the Sierpiński triangle (see Fig. 1). It is strongly self-homeomorphic by Theorem 3.5 and pointwise self-homeomorphic by Theorem 3.6 and it is not strongly pointwise self-homeomorphic as observed in Example 2.7.

Example 3.10. Divide I^2 into nine isometric squares, and define $g_n : I^2 \rightarrow B_n$, where B_1, \dots, B_8 are all of the squares except the middle one as standard linear homeomorphisms. Then $\mathcal{F}(I^2, \{g_1, \dots, g_8\})$ is the Sierpiński universal plane curve. It is strongly self-homeomorphic and pointwise self-homeomorphic by Theorems 3.5 and 3.6 respectively, but it is also strongly pointwise self-homeomorphic as a consequence of the Whyburn characterization theorem (see [10, Theorem 3, p. 322] or Theorem A at the beginning of Section 5 here).

4. Points at which a space is pointwise self-homeomorphic

In this section we study sets of points at which a given space X is pointwise self-homeomorphic or strongly pointwise self-homeomorphic. We denote these sets

by $P(X)$ and $PS(X)$ respectively. Note that $PS(X) \subseteq P(X)$. We start with the following theorem.

Theorem 4.1. *Let X be a compact self-homeomorphic space. Then $P(X)$ is an uncountable dense set.*

Proof. First we show that $P(X)$ contains a Cantor set, so it is uncountable. By the definition of self-homeomorphic spaces there are two disjoint homeomorphic copies X_0 and X_1 of X with $\text{diam}(X_i) < 1$ for $i \in \{0, 1\}$. Since X_0 and X_1 are homeomorphic to X , there are disjoint subsets X_{00}, X_{01}, X_{10} and X_{11} such that, for $i_1, i_2 \in \{0, 1\}$, each $X_{i_1 i_2} \subseteq X_{i_1}$, $\text{diam}(X_{i_1 i_2}) < \frac{1}{2}$ and $X_{i_1 i_2}$ is homeomorphic to X . Continuing in this manner we can define by induction sets $X_{i_1 i_2 \dots i_k}$, where $i_j \in \{0, 1\}$, for $j \leq k$, and $k \in \{1, 2, \dots\}$ which are homeomorphic to X and satisfy the following conditions:

$$X_{i_1 i_2 \dots i_{k+1}} \subseteq X_{i_1 i_2 \dots i_k}, \tag{4.1.1}$$

$$X_{i_1 i_2 \dots i_k 0} \cap X_{i_1 i_2 \dots i_k 1} = \emptyset, \tag{4.1.2}$$

$$\text{diam}(X_{i_1 i_2 \dots i_k}) < \frac{1}{k}. \tag{4.1.3}$$

Then, for any infinite sequence i_1, i_2, \dots the intersection $\bigcap_{k=1}^{\infty} X_{i_1 i_2 \dots i_k}$ is a one-point set (because of the compactness of X). Denote the union of all those intersections by A . The space X is pointwise self-homeomorphic at each point of A . The set A is homeomorphic to the Cantor set, so it is uncountable. Also, because we can start our construction in an arbitrarily small open set U , we can find such a set $A \subseteq U$, for any U . Therefore $P(X)$ is dense and the proof is complete. \square

The next example shows that the assumption of compactness is essential in the above theorem.

Example 4.2. There is a strongly self-homeomorphic space which is not pointwise self-homeomorphic at any point.

Proof. Let $a_n = \langle 1/n, 0 \rangle$, for $n \in \{1, 2, \dots\}$, and let $a_0 = \langle 0, 0 \rangle$. Denote by I_n the straight line segment with endpoints a_n and t , where $t = \langle 0, 1 \rangle$ and $n \in \{0, 1, \dots\}$. Then $X_1 = \bigcup_{n=0}^{\infty} I_n$ is the well-known harmonic fan. Let t_n , for $n \in \{1, 2, \dots\}$ be the midpoint of the segment I_n . Attach a harmonic fan with the top a_n and having the arc $t_n a_n$ as the limit arc. Similarly, attach a harmonic fan with the top t_n and having the arc $t t_n$ as the limit arc. The obtained continuum is X_2 . For the next approximation divide each maximal free arc of X_2 into two equal parts and attach harmonic fans to every such a part in the same manner, i.e., in the opposite direction than harmonic fans were attached in the previous step. The obtained continuum is X_3 . We construct X_4, X_5, \dots analogously, and the union $\bigcup_{n=1}^{\infty} X_n$ is the space X . It is connected, but not compact. It is strongly self-homeomorphic by

its construction. To show $P(X) = \emptyset$ choose any point $p \in X$. If p is not a ramification point, then small connected sets containing p are contained in an arc. If p is a ramification point, then small connected sets are contained in a harmonic fan with the top p . In any case it is not homeomorphic to X . \square

Now we will prove, for strongly self-homeomorphic spaces, a stronger version of Theorem 4.1.

Theorem 4.3. *Let X be a strongly self-homeomorphic space. Then $PS(X)$ is a residual G_δ -set. In particular, if X is topologically complete, then $PS(X)$ is a dense G_δ -set.*

Proof. Define $D_n(X)$, for $n \in \{1, 2, \dots\}$, to be the set of all points x of X such that there is a homeomorphic copy A of X with $x \in \text{int} A \subseteq X$, and $\text{diam}(A) < 1/n$. Thus $D_m(X) \subseteq D_n(X)$ for $m > n$ and every D_n is an open set in X . Moreover, by the definition of a strongly self-homeomorphic space, $D_n(X)$ is dense for every n . Now observe that $PS(X) = \bigcap_{n=1}^\infty D_n(X)$, so $PS(X)$ is a residual G_δ -set and the proof is complete. \square

We remark that because of the inclusion $PS(X) \subseteq P(X)$ we have the following.

Corollary 4.4. *If the space X is topologically complete and strongly self-homeomorphic, then $P(X)$ is a residual set in X .*

Example 4.5. There is a self-homeomorphic continuum X such that $P(X)$ is an F_σ -set of the first category.

