

# Metrics defined via discrepancy functions <sup>☆</sup>

Włodzimierz J. Charatonik <sup>\*</sup>, Matt Insall

*Department of Mathematics and Statistics, University of Missouri – Rolla, 1870 Miner Circle, Rolla, MO 65409-0020, USA*

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## Abstract

We introduce the notion of a *discrepancy function*, as an extended real-valued function that assigns to a pair  $(A, U)$  of sets a nonnegative extended real number  $\omega(A, U)$ , satisfying specific properties. The pairs  $(A, U)$  are certain pairs of sets such that  $A \subseteq U$ , and for fixed  $A$ , the function  $\omega$  takes on arbitrarily small nonnegative values as  $U$  varies. We present natural examples of discrepancy functions and show how they can be used to define traditional pseudo-metrics, quasimetrics and metrics on hyperspaces of topological spaces and measure spaces.

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## 1. Definitions and notation

Let  $X$  be a set and let  $\mathcal{P}(X)$  denote its power set, i.e.  $\mathcal{P}(X)$  is the set of all subsets of  $X$ . Let  $\mathcal{H} \subset \mathcal{P}(X)$ . We will refer to  $\mathcal{H}$  as “the hyperspace”, and we will define distance functions of various ilks on  $\mathcal{H}$ . For each  $A \in \mathcal{H}$ , let

$$\mathcal{F}(A) \subseteq \{U \in \mathcal{P}(X) \mid A \subseteq U\}$$

be given. (That is,  $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{P}(X)$ .) Also, let

$$\mathcal{D}(\mathcal{H}, \mathcal{F}) = \{(A, U) \mid U \in \mathcal{F}(A)\},$$

and let an extended real-valued function  $\omega : \mathcal{D}(\mathcal{H}, \mathcal{F}) \rightarrow [0, \infty]$  be given. We call the function  $\omega$  a *discrepancy function* provided that it satisfies

$$(df1) \quad \inf\{\omega(A, U) \mid U \in \mathcal{F}(A)\} = 0.$$

In this case, the number  $\omega(A, U)$  is said to be the *discrepancy between  $A$  and  $U$* .

A *t-discrepancy function* satisfies the following, in addition to (df1):

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<sup>\*</sup> Corresponding author.

*E-mail address:* [wjcharat@umr.edu](mailto:wjcharat@umr.edu) (W.J. Charatonik).

(df2) If  $A \subseteq B \subseteq C$ ,  $U \in \mathcal{F}(A)$ ,  $V \in \mathcal{F}(B)$ ,  $B \subseteq U$  and  $C \subseteq V$ , then there is  $W \in \mathcal{F}(A)$  such that  $C \subseteq W$  and  $\omega(A, U) + \omega(B, V) \geq \omega(A, W)$ .

A *proper discrepancy function* is a discrepancy function  $\omega$  which also satisfies the following property:

(df3) If  $A, B \in \mathcal{H}$  and  $A \subsetneq B$ , then  $\inf\{\omega(A, U) \mid B \subseteq U\} > 0$ .

Of course, a function  $\omega$  satisfying (df1)–(df3) is a *proper  $t$ -discrepancy function*.

We will use these discrepancy functions to construct distance functions on the set  $\mathcal{H}$ . To this end, define, for  $A, B \in \mathcal{H}$ :

$$s_\omega(A, B) = \inf\{\omega(A, U) \mid B \subseteq U\}$$

and

$$d_\omega(A, B) = \max\{s_\omega(A, B), s_\omega(B, A)\}.$$

We will show that if  $\omega$  is a discrepancy function, then  $d_\omega$  is a pseudo-symmetric, and that if  $\omega$  is a proper discrepancy function, then  $s_\omega$  is a quasisymmetric and  $d_\omega$  is a symmetric. In case  $\omega$  is a  $t$ -discrepancy function,  $s_\omega$  is a pseudo-quasimetric and  $d_\omega$  is a pseudo-metric. Finally, if  $\omega$  is a proper  $t$ -discrepancy function, then  $s_\omega$  is a quasimetric and  $d_\omega$  is a metric. Thus we recall for the reader the definitions of these notions. For symmetric and their generalizations, our definitions are natural generalizations of those described in [2]. Let  $\rho: \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty]$ , and consider the following conditions on  $\rho$ :

- (i)  $\rho(A, A) = 0$ ,
- (ii)  $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$ ,
- (iii)  $\rho(A, B) = 0 = \rho(B, A)$  implies  $A = B$ ,
- (iv)  $\rho(A, B) = \rho(B, A)$ .

The function  $\rho$  is a *pseudo-symmetric* if it satisfies (i) and (iv). A pseudo-symmetric is a *symmetric* provided that it satisfies also property (iii).

The function  $\rho$  is a *pseudo-quasimetric* provided that it satisfies (i) and (ii). A pseudo-quasimetric  $\rho$  is a *quasimetric* if it also satisfies (iii).

A *pseudo-metric* is a pseudo-quasimetric  $\rho$  that satisfies (iv), and finally, a *metric* is a function that satisfies (i) through (iv).

We will have occasion to discuss measure spaces in the sequel. For simplicity, throughout this work, when we refer to a measure space  $X$ , we mean a measure space  $(X, \lambda)$  with a metric topology, such that, given  $\varepsilon > 0$ ,  $\inf\{\lambda(\mathcal{B}(x, \varepsilon)) \mid x \in X\} > 0$ , where for  $x \in X$ ,  $\mathcal{B}(x, \varepsilon)$  is the ball of radius  $\varepsilon$  centered at  $x$ .

Our discrepancy functions can also be applied in the setting of Approach Spaces, a topic pioneered by Lowen (cf. [4]). Let us recall here the definition of such a space. Let  $X$  be a set and let  $\mathcal{H} \subseteq \mathcal{P}(X)$ . Let  $d: X \times \mathcal{H} \rightarrow [0, \infty]$ . Then the triple  $(X, \mathcal{H}, d)$  is an *approach space* provided that the following hold:

- (a1) For each  $x \in X$ ,  $\{x\} \in \mathcal{H}$  and  $d(x, \{x\}) = 0$ ;
- (a2)  $\emptyset \in \mathcal{H}$  and for each  $x \in X$ ,  $d(x, \emptyset) = \infty$ ;
- (a3)  $\mathcal{H}$  is closed under unions and for each  $x \in X$  and any  $A, B \in \mathcal{H}$ ,  $d(x, A \cup B) = \min\{d(x, A), d(x, B)\}$ ;
- (a4) Given  $A \in \mathcal{H}$  and  $\varepsilon > 0$ , let  $A^{(\varepsilon)} = \{x \in X \mid d(x, A) \leq \varepsilon\}$ . Then for each  $\varepsilon \in [0, \infty]$  and  $A \in \mathcal{H}$ ,  $A^{(\varepsilon)} \in \mathcal{H}$ , and for each  $x \in X$ ,  $d(x, A) \leq d(x, A^{(\varepsilon)}) + \varepsilon$ .

(Note: Our definition varies slightly from the one given by Lowen, as Lowen requires  $\mathcal{H} = \mathcal{P}(X)$ , and considers the pair  $(X, d)$  to be the approach space.)

We shall have need of the Vietoris topology on hyperspaces. We recall here the associated notations and definitions. For more details, we refer to [3]. Let  $(X, \tau)$  be a topological space. We denote by  $2^X$  the collection of nonempty closed subsets of  $X$ . The *Vietoris topology* on  $2^X$  is the smallest topology  $\tau_v$  satisfying the following properties:

- (v1) If  $U \in \tau$ , then  $\{A \in 2^X \mid A \subseteq U\} \in \tau_v$ ;

(v2) If  $U \in \tau$ , then  $\{A \in 2^X \mid A \cap U \neq \emptyset\} \in \tau_v$ .

The Vietoris topology on a subset  $Y$  of  $2^X$  is the subspace topology on  $Y$  as a subspace of the space  $(2^X, \tau_v)$ .

We recall here a natural base for the Vietoris topology on  $2^X$ , for we shall use it in the sequel. Given subsets  $A_1, \dots, A_n$  of  $X$ , we denote by  $\langle A_1, \dots, A_n \rangle$  the set

$$\left\{ A \in 2^X \mid A \subseteq \bigcup_{j=1}^n A_j \text{ and for each } j, A \cap A_j \neq \emptyset \right\}.$$

Now, let

$$\mathcal{B}_v = \{ \langle U_1, \dots, U_n \rangle \mid U_1, \dots, U_n \in \tau \}.$$

Then  $\mathcal{B}_v$  is a base for the topology  $\tau_v$ .