Proof. In order to construct the example we need to recall the following. If X is any continuum and $p \in X$ we can replace p by any continuum Y . Precisely, there is a continuum Z and a map $f: Z \rightarrow X$ such that $f^{-1}(p) = Y$, Y is nowhere dense in Z , and $f^{-1}(q)$ is a one-point set for $q \neq p$ [1, Theorem, p. 35].

Denote by F the Cantor fan, i.e., the cone over the Cantor set. Let $\{a_1, a_2, \dots\}$ be a dense subset of F that does not include the top of F . Now replace a_1 by a Hilbert cube H_1 as in the statement above, so H_1 is nowhere dense in the new continuum X_1 , and let $f_1: X_1 \rightarrow F$ be the required mapping. Next, replace $f_1^{-1}(a_2)$ by a Hilbert cube H_2 which is nowhere dense in the new continuum X_2 , and define $f_2: X_2 \rightarrow X_1$ as above. Continuing this process we can define an inverse sequence $F \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \dots$ such that the inverse image of a point under f_n is a point or a Hilbert cube. Let X be the limit of the inverse sequence. We can visualize X as replacing each point of the set $\{a_1, a_2, \dots\}$ by a Hilbert cube, which gets smaller and smaller, are nowhere dense, and do not intersect.

The continuum X is self-homeomorphic because each open set contains a Hilbert cube which, as a universal continuum, contains a homeomorphic copy of any continuum, in particular of X itself.

On the other hand, $P(X)$ is a union of countably many nowhere dense Hilbert cubes, so it is an F_σ -set of the first category. \square

5. Structure of the set of local cut points

Let us start recalling the Whyburn characterization theorem for the Sierpiński universal plane curve [10, Theorem 3, p. 322].

Theorem A (Whyburn). *A locally connected one-dimensional plane continuum with no local cut points is homeomorphic to the Sierpiński universal plane curve.*

The above theorem leads one to ask whether the structure of the sets of local cut points completely determines any self-homeomorphic plane curve. We answer the question negatively by showing two curves constructed using the procedure described in Section 3 with the same structure of the set of local cut points.

Let us precisely state the needed definition. Denote by $LC(X)$ the set of all local cut points of a space X .

Definition 5.1. We say that two continua X and Y have the same structure of the set of local cut points if there is a homeomorphism $\alpha : LC(X) \rightarrow LC(Y)$ such that a set $A \subseteq LC(X)$ disconnects X if and only if $\alpha(A)$ disconnects Y and the number of components of $X \setminus A$ is the same as the number of components of $Y \setminus \alpha(A)$.

Example 5.2. There are two nonhomeomorphic plane curves U and H having the same structure of the sets of local cut points.

Proof. To construct U divide I^2 into nine isometric squares. Consider all of the squares except the middle one and one above the middle one. Denote these seven squares by A_1, A_2, \dots, A_7 . Then $\bigcup_{n=1}^7 A_n$ forms the letter U . Let $f_n : I^2 \rightarrow A_n$, for $n \in \{1, 2, \dots, 7\}$, be the standard linear homeomorphism of I^2 onto A_n . Then $U = \mathcal{F}(I^2, \{f_1, f_2, \dots, f_7\})$ (see Fig. 2).

Similarly, divide I^2 into nine isometric squares and let B_1, B_2, \dots, B_7 be all of the squares except the middle upper square and the middle lower one, so $\bigcup_{n=1}^7 B_n$ forms the letter H . As before, let $g_n : I^2 \rightarrow B_n$ be the standard linear homeomorphism of I^2 onto B_n . Put $H = \mathcal{F}(I^2, \{g_1, g_2, \dots, g_7\})$ (see Fig. 3). Then U and H are strongly self-homeomorphic and pointwise self-homeomorphic curves by Theorems 3.5 and 3.6 respectively. We will show that they have the same structures of the sets of local cut points and that they are not homeomorphic.

We start with the description of the homeomorphism α as in Definition 5.1, which can be uniquely determined by the following conditions:

$$\text{if } \alpha(\langle x, y \rangle) = \langle p, q \rangle, \text{ then } x = p, \tag{5.2.1}$$

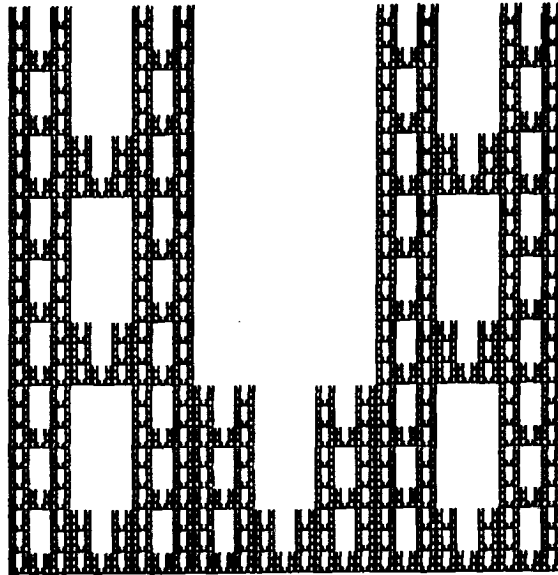


Fig. 2.

and

if $y_1 < y_2$ and $\alpha(\langle x, y_1 \rangle) = \langle x, q_1 \rangle$, $\alpha(\langle x, y_2 \rangle) = \langle x, q_2 \rangle$,

then $q_1 < q_2$. (5.2.2)

One can check that such defined α is a homeomorphism of $LC(U)$ onto $LC(H)$ and satisfies the required conditions.

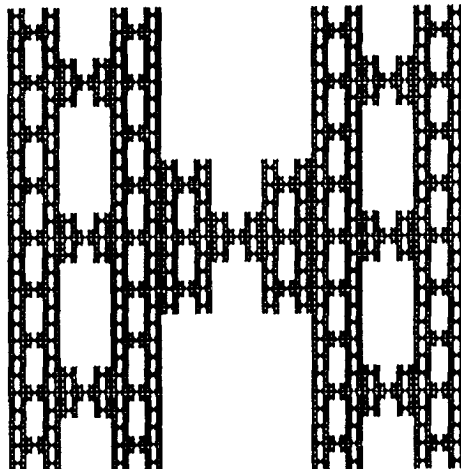


Fig. 3.