In some applications, we shall construct discrepancy functions from Whitney Maps. Let us recall here this concept. Let  $X$  be a topological space, and let  $2^X$  denote the hyperspace of closed subsets of  $X$ . Let  $\mu$  be a real-valued function on  $2^X$  with the following properties:

- (w1)  $\mu : 2^X \rightarrow [0, \infty)$ ;
- (w2) For each  $x \in X$ ,  $\mu(\{x\}) = 0$ ;
- (w3) For  $A, B \in 2^X$ , if  $A \subsetneq B$ , then  $\mu(A) < \mu(B)$ .

## 2. The main theorem and some examples

We will show how to use discrepancy functions  $\omega$  to generate pseudo-symmetrics, pseudo-quasimetrics, etc. Let us first exhibit some natural examples of discrepancy functions.

### 2.1. Examples of discrepancy functions

**Example 2.1.** Let  $X$  be a measure space, and let  $\mathcal{H}$  be the collection of measurable subsets of  $X$ . For  $A \in \mathcal{H}$ , let  $\mathcal{F}(A) = \{U \in \mathcal{H} \mid A \subseteq U\}$ . Let  $\lambda$  denote the measure with which  $X$  is equipped, and define  $\omega(A, U) = \lambda(U) - \lambda(A)$ . Then  $\omega$  is a  $t$ -discrepancy function, but it is (typically) not a proper  $t$ -discrepancy function. We shall call it a *measure  $t$ -discrepancy function*.

**Example 2.2.** As in the preceding example, let  $X$  be a measure space with measure  $\lambda$ , but now let  $\mathcal{H}$  be the collection of nonempty closed subsets of  $X$  that are of finite measure. For  $\varepsilon \in [0, \infty)$  and  $A \in \mathcal{H}$ , set  $N(A, \varepsilon) = \bigcup \{ \mathcal{B}(x, \varepsilon) \}$ . Then for each set  $A \in \mathcal{H}$ , let  $\mathcal{F}(A) = \{N(A, \varepsilon) \mid \varepsilon > 0\}$ . Define, as before,  $\omega(A, U) = \lambda(U) - \lambda(A)$ . Then  $\omega$  is a discrepancy function. Indeed, in this case,  $\omega$  is a proper discrepancy function.

**Example 2.3.** The *discrete* discrepancy function is given by

$$\omega(A, U) = \begin{cases} 1 & \text{if } A \subsetneq U, \\ 0 & \text{if } A = U. \end{cases}$$

In this case,  $\mathcal{H} = \mathcal{P}(X) \setminus \{\emptyset\}$  and  $\mathcal{F}(A) = \{U \subseteq X \mid A \subseteq U\}$ . This function  $\omega$  is a proper  $t$ -discrepancy function.

**Example 2.4.** Let  $X$  be a measure space with measure  $\lambda$ , and let  $\mathcal{H}$  be the collection of closed nonempty sets of finite measure. For  $A \in \mathcal{H}$  and  $\varepsilon : A \rightarrow (0, \infty)$ , let

$$N(A, \varepsilon) = \bigcup \{ \mathcal{B}(x, \varepsilon(x)) \mid x \in A \}.$$

For each  $A \in \mathcal{H}$ , set

$$\mathcal{F}(A) = \{N(A, \varepsilon) \mid \varepsilon : A \rightarrow (0, \infty)\}.$$

Define, for  $A \in \mathcal{H}$  and  $U \in \mathcal{F}(A)$ ,

$$\omega(A, U) = \lambda(U) - \lambda(A).$$

Then this function  $\omega$  is a proper discrepancy function. We call it a *Hausdorff–Lebesgue* discrepancy function.

**Example 2.5.** Let  $X = \mathbb{R}^n$ ,  $n \geq 1$ , and let  $m = \lambda(\mathcal{B}(\vec{0}, 1))$  be the Lebesgue measure of the unit ball in  $X$ . Define  $\mathcal{H}$  and  $\mathcal{F}$  as in the previous example, and set

$$\omega_1(A, U) = \left[ \frac{1}{m} \lambda(U) \right]^{\frac{1}{n}} - \left[ \frac{1}{m} \lambda(A) \right]^{\frac{1}{n}},$$

$$\omega_2(A, U) = \left[ \frac{1}{m} (\lambda(U) - \lambda(A)) \right]^{\frac{1}{n}}.$$

Then  $\omega_1$  and  $\omega_2$  are proper discrepancy functions.

**Problem 2.1.** If  $X$ ,  $\mathcal{H}$ ,  $\mathcal{F}$ ,  $\omega_1$ , and  $\omega_2$  are as above, is  $\omega_1$  a proper  $t$ -discrepancy function? Is  $\omega_2$  a proper  $t$ -discrepancy function? If the above questions have positive answers, then, as we will see,  $d_{\omega_1}$  and  $d_{\omega_2}$  will be new metrics on  $\mathcal{H}$ , the hyperspace of closed, nonempty sets of finite measure.

**Example 2.6.** In [1], Whitney maps were considered, satisfying the following additional condition:

$$(w4) \quad A, B, C \in 2^X, A \subseteq B \implies \mu(B \cup C) - \mu(A \cup C) \leq \mu(B) - \mu(A).$$

Given such a Whitney map,  $\mu$ , set  $\mathcal{H} = 2^X$ , and for each  $A \in \mathcal{H}$ , set  $\mathcal{F}(A) = \{U \in \mathcal{H} \mid A \subseteq U\}$ , where  $X$  is a given topological space. Define  $\omega$  by  $\omega(A, U) = \mu(U) - \mu(A)$ , for  $A \in \mathcal{H}$  and  $U \in \mathcal{F}(A)$ . Then  $\omega$  is a proper  $t$ -discrepancy function. (Note that if (w4) is not satisfied, then  $\omega$  is still a proper discrepancy function.)

**Example 2.7.** Let  $(X, d)$  be an arbitrary metric space, let  $\mathcal{H} = \{A \in 2^X \mid A \text{ is bounded}\}$ , and for  $A \in \mathcal{H}$ , set

$$\mathcal{F}(A) = \{U \in \mathcal{H} \mid A \subseteq U\}.$$

Define  $\omega$  on  $\mathcal{D}(\mathcal{H}, \mathcal{F})$  by the formula

$$\omega(A, U) = \sup\{d(x, A) \mid x \in U\}.$$

Then  $\omega$  is a proper  $t$ -discrepancy function, and we call it a *Hausdorff ( $t$ -)discrepancy function*.

**Example 2.8.** Let  $(X, \mathcal{H}, d)$  be an approach space (e.g., let  $\mathcal{H}$  be either  $2^X$  or  $\mathcal{P}(X)$ , where  $(X, d)$  is an approach space in the sense of Lowen). Set  $\mathcal{F}(A) = \{U \in \mathcal{H} \mid A \subseteq U\}$ , for each  $A \in \mathcal{H}$ . Define  $\omega$  on  $\mathcal{D}(\mathcal{H}, \mathcal{F})$  by

$$\omega(A, U) = \sup\{d(x, A) \mid x \in U\}.$$

Then  $\omega$  is a  $t$ -discrepancy function, but it need not be proper. We will call it a *Hausdorff ( $t$ -)discrepancy function for the approach space  $(X, \mathcal{H}, d)$* .

## 2.2. Statement of the main theorem and examples of metrics

**Theorem 2.1.** *Let  $\omega$  be a  $t$ -discrepancy function. Then  $s_\omega$  is a pseudo-quasimetric and  $d_\omega$  is a pseudo-metric. If  $\omega$  is a proper  $t$ -discrepancy function, then  $s_\omega$  is a quasimetric and  $d_\omega$  is a metric.*

We will leave it to the reader to state and prove the corresponding result that relates discrepancy functions and proper discrepancy functions with pseudo-symmetrics, etc.

Before proving this theorem, we will list some of its consequences.

**Example 2.9.** Let  $\omega$  be a measure  $t$ -discrepancy function on a measure space  $X$ , as defined in Example 2.1. Then  $s_\omega$  is a pseudo-quasimetric, and  $d_\omega$  is a pseudo-metric, defined on the space of measurable subsets of  $X$ . We will show that  $d_\omega$  is topologically equivalent to the known pseudo-metric  $\rho$  that is defined by

$$\rho(A, B) = \lambda(A \cup B) - \lambda(A \cap B).$$

**Proof.** To this end, let  $A_n \xrightarrow{\rho} A$ . We need to show that  $d_\omega(A_n, A) \rightarrow 0$ . Observe that since  $d_\omega(A_n, A) = \max\{s_\omega(A_n, A), s_\omega(A, A_n)\}$ , this means we must show that

$$s_\omega(A_n, A) \rightarrow 0 \quad \text{and} \quad s_\omega(A, A_n) \rightarrow 0.$$

Note that

$$s_\omega(A_n, A) = \lambda(A_n \cup A) - \lambda(A)$$

and

$$s_\omega(A, A_n) = \lambda(A_n \cup A) - \lambda(A_n).$$

Thus we must show that

$$\lambda(A_n \cup A) - \lambda(A) \rightarrow 0$$

and

$$\lambda(A_n \cup A) - \lambda(A_n) \rightarrow 0.$$

Consider the first of these. We have

$$0 \leq \lambda(A_n \cup A) - \lambda(A_n) \leq \lambda(A_n \cup A) - \lambda(A_n \cap A) = \rho(A_n, A),$$

so it follows that

$$s_\omega(A_n, A) \rightarrow 0.$$

The proof that  $s_\omega(A, A_n) \rightarrow 0$  is very similar.