To see that U and H are topologically distinct assume there is a homeomorphism $h:U \rightarrow H$. Denote by A_x the set of all points of U having the first coordinate equal to x , i.e., $A_x = U \cap (\{x\} \times [0, 1])$. Similarly, $B_x = H \cap (\{x\} \times [0, 1])$. First observe that $A_{1/2}$ and $B_{1/2}$ are one-point sets consisting of a cut point only, so $h(A_{1/2}) = B_{1/2}$. Now look at $A_{1/6}$. It is a three-point set with the property that the component of $U \setminus A_{1/6}$ that does not contain $A_{1/2}$ contains no three-point set disconnecting U . There are two subset of H with the above property: $B_{1/6}$ and $B_{5/6}$. Because of the symmetry of H we can choose $h(A_{1/6}) = B_{1/6}$. Since $A_{1/2} \cup A_{1/6}$ separates U into three components and $h(A_{1/2}) = B_{1/2}$ and $h(A_{1/6}) = B_{1/6}$, we have $h(U \cap ([\frac{1}{6}, \frac{1}{2}] \times [0, 1])) = H \cap ([\frac{1}{6}, \frac{1}{2}] \times [0, 1])$. Further, $A_{7/18}$ is the only three-point subset of $U \cap ([\frac{1}{6}, \frac{1}{2}] \times [0, 1])$ with the property that there is a component of $U \cap ([\frac{1}{6}, \frac{1}{2}] \times [0, 1]) \setminus A_{7/18}$ containing no three-point set disconnecting U . Thus $h(A_{7/18}) = B_{7/18}$. We can continue this process to prove that $h(A_x) = B_x$ whenever A_x is a three-point set disconnecting U .

Now we consider nine-point sets which disconnect U . Among them $A_{1/18}$ and $A_{17/18}$ are the only two with the property that there is a component of $U \setminus A_{1/18}$ or $U \setminus A_{17/18}$ containing no nine-point set disconnecting U . This fact together with the previously established images give that $h(A_{1/18}) = B_{1/18}$ and $h(A_{17/18}) = B_{17/18}$. Continuing in this manner, we see that $h(A_x) = B_x$ for any A_x which is a nine-point set. We can repeat the above argument for 27-point sets, 81-point sets, and in general for each set A_x containing 3^n points, for $n \in \{1, 2, \dots\}$, proving that $h(A_x) = B_x$ for all such A_x . Because $\{x: A_x \text{ contains } 3^n \text{ points for some natural number } n\}$ is dense in $[0, 1]$, we have, by continuity of h , that $h(A_x) = B_x$ for all $x \in [0, 1]$.

Now $A_{1/3} = \{\frac{1}{3}\} \times [0, 1]$ and because $h(A_{1/3}) = B_{1/3}$, we have $h(\langle \frac{1}{3}, 0 \rangle) = \langle \frac{1}{3}, 0 \rangle$ or $h(\langle \frac{1}{3}, 0 \rangle) = \langle \frac{1}{3}, 1 \rangle$. Choose $x_n \rightarrow \frac{1}{3}$ such that $x_n > \frac{1}{3}$ and A_{x_n} is the whole interval $\{x_n\} \times [0, \frac{1}{3}]$. Then endpoints of A_{x_n} must be mapped onto endpoints of B_{x_n} , so $h(\langle x_n, 0 \rangle) = \langle x_n, \frac{1}{3} \rangle$ or $h(\langle x_n, 0 \rangle) = \langle x_n, \frac{2}{3} \rangle$. Hence $\lim_{x \rightarrow \frac{1}{3}} h(\langle x_n, 0 \rangle) = \langle \frac{1}{3}, \frac{1}{3} \rangle$ or $\lim_{x \rightarrow \frac{1}{3}} h(\langle x_n, 0 \rangle) = \langle \frac{1}{3}, \frac{2}{3} \rangle$. In either case $\lim_{x \rightarrow \frac{1}{3}} h(\langle x_n, 0 \rangle)$ is not equal to $h(\langle \frac{1}{3}, 0 \rangle)$. Thus h is not continuous. This contradiction finishes the proof. \square

One can show that neither of the two curves is strongly pointwise self-homeomorphic. The points at which they are not are common points of four copies of U or H . For example, the point $\langle \frac{1}{3}, \frac{2}{9} \rangle$ in U or $\langle \frac{1}{3}, \frac{4}{9} \rangle$ in H . This leads to the question of whether an example analogous to Example 5.2 can be constructed which is strongly pointwise self-homeomorphic. More precisely we have the following problem.

Problem 5.3. Assume X and Y are two strongly pointwise self-homeomorphic, planar, one-dimensional continua with the same structure of the sets of local cut points. Do X and Y have to be homeomorphic?

Observe that an argument analogous to the one above shows that the only two homeomorphisms of U onto itself are the identity and the reflection about the vertical line containing the cut point. If we consider half of the space U , namely

$U \cap ([0, \frac{1}{2}] \times [0, 1])$, then we get the example of a strongly self-homeomorphic, pointwise self-homeomorphic continuum with the identity as the only autohomeomorphism. This shows that self-homeomorphic continua can be very different from homogeneous continua.

6. Dendrites

In this section we discuss self-homeomorphic dendrites. We introduce a subclass of dendrites of special interest, those having all orders of ramification points on every arc. These dendrites are generalizations of universal dendrites of a given order.

By a dendrite we mean a locally connected continuum containing no simple closed curves. It is known that every subcontinuum of a dendrite is a dendrite, so dendrites are hereditarily locally connected. For a dendrite X , the order of a point $x \in X$ is the number of components of $X \setminus \{x\}$. It is denoted $\text{ord}(x)$. If there are infinitely many components of $X \setminus \{x\}$ we say $\text{ord}(x) = \omega$, where $\omega > n$ for every natural number n . The definition of order of a point in dendrites coincides with the Menger–Urysohn concept of order [7, §51, I, p. 274], and with the definition of order of the point x as the number of arcs intersecting exactly in their common endpoint x . Points of order one are called endpoints, and points of order three or more are called ramification points.

To prove the next theorem we need the following lemma.

Lemma 6.1. *Let A_1, A_2, \dots and B_1, B_2, \dots be two families of subsets of $(0, 1)$ such that:*

$$\text{for each } i \in \{1, 2, \dots\} \text{ the sets } A_i \text{ and } B_i \text{ are both countable dense subsets of } (0, 1) \text{ or are both empty,} \tag{6.1.1}$$

and

$$\text{for } i \neq j \text{ we have } A_i \cap A_j = \emptyset = B_i \cap B_j. \tag{6.1.2}$$

Then there is a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ with $h(A_i) = B_i$ for each i .

Proof. For completeness we sketch a standard zig-zag proof of the lemma. Let $\bigcup_{i=1}^{\infty} A_i = A = \{a_1, a_2, \dots\}$ and $\bigcup_{i=1}^{\infty} B_i = B = \{b_1, b_2, \dots\}$. We will rearrange the order of points in A and B in such a way that

$$A = \{a'_1, a'_2, \dots\} \quad \text{and} \quad B = \{b'_1, b'_2, \dots\}, \tag{6.1.3}$$

$$a'_m < a'_n \iff b'_m < b'_n, \tag{6.1.4}$$

$$a'_m \in A_i \iff b'_m \in B_i. \tag{6.1.5}$$

We start with $a'_1 = a_1$ and $b'_1 = b_1$. Then let $b'_2 = b_2$ and let a'_2 be the first point in the sequence a_1, a_2, \dots satisfying (6.1.4) and (6.1.5) for $m, n \leq 2$. This is possible because of the density properties of the A_i . Assume we have defined $a'_1, a'_2, \dots, a'_{2k}$ and $b'_1, b'_2, \dots, b'_{2k}$ satisfying (6.1.4) and (6.1.5). Let a'_{2k+1} be the first point in the

sequence a_1, a_2, \dots not in the set $\{a'_1, a'_2, \dots, a'_{2k}\}$ and let b'_{2k+1} be the first point in the sequence b_1, b_2, \dots satisfying (6.1.4) and (6.1.5) for $m, n < 2k + 1$. Then b'_{2k+2} is the first point in the sequence b_1, b_2, \dots not in the set $\{b'_1, b'_2, \dots, b'_{2k+1}\}$ and a'_{2k+2} is the first point in the sequence a_1, a_2, \dots satisfying (6.1.4) and (6.1.5) for $m, n \leq 2k + 1$. So a'_n and b'_n are defined for all n . Now let $h: [0, 1] \rightarrow [0, 1]$ be the only homeomorphism satisfying $h(a'_n) = b'_n$. This finishes the proof of the lemma. \square

The following theorem generalizes a fact observed in [5, (6), p. 490] for the case when A is a one-point set.

Theorem 6.2. *Let A be a subset of $\{3, 4, \dots, \omega\}$. Let X and Y be the two dendrites such that:*

$$\text{if } a \text{ is a ramification point of } X \text{ or } Y, \text{ then } \text{ord}(a) \in A, \tag{6.2.1}$$

and

$$\text{for any arc } I \text{ contained in } X \text{ or } Y, \text{ and for any } n \in A \text{ there} \\ \text{is a ramification point } a \in I \text{ with } \text{ord}(a) = n. \tag{6.2.2}$$

Then X and Y are homeomorphic. Moreover, the homeomorphism $h: X \rightarrow Y$ can be defined in such a way that for any distinct endpoints $e_1, e_2 \in E(X)$ and $f_1, f_2 \in E(Y)$ we have $h(e_1) = f_1$ and $h(e_2) = f_2$.

Proof. First observe that the condition (6.2.2) and the fact that X , as a hereditarily locally connected continuum, does not contain any continuum of convergence, imply that the set $E(X)$ of endpoints of X is dense in X . Let $\{e_1, e_2, \dots\}$ be a countable dense subset of $E(X)$, and similarly, let $\{f_1, f_2, \dots\}$ be a countable dense subset of $E(Y)$. We will construct a homeomorphism $h: X \rightarrow Y$. First, let $h(e_1) = f_1$ and $h(e_2) = f_2$. Let I_1 be the arc e_1e_2 . By Lemma 6.1 there is a homeomorphism $h: e_1e_2 \rightarrow f_1f_2$ such that the order of any ramification point $r \in e_1e_2$ in X is equal to the order of the image $h(r)$ in Y . Also define $J_1 = f_1f_2$. Now let J_2 be the irreducible arc between f_3 and J_1 . Let $h^{-1}(f_3) = e_3$ and let s_1 be the only common point of J_2 and J_1 . Define $r_1 = h^{-1}(s_1)$ and $I_2 = e_3r_1$. Note that, by the construction of the homeomorphism h on e_1e_2 , we have $\text{ord}(r_1) = \text{ord}(s_1)$. Now we use the lemma again to construct the homeomorphism h on I_2 so that h sends ramification points of I_2 onto ramification points of the same order on J_2 . The description of the next step is necessary. Let I_3 be the irreducible arc between e_4 and $I_1 \cup I_2$, and let r_2 be the only common point of I_3 and $I_1 \cup I_2$. The position of r_2 can vary. We can have $r_2 = r_1$ or $r_2 \in r_1e_i$ for some $i \in \{1, 2, 3\}$. Let $s_2 = h(r_2)$ (h having already been defined on $I_1 \cup I_2$). Now define $h(e_4)$ to be the first point in the sequence f_4, f_5, \dots such that s_2 is the only point in the intersection $h(e_4)s_2 \cap (J_1 \cup J_2)$. The existence of such a point is a consequence of the density of $\{f_1, f_2, \dots\}$. Let $J_3 = h(f_4)s_2$. We use the lemma to define h on I_3 in such a way that orders of ramification points are preserved. Now we define J_4 as an arc in Y with one endpoint in $J_1 \cup J_2 \cup J_3$ and the other endpoint being the first

point of the sequence f_1, f_2, \dots which is not in the image of $I_1 \cap I_2 \cap I_3$. We define h^{-1} on J_4 so that it preserves orders of ramification points and such that the position of $h^{-1}(J_4) = I_4$ in X is the same as the position of J_4 in Y . Then we define I_5, J_6, I_7, \dots and so on in the same manner. Now we have defined h on a dense subset of X . We know h preserves orders of ramification points, so it can be uniquely extended to a homeomorphism of X onto Y . This finishes the construction of a homeomorphism between X and Y . \square

Let us collect some of the consequences of Theorem 6.2. We start with a definition.