For the converse, suppose that

$$d_\omega(A_n, A) \rightarrow 0,$$

i.e.  $\lambda(A_n \cup A) - \lambda(A) \rightarrow 0$  and  $\lambda(A_n \cup A) - \lambda(A_n) \rightarrow 0$ . Since

$$\lambda(A_n \cup A) - \lambda(A_n \cap A) = \lambda(A_n \cup A) - \lambda(A) + \lambda(A_n \cup A) - \lambda(A_n),$$

it follows that  $\rho(A_n, A) \rightarrow 0$ .  $\square$

**Example 2.10.** Set

$$\mathcal{H} = \{A \in \mathcal{P}(X) \mid A \neq \emptyset, A \text{ is closed and, } A \text{ has finite measure}\},$$

and

$$\mathcal{F}(A) = \{N(A, \varepsilon) \mid \varepsilon > 0\},$$

where for  $A$  and  $\varepsilon$ ,

$$N(A, \varepsilon) = \bigcup \{\mathcal{B}(x, \varepsilon) \mid x \in A\}.$$

Let  $\omega(A, U) = \lambda(U) - \lambda(A)$ , for  $U \in \mathcal{F}(A)$ . Then  $d_\omega$  is a symmetric. Moreover, for the bounded sets in  $X$ ,  $d_\omega$  is topologically strictly stronger than the Hausdorff metric,  $h$ ; i.e. if  $A_n, A$  are bounded closed subsets of  $X$ , then  $A_n \xrightarrow{d_\omega} A$  implies  $A_n \xrightarrow{h} A$ , while the converse is usually false. Let  $\rho$  be as in Example 2.9. Then  $d_\omega$  is also strictly stronger than  $\rho$ . However, the topology of  $d_\omega$  is equivalent to the intersection of the Vietoris topology (generated by the Hausdorff metric) with the symmetric-difference pseudo-metric topology generated by  $\rho$ , on the domain of compact subsets of  $X$ .

**Proof.** To see that  $d_\omega$  is topologically stronger than the Hausdorff metric,  $h$ , let  $A_n \xrightarrow{d_\omega} A$ . Then  $s_\omega(A_n, A) \rightarrow 0$  and  $s_\omega(A, A_n) \rightarrow 0$ . Suppose it is not the case that  $A_n \xrightarrow{h} A$ . Then for some fixed  $\varepsilon$  (taking a subsequence if necessary),  $h(A_n, A) > \varepsilon$  for all  $n$ . For each  $n$ , then, either there is  $x_n \in A_n$  such that  $x_n \notin N(A, \varepsilon)$  or there is  $x_n \in A_n$  such that  $\mathcal{B}(x_n, \varepsilon) \cap A_n = \emptyset$ . We may assume that the first holds for all  $n$  or the second holds for all  $n$ , by taking subsequences. In the first case, since  $N(A, \varepsilon) \setminus A$  is an open set, we have

$$\begin{aligned} s_\omega(A, A_n) &= \inf\{\omega(A, N(A, \delta)) \mid A_n \subseteq N(A, \delta)\} \\ &= \inf\{\lambda(N(A, \delta) - \lambda(A)) \mid A_n \subseteq N(A, \delta)\} \geq \lambda(N(A, \varepsilon)) - \lambda(A) > 0. \end{aligned}$$

In the second case, because of our assumptions about the interaction of the measure with the topology, we have

$$\begin{aligned} s_\omega(A_n, A) &= \inf\{\omega(A_n, N(A_n, \delta)) \mid A \subseteq N(A_n, \delta)\} = \inf\{\lambda(N(A_n, \delta) - \lambda(A_n)) \mid A \subseteq N(A_n, \delta)\} \\ &\geq \inf\{\lambda(N(A_n, \varepsilon) - \lambda(A_n)) \mid A \subseteq N(A_n, \varepsilon)\} > 0. \end{aligned}$$

This contradicts the conditions  $s_\omega(A_n, a) \rightarrow 0$  and  $s_\omega(A, A_n) \rightarrow 0$ . Now, to see that the inclusion between these topologies is strict, it is enough to observe that there are sets  $A_n$ ,  $n \in \mathbb{N}$ , and  $A$ , such that  $A_n \xrightarrow{h} A$ , but  $A_n \not\xrightarrow{d_\omega} A$ . In fact, this is clear: Let  $A_n = \{\frac{p}{2^n} \mid p \in \mathbb{N} \cap [0, 2^n]\}$  and  $A = [0, 1]$ , in the reals.