Definition 6.3. For a given set $A \subseteq \{3, 4, \dots, \omega\}$, we denote by D_A any dendrite satisfying (6.2.1) and (6.2.2). According to Theorem 6.2 the dendrite D_A is topologically unique. If $A = \{n\}$ for some number n , we write D_n instead of $D_{\{n\}}$.

Theorem 6.4. For any $A \subseteq \{3, 4, \dots, \omega\}$, the dendrite D_A is strongly pointwise self-homeomorphic.

Proof. Obviously one can find small neighborhoods of a point satisfying (6.2.1) and (6.2.2). \square

Corollary 6.5. There are uncountably many topologically different strongly pointwise self-homeomorphic dendrites.

Now we will discuss some universality properties of D_A . Recall that a space X is universal in a class \mathcal{E} of spaces if X contains a homeomorphic copy of any member of \mathcal{E} . It is known that D_n is universal in the class of dendrites having orders of ramification points at most n . In particular D_ω is universal in the class of dendrites. Using this fact we can prove the following two theorems.

Theorem 6.6. If $\omega \in A$, then D_A is universal in the class of dendrites.

Theorem 6.7. If A is finite, and n is the greatest element of A , then D_A is universal in the class of dendrites having orders of ramification points at most n .

Using a similar embedding procedure as in the proof of universality of D_n the following theorem can be proven. The proof is left to the reader.

Theorem 6.8. If A is an infinite set, $\omega \notin A$, then D_A is universal in the class of dendrites having finite orders of ramification points.

Recall that a mapping is called open if the image of an open set is open. Generalizing the ideas in [5, Theorem 2, p. 492] we prove the following theorem.

Theorem 6.9. *There is an open mapping of D_A onto D_B if and only if the sets A and B satisfy the following conditions:*

$$\text{if } A \text{ and } B \text{ are finite then } \max A \geq \max B, \tag{6.9.1}$$

$$\min A \geq \min B, \tag{6.9.2}$$

$$\text{if } B \text{ is infinite then } A \text{ is infinite,} \tag{6.9.3}$$

$$\omega \in A \text{ if and only if } \omega \in B. \tag{6.9.4}$$

Proof. Assume there is an open mapping f of D_A onto D_B . Conditions (6.9.1) and (6.9.3) are consequences of the fact that open mappings do not increase the order of ramification points [9, VIII, (7.31) Corollary, p. 147], and (6.9.4) follows from the fact that the image of a point of order ω is again a point of order ω [5, Lemma, p. 489].

To prove (6.9.2) denote by k the minimum of A , and assume orders of ramification points in D_B are greater than k . Choose two points p and q of D_A such that $f(p) \neq f(q)$. Then one can find a copy C of D_k in D_A with p and q in C and such that endpoints of C are endpoints of D_A . The image $f(C)$ is a subcontinuum of D_B having points of order not greater than k (in $f(C)$), so it must be an arc or a point. Because endpoints of C form a dense subset of C , the image of the set of endpoints of C is a dense subset of $f(C)$ composed of endpoints only. This implies that $f(C)$ is a one-point set and contradicts the fact that $f(p) \neq f(q)$.

Now assume D_A and D_B satisfy conditions (6.9.1)–(6.9.4). Then we can define sequences n_1, n_2, \dots and m_1, m_2, \dots such that $A = \{n_1, n_2, \dots\}$, $B = \{m_1, m_2, \dots\}$, and $m_i < n_i$ for each $i \in \{1, 2, \dots\}$, $n_i = \omega$ if and only if $m_i = \omega$, and every member of A and B appears infinitely many times in the appropriate sequence. We construct inverse sequences (X_i, f_i) and (Y_i, g_i) such that D_A is the inverse limit of (X_i, f_i) and D_B is the inverse limit of (Y_i, g_i) . Let X_1 be the unit interval with midpoint x_1 . For $n \in A$, $n \neq \omega$, let C_2 be a simple $(n - 2)$ -od with vertex v_2 and length of each arc from v_2 to an endpoint equal $1/4$. If $n_1 = \omega$, let C_2 be the ω -od $\bigcup_{k=1}^{\infty} I_k$, where I_k is an arc of length $1/(4k)$ for $k \in \{1, 2, \dots\}$ and all of the I_k intersect in a common endpoint v_2 , called the vertex, and are otherwise disjoint. Let X_2 be the one-point union of C_2 and X_1 obtained by identifying x_1 to v_2 . Define $f_1: X_2 \rightarrow X_1$ by letting $f_1|_{X_1}$ be the identity and $f_1(x) = x_2$ for any $x \in C_2$. Then X_2 is a dendrite with one ramification point of order n_1 . In a similar fashion we form X_3 by attaching copies of a simple $(n_2 - 2)$ -od (or ω -od if $n_2 = \omega$) C_3 to a midpoint of each maximal free arc in X_2 , so each attached $(n_2 - 2)$ -od has arcs from the vertex to an endpoint of length $1/16$ each, and ω -od has arcs of length $1/(16k)$ for $k \in \{1, 2, \dots\}$. Define $f_2: X_3 \rightarrow X_2$ by letting $f_2|_{X_2}$ be the identity and f_2 mapping each copy of C_3 to its vertex. Continuing in this manner we obtain an inverse sequence of dendrites and surjective bonding mappings (X_i, f_i) with inverse limit homeomorphic to D_A .

We construct the inverse sequence (Y_i, f_i) analogously using the sequence m_1, m_2, \dots in place of n_1, n_2, \dots . Note that because of the construction each X_{i+1} contains a copy of X_i and each Y_{i+1} contains a copy of Y_i .

Let $h_1: X_1 \rightarrow Y_1$ be the identity mapping and extend h_1 to a map $h_2: X_2 \rightarrow Y_2$ by mapping $(m_1 - 2)$ of the length- $\frac{1}{4}$ arcs in C_2 identically onto the $(m_1 - 2)$ -od in Y_2 and mapping the remaining $(n_1 - m_1)$ arcs of the length $\frac{1}{4}$ in C_2 onto some arc of the $(m_1 - 2)$ -od in Y_2 . If $m_1 = n_1 = \omega$ we map the ω -od in X homeomorphically onto the ω -od in Y_2 . To extend the map h_2 to a map $h_3: X_3 \rightarrow Y_3$ choose $(m_2 - 2)$ arcs of the length $1/16$ in each copy of C_3 in X_3 and map them homeomorphically onto arcs of the $(m_2 - 2)$ -od in Y_3 . Remaining $(n_2 - m_2)$ arcs of the length $1/16$ in each copy of C_3 in X_3 are mapped onto some arc of the $(n_2 - 2)$ -od in Y_3 . In this way we obtain the following commutative diagram.

$$\begin{array}{ccccccc}
 X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{f_3} & \cdots & \longleftarrow & D_A \\
 \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h \\
 Y_1 & \xleftarrow{g_1} & Y_2 & \xleftarrow{g_2} & Y_3 & \xleftarrow{g_3} & \cdots & \longleftarrow & D_B
 \end{array}$$

One can check that all the maps h_n are open and that for any points $x_n \in X_n$ and $y_{n+1} \in Y_{n+1}$ satisfying $h_n(x_n) = g_n(y_{n+1})$ we have $h_n^{-1}(y_{n+1}) \subseteq f_n^{-1}(x_n)$. Thus all the assumptions of [6, Theorem 3, p. 58] are satisfied, and so the limit map h is open. This finishes the proof. \square

Corollary 6.10. *Open images of D_A are homeomorphic to D_A if and only if A is a nonempty subset of $\{3, \omega\}$.*

Proof. First assume every open image of D_A is homeomorphic to D_A . Then $A \subseteq \{3, \omega\}$ is a consequence of Theorem 6.9.

Now assume that A is a proper subset of $\{3, \omega\}$. It was proven in [4] that an open image is homeomorphic to D_A .

Finally, suppose $A = \{3, \omega\}$, and assume we have an open mapping of D_A onto a continuum Y . Because an open image of a dendrite is a dendrite [9, VIII, (7.7), p. 148], open maps do not increase orders of ramification points [9, VIII, (7.31) Corollary, p. 147], and images of points of order ω are points of order ω [5, Lemma, p. 489], Y is a dendrite with ramification points of order 3 and ω only. Points of order ω must be on every arc, because they are on every arc in D_A . Moreover, one can use the same argument as in the proof of (6.9.2) that points of order 3 must be on every arc in Y . Thus Y satisfies (6.2.1) and (6.2.2), so it is homeomorphic to D_A . \square

Example 6.11. There is a strongly pointwise self-homeomorphic dendrite which is not of the form D_A for some $A \subseteq \{3, 4, \dots, \omega\}$.

Proof. The dendrite pictured in Fig. 4 has points of order four on vertical arcs and points of order three on some horizontal arcs, so it does not satisfy condition (6.2.2). One can check that it is strongly pointwise self-homeomorphic. \square

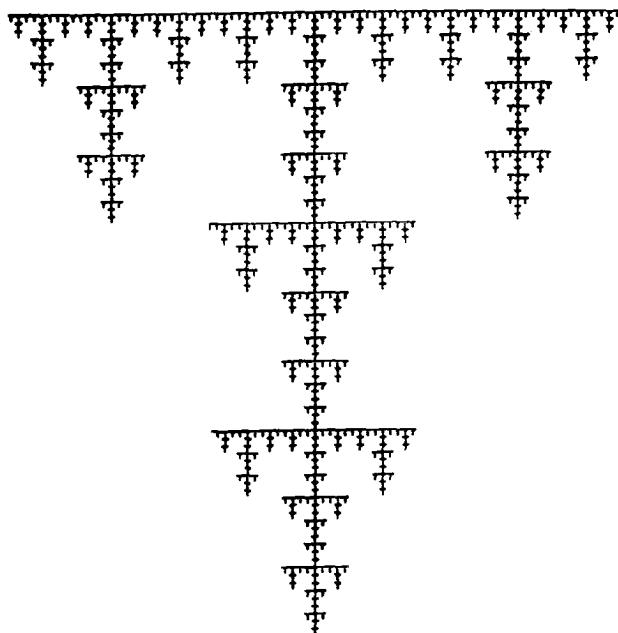


Fig. 4.

We next turn our attention to implications between different types of self-homeomorphic dendrites, i.e., we discuss whether we can add any implications to those in Theorem 2.5 under the additional assumption that X is a dendrite.

Our next example for dendrites is analogous to Example 2.7.

Example 6.12. There is a strongly self-homeomorphic and pointwise self-homeomorphic dendrite which is not strongly pointwise self-homeomorphic.

Proof. To describe the example, we start with a universal dendrite of order three, constructed in such a way that the longest arc lies on the x -axis of the plane. We attach vertical arcs going into the upper half of the plane and to these vertical arcs we attach horizontal arcs going to the right from the vertical arcs. Continuing in this manner, we build a homeomorphic copy of D_3 such that from every horizontal arc there are vertical arcs going upward, and from every vertical arc there are horizontal arcs going to the right. Denote the described dendrite by A . Now let A' be the reflection of A across the x -axis. The union, $A \cup A'$ is a dendrite having points of order four on the arc contained in the x -axis and such that all other ramification points are of order three.