To see that  $d_\omega$  is strictly stronger than  $\rho$ , let  $A_n \xrightarrow{d_\omega} A$  in  $\mathcal{H}$ . Then for each  $n$ , there is a function  $\varepsilon_n : A \rightarrow (0, \infty)$  such that

$$A_n \subseteq N(A, \varepsilon_n),$$

and for this sequence,  $\varepsilon_n$ , we have  $\lambda(N(A, \varepsilon_n)) - \lambda(A) \rightarrow 0$  as  $n \rightarrow \infty$ . But then, since  $A \subseteq (A_n \cup A) \subseteq N(A, \varepsilon_n)$ , we have

$$\lambda(A_n \cup A) - \lambda(A) \rightarrow 0$$

as  $n \rightarrow \infty$ . Also, since  $A_n \xrightarrow{d_\omega} A$ , for each  $n \in \mathbb{N}$ , there is a function  $\varepsilon'_n : A_n \rightarrow (0, \infty)$  such that

$$A \subseteq N(A_n, \varepsilon'_n),$$

and for this sequence,  $\varepsilon'_n$ , we have  $\lambda(N(A_n, \varepsilon'_n)) - \lambda(A_n) \rightarrow 0$  as  $n \rightarrow \infty$ . But then, since  $A_n \subseteq A_n \cup A \subseteq N(A_n, \varepsilon'_n)$ , it follows that

$$\lambda(A_n \cup A) - \lambda(A_n) \rightarrow 0,$$

as  $n \rightarrow \infty$ . Consequently,

$$\lambda(A_n \cup A) - \lambda(A_n \cup A) = \lambda(A_n \cup A) - \lambda(A) + \lambda(A_n \cup A) - \lambda(A_n) \rightarrow 0.$$

Since  $\rho$  is a pseudo-metric but not a metric, we see then that the Hausdorff–Lebesgue topology is stronger than the topology generated by  $\rho$ .

Now we will show that the Hausdorff–Lebesgue topology is actually the intersection of the Vietoris topology with the topology generated by  $\rho$ . To do this, we will show that convergence in both the Vietoris topology and the topology generated by  $\rho$  implies convergence in the Hausdorff–Lebesgue topology. To see this, suppose that  $A_n \xrightarrow{\rho} A$  and  $A_n \xrightarrow{h} A$ . We wish to show that  $A_n \xrightarrow{d_\omega} A$ . Thus for each  $n \in \mathbb{N}$ , let  $\varepsilon_n$  be a constant in  $(0, \infty)$  such that

$$A_n \subseteq N(A, \varepsilon_n),$$

and such that, as  $n \rightarrow \infty$ ,

$$\varepsilon_n \rightarrow 0.$$

Then

$$\lambda(N(A, \varepsilon_n)) \rightarrow \lambda\left(\bigcap_{n=1}^{\infty} N(A, \varepsilon_n)\right) = \lambda(A).$$

Thus,

$$\lambda(N(A, \varepsilon_n)) - \lambda(A) \rightarrow 0.$$

To complete the proof, we will show that as  $n \rightarrow \infty$ ,

$$\lambda(N(A_n, \varepsilon_n)) - \lambda(A_n) \rightarrow 0.$$

First, observe that  $N(A, \varepsilon_n) \subseteq N(A, 2\varepsilon_n)$ , and

$$\lambda(A_n \cup A) - \lambda(A_n \cap A) \geq \lambda(A_n \cup A) - \lambda(A) \geq |\lambda(A - n) - \lambda(A)| \geq 0.$$