At every endpoint of a horizontal arc of $A \cup A'$ we attach a copy of D_5 in such a way that all copies of D_5 are disjoint, the only point of intersection of any D_5 with $A \cup A'$ is the endpoint of the horizontal arc where it is attached, and that point is of order two. The dendrite so obtained is our example.

To show that X is strongly self-homeomorphic observe that every open set U contains a copy C of D_5 and that C is the one-point union, $C_1 \cup C_2$, of two copies of D_5 . Now take a homeomorphism h that maps C onto C_1 and $\text{cl}(X \setminus C)$ onto C_2 . Then $h(X)$ has nonempty interior.

To show that X is pointwise self-homeomorphic, we need the following lemma and its corollary. The proof of the lemma is analogous to the proof of Theorem 6.2.

Lemma 6.13. *Let P and Q be dendrites homeomorphic to D_3 , and let $E = \{e_1, e_2, \dots\}$ and $F = \{f_1, f_2, \dots\}$ be dense subsets of endpoints of P and Q respectively. Then there is a homeomorphism $h : P \rightarrow Q$ such that $h(E) = F$ and $f(e_1) = f_1$.*

Corollary 6.14. *Let P and Q be constructed by attaching infinitely many copies of D_5 to D_3 in such a way that copies of D_5 are mutually disjoint, form a null sequence and are attached to an endpoint of D_3 in such a way that the intersection of D_3 and any copy of D_5 is a point of order two. Moreover, let e be a point in $D_3 \subseteq P$ which is an endpoint in both D_3 and in P , and let f be a point in $D_3 \subseteq Q$ which is an endpoint in both D_3 and in Q . Then there is a homeomorphism $h : P \rightarrow Q$ with $h(p) = q$.*

Proof of Example 6.12 continued. We are now ready to prove that X is pointwise self-homeomorphic. Let p be any point of X and let U be any neighborhood of p in X . Denote by C any copy of D_5 which is attached to some point of A or A' , is contained in U , and does not intersect the arc $A \cap A'$. Denote by a the common point of C and $A \cup A'$. Choose a point b of order two in $(A \cup A') \setminus (A \cap A')$. To construct the embedding h of X into U we start with $h(b) = a$. Let X_1 be the component of $X \setminus \{b\}$ which contains $A \cap A'$. Then $X_1 \cup \{b\}$ is a dendrite with orders of points not more than five, so we can define $h|_{X_1} : X_1 \rightarrow C$ so that h is continuous on $X_1 \cup \{b\}$. Denote by X_2 the other component of $X \setminus \{b\}$. Then $X_2 \cup \{b\}$ is a dendrite satisfying the assumptions of Corollary 6.14, and we can find a dendrite Q satisfying the same conditions which is contained in U , contains points a and p , and is disjoint with $h(X_1)$. Therefore there is a homeomorphism $h|_{X_2 \cup \{b\}} : X_2 \cup \{b\} \rightarrow Q$ with $h(b) = a$. This completes the construction of the embedding h which sends X into U and shows that X is pointwise self-homeomorphic.

Next we will prove that X is not strongly pointwise self-homeomorphic. Denote by I the arc $A \cap A'$, and let p be a point of order four in I . Let U be a neighborhood of p in X not containing endpoints of I , and assume that there is an embedding h sending X into U with $p \in \text{int } h(U)$. Because p is in the interior of $h(X)$, the order of p in $h(X)$ is four, so the order of $h^{-1}(p)$ is also four. This fact implies that $h^{-1}(p) \in I$, because all points of order four are in I . Now, $h(I)$ is an arc in U having points of order two and four (in $h(X)$) only, and points of order four are dense in $h(I)$. Moreover, $p \in h(I)$, so $h(I) \subseteq I$ since every arc containing p and going out of I has some subarcs with points of order two and three only. Note that I , as a horizontal arc, has a copy of D_5 attached at each of its

endpoints. There are no such copies attached to $h(I)$ in U and we have a contradiction. \square

In the investigation of self-homomorphic dendrites, we observe that for most of them, in particular for all D_A , the required embedding can be chosen to have a given point, or even a given neighborhood, fixed. This could lead to a new property, stronger than being strongly pointwise self-homeomorphic. The next theorem states some equivalent definitions of the new property.

Theorem 6.15. *Let X be a dendrite. The following conditions are equivalent:*

$$\text{for any point } p \text{ of order two, any neighborhood } U \text{ of } p, \text{ and any (one of two) component } K \text{ of } X \setminus \{p\} \text{ there is an embedding } h: K \cup \{p\} \rightarrow K \cup \{p\} \text{ with } h(p) = p \text{ and } h(K \cup \{p\}) \subseteq U; \tag{6.15.1}$$

$$\text{for any point } p, \text{ any neighborhood } U \text{ of } p, \text{ and any component } K \text{ of } X \setminus \{p\} \text{ there is an embedding } h: K \cup \{p\} \rightarrow K \cup \{p\} \text{ with } h(p) = p \text{ and } h(K \cup \{p\}) \subseteq U; \tag{6.15.2}$$

$$\text{for any point } p, \text{ any neighborhood } U \text{ of } p, \text{ there is an embedding } h: X \rightarrow U \text{ with } h(p) = p \text{ and } p \in \text{int } h(X); \tag{6.15.3}$$

$$\text{for any point } p \text{ and any neighborhood } U \text{ of } p \text{ there is a neighborhood } V \text{ of } p \text{ with } V \subseteq U \text{ and an embedding } h: X \rightarrow U \text{ satisfying } h|_V = \text{id}_V. \tag{6.15.4}$$

Proof. The sequence of implications $(6.15.4) \Rightarrow (6.15.3) \Rightarrow (6.15.2) \Rightarrow (6.15.1)$ is a simple consequence of definitions and each of the implications is true even without the assumption that X is a dendrite. Thus, it is enough to prove the implication $(6.15.1) \Rightarrow (6.15.4)$. Toward this aim, assume a dendrite X satisfies (6.15.1) and let U be a neighborhood of a point $p \in X$. Define V to be an open set containing p such that $\text{cl}(V) \subseteq U$ and the boundary of V consists of finitely many points of order two, say $\text{bd } V = \{a_1, a_2, \dots, a_n\}$. Now, for any point a_i we denote by K_i the component of $X \setminus \{a_i\}$ that does not contain the point p . According to our assumption, there is an embedding $h_i: K_i \cup \{a_i\} \rightarrow K_i \cup \{a_i\}$ with $h_i(a_i) = a_i$ and $h_i(K_i \cup \{a_i\}) \subseteq U$. Define $h: X \rightarrow X$ by $h|_{K_i} = h_i$ and $h|_{\text{cl } V} = \text{id}_{\text{cl } V}$. One can now check that h satisfies the conditions required in (6.15.4). \square

The following proposition is a consequence of the definitions.

Proposition 6.16. *Let X be an arbitrary continuum. Then each of the conditions (6.15.3) and (6.15.4) implies that X is strongly pointwise self-homeomorphic.*

The next theorem shows that in general (without the assumption that X is a dendrite) condition (6.15.4) is a strong assumption.

Theorem 6.17. *Assume a plane continuum X satisfies (6.15.4). Then X is a dendrite or X contains the Sierpiński universal plane curve.*

Proof. Assume X is not a dendrite. Because X is locally connected it must contain a simple closed curve C . Let p be any point of C . Define A to be the set of all points x such that there is a simple closed curve C_x containing x such that $C_x \cap C$ contains an interval with the point p in its interior. Because C_x is the union of simple closed curves with p in its intersection, A is connected. The closure $\text{cl } A$ is a continuum containing no cut points (in fact p is not a cut point, because p is an interior point of an arc contained in $C_x \cap C$ for any x). If $\text{cl } A$ is a two-dimensional continuum, it contains a disk, and therefore it contains a Sierpiński universal plane curve. Thus we can assume $\text{cl } A$ is one dimensional. In this case we will use Whyburn's characterization (see Theorem A) to prove $\text{cl } A$ is a Sierpiński universal plane curve. By the characterization it is enough to show that $\text{cl } A$ does not contain local cut points. Assume the contrary. Choose a local cut point q of $\text{cl } A$, and let U be a neighborhood of q in X such that $p \notin U$ and that q is a cut point of $U \cap \text{cl } A$. Suppose there is a neighborhood V of q and an embedding $h: X \rightarrow U$ satisfying (6.15.4). Choose two points a and b belonging to different components of $(V \cap \text{cl } A) \setminus \{q\}$ and being close enough to q so there are arcs I_a and I_b joining a and q and b and q respectively with $I_a \cup I_b \subseteq U$. Then $C_a \cup C_b \cup I_a \cup I_b$ is a continuum without cut points containing a, b, p and q . Its image under h must be a continuum contained in U , containing $a, b, h(p)$ and q and without cut points. This contradicts the fact that a and b are in different components of $(V \cap \text{cl } A) \setminus \{q\}$. \square

Now we will show that Sierpiński universal plane curve satisfies (6.15.4). Toward this goal, we need a slightly stronger version of Whyburn's theorem. The proof of the stronger version is in fact in [10, proof of Theorem 3, p. 322].

Theorem B (Whyburn). *Let S and S' be two Sierpiński universal plane curves with outer boundaries C and C' respectively. Then any homeomorphism of C onto C' can be extended to a homeomorphism of S onto S' .*

As a corollary we can obtain an even more general result.

Theorem 6.18. *Let S and S' be two Sierpiński universal plane curves and let C and C' be any two boundaries of components of the complements of S and S' in the plane, respectively. Then any homeomorphism of C onto C' can be extended to a homeomorphism of S onto S' .*

Proof. Embed S and S' into the complex plane in such a way that the origin is inside C and C' . The inversion $z \rightarrow 1/z$ is a homeomorphism on S and S' and sends C and C' onto outer boundaries. Now apply the Whyburn theorem above. \square

Theorem 6.19. *The Sierpiński universal plane curve satisfies (6.15.4).*

Proof. Assume S is a Sierpiński universal plane curve embedded in \mathbb{R}^2 . Let p be any point of S , and let U be any open in S neighborhood of p . Choose open connected neighborhoods U' and V of p such that $p \in V \subseteq \text{cl } V \subseteq U' \subseteq \text{cl } U' \subseteq U$ and $U' \setminus V$ is connected. According to the Whyburn's characterization (Theorem A) the sets $S_1 = S \setminus V$ and $S_2 = \text{cl } U' \setminus V$ are homeomorphic to S . Denote by C and D the simple closed curves that are the boundaries of the components of $\mathbb{R}^2 \setminus S_1$ and of $\mathbb{R}^2 \setminus S_2$, respectively, containing V . Note that $\text{bd } V \subseteq C \cap D$. Let $f: C \rightarrow D$ be any homeomorphism that is the identity on $C \cap D$. By Theorem 6.18 there is a homeomorphism $g: S_1 \rightarrow S_2$ such that $g|_C = f$. Finally, define an embedding $h: S \rightarrow U$ by $h(x) = g(x)$ for $x \in S_1$ and $h(x) = x$ for $x \in V$ and observe that h is the embedding satisfying (6.15.4) with $X = S$ as required. This finishes the proof. \square

We close this paper with posing some problems. The first four of them ask if there are any implications for dendrites in the diagram of Theorem 2.5 other than ones established there (compare Example 6.12).

Problem 6.20. If X is a self-homeomorphic dendrite, is X strongly self-homeomorphic?

Problem 6.21. If X is a self-homeomorphic dendrite, is X pointwise self-homeomorphic?

Problem 6.22. If X is a pointwise self-homeomorphic dendrite, is X strongly self-homeomorphic?

Problem 6.23. If X is a strongly self-homeomorphic dendrite, is X pointwise self-homeomorphic?

Problem 6.24. In Theorem 6.17 can condition (6.15.4) be replaced by (6.15.3)?

Problem 6.25. Does every strongly pointwise self-homeomorphic continuum (dendrite) satisfy (6.15.3) or (6.15.4)?

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