(Note that the inequality  $\lambda(A_n \cup A) - \lambda(A) \geq |\lambda(A - n) - \lambda(A)|$  follows from considering the cases  $\lambda(A) \leq \lambda(A_n)$  and  $\lambda(A_n) \leq \lambda(A)$ : in the first case, we know that  $\lambda(A_n \cup A) - \lambda(A) \geq \lambda(A_n) - \lambda(A) \geq 0$ , while in the second case, we know that  $\lambda(A_n \cup A) - \lambda(A) \geq \lambda(A) - \lambda(A_n) \geq 0$ .) Since  $\lambda(A_n \cup A) - \lambda(A_n \cap A) \rightarrow 0$ , it follows that  $\lambda(A_n) - \lambda(A) \rightarrow 0$ . Thus we have

$$0 \leq \lambda(N(A_n, \varepsilon_n)) - \lambda(A_n) \leq \lambda(N(A, 2\varepsilon_n)) - \lambda(A_n) \rightarrow \lambda(A) - \lambda(A) = 0,$$

and so, of course,  $\lambda(N(A_n, \varepsilon_n)) - \lambda(A_n) \rightarrow 0$ , which entails  $A_n \xrightarrow{d_\omega} A$ , as desired.  $\square$

**Example 2.11.** Let  $\omega$  be the discrete  $t$ -discrepancy function, as defined in Example 2.3:

$$\omega(A, U) = \begin{cases} 1 & \text{if } A \subsetneq U, \\ 0 & \text{if } A = U, \end{cases}$$

where

$$\mathcal{H} = \mathcal{P}(X) \setminus \{\emptyset\}$$

and

$$\mathcal{F}(A) = \{U \subseteq X \mid A \subseteq U\}.$$

Then  $s_\omega$  is a pseudo-metric, described by

$$s_\omega(A, B) = \begin{cases} 0 & \text{if } A \subseteq B, \\ 1 & \text{otherwise,} \end{cases}$$

and  $d_\omega$  is a discrete metric.

**Example 2.12.** Let  $\omega$  be the Hausdorff–Lebesgue discrepancy function, as in Example 2.4. Then  $d_\omega$  is a pseudo-symmetric (on the sets of finite measure), but its restriction to the closed sets of finite measure is a metric. Its restriction to the closed and bounded sets is topologically stronger than the Hausdorff metric. The Vietoris topology on the closed sets is an extension of the Hausdorff metric topology, but we will show that the Hausdorff–Lebesgue topology and the Vietoris topology are incomparable in the lattice of topologies, even on the common domain consisting of the closed sets of finite measure.

**Proof.** Our previous example shows that the Vietoris topology is not finer than the Hausdorff–Lebesgue topology. On the other hand, let

$$B_n = \left\{ \left( x, \frac{1}{nx^2} \right) \mid x \geq 1 \right\}$$

and

$$B = [1, \infty) \times \{0\}.$$

We will show that  $B_n \rightarrow B$  in the Hausdorff–Lebesgue symmetric, but not in the Vietoris topology. To see this, define

$$U_n = \bigcup \left\{ \mathcal{B} \left( x, \frac{2}{nx^2} \right) \mid x \geq 1 \right\}.$$

Then

$$d_\omega(B_n, B) \leq \lambda(U_n) = \frac{\pi}{2} \left( \frac{2}{n} \right)^2 + \int_1^\infty \frac{2}{nx^2} dx \rightarrow 0.$$

However, if we define

$$V = \left\{ (x, y) \mid x > \frac{1}{2}, |y| < \frac{1}{x^3} \right\},$$

then  $\langle V \rangle$  is a Vietoris-open neighborhood of  $B$ , and no  $B_n$  is in  $\langle V \rangle$ . It follows that  $B_n \not\rightarrow B$  in the Vietoris topology, i.e. the Hausdorff–Lebesgue topology is not finer than the Vietoris topology, as claimed.  $\square$

**Example 2.13.** Let  $\mu$  be the Whitney map, as in Example 2.6, and let  $\omega$  be the associated  $t$ -discrepancy function:

$$\omega(A, U) = \mu(U) - \mu(A).$$

Then  $s_\omega$  is a quasimetric and  $d_\omega$  is a metric, as described in [1].

**Example 2.14.** Let  $(X, d)$  be a metric space, as in Example 2.7, and let  $\omega$  be the Hausdorff  $t$ -discrepancy function on the closed and bounded subsets of  $X$ :

$$\omega(A, U) = \sup\{d(x, A) \mid x \in U\}.$$

Then  $s_\omega$  is a quasimetric and  $d_\omega$  is the Hausdorff metric.

**Example 2.15.** Let  $(X, \mathcal{H}, d)$  be an approach space, as in Example 2.8, and let  $\omega$  be the Hausdorff  $t$ -discrepancy function for the approach space  $(X, \mathcal{H}, d)$ :

$$\omega(A, U) = \sup\{d(x, A) \mid x \in U\}.$$

Then  $s_\omega$  is a pseudo-quasimetric and  $d_\omega$  is a pseudo-metric.

### 3. Proof of the main theorem and generalizations

Now we will prove our main theorem and describe extensions of these ideas. Thus let  $\omega$  be a  $t$ -discrepancy function. We wish to show first that  $s_\omega$  is a pseudo-quasimetric. Condition (i) for a pseudo-quasimetric is a consequence of condition (df1) for a  $t$ -discrepancy function. Condition (ii) follows from the associated condition for  $\omega$ . Next, to see that  $d_\omega$  is a pseudo-metric, observe that the symmetry condition is built into the definition of  $d_\omega$ .

Now, assume that  $\omega$  is a proper  $t$ -discrepancy function. Then condition (df3) implies that  $s_\omega$  is a quasimetric and  $d_\omega$  is a metric. This proves our main theorem.

For generalizations of our results, we consider functions  $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  that satisfy the following conditions:

(a)  $f$  is monotone in each variable: If  $x_1 \leq x_2$  and  $y_1 \leq y_2$ , then

$$f(x_1, y_1) \leq f(x_2, y_2).$$

(b)  $f$  is *subadditive* in the following sense:

$$f(x + y, z + w) \leq f(x, z) + f(y, w).$$

(c)  $f$  is *symmetric*:  $f(x, y) = f(y, x)$ .

(d)  $f$  is *nondegenerate*:  $f(x, y) = 0$  iff  $x = y = 0$ .

**Theorem 3.1.** Let  $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  satisfy (a)–(d), and let  $s$  be a pseudo-metric. Define  $d$  by

$$d(x, y) = f(s(x, y), s(y, x)).$$

Then  $d$  is a pseudo-metric. If  $s$  is a quasimetric, then  $d$  is a metric.

**Proof.** Symmetry of  $d$  follows from its definition and symmetry of  $f$ . The triangle inequality for  $d$  is verified as follows:

$$\begin{aligned} d(x, z) &= f(s(x, z), s(z, x)) \leq f(s(x, y) + s(y, z), s(z, y) + s(y, x)) \\ &\leq f(s(x, y), s(y, x)) + f(s(y, z), s(z, y)) = d(x, y) + d(y, z). \end{aligned}$$

It is clear that  $d(x, x) = 0$ , so  $d$  is a pseudo-metric, as claimed. If  $s$  is a quasimetric, and if  $d(x, y) = 0$ , then

$$f(s(x, y), s(y, x)) = 0,$$

so that by the nondegeneracy of  $f$ , we have

$$s(x, y) = 0 = s(y, x),$$

and then it follows that  $x = y$ . Hence  $d$  is a metric in this case.  $\square$

To see how this provides for generalizations of our main theorem, observe that the function  $f$  defined by  $f(x, y) = \max\{x, y\}$  satisfies (a)–(d), but so does, for example, the function  $f$  defined by  $f(x, y) = x + y$ . In general, in our main theorem,  $\max$  may be replaced by any function  $f$  satisfying properties (a)–(d).

**Corollary 3.1.** *Let  $\omega$  be a  $t$ -discrepancy function, and define  $d$  by*

$$d(A, B) = f(s_\omega(A, B), s_\omega(B, A)),$$

where  $f$  satisfies properties (a)–(d). Then  $d$  is a pseudo-metric, and if  $\omega$  is a proper  $t$ -discrepancy function, then  $d$  is a metric.

**Example 3.1.** Let  $(X, d)$  be a metric space, and let

$$\mathcal{H} = \{A \in 2^X \mid A \text{ is bounded}\}.$$

Let  $s$  be the quasimetric on  $\mathcal{H}$  that is given by

$$s(A, B) = \inf\{\sup\{d(x, A) \mid x \in U\} \mid B \subseteq U\}.$$

If  $f$  satisfies properties (a)–(d) and we define  $\delta$  by

$$\delta(A, B) = f(s(A, B), s(B, A)),$$

then  $\delta$  is a metric analogous to the Hausdorff metric on  $\mathcal{H}$ . For example, if

$$\delta(A, B) = s(A, B) + s(B, A),$$

or if

$$\delta(A, B) = \sqrt{s(A, B)^2 + s(B, A)^2},$$

then  $\delta$  is a metric on  $\mathcal{H}$  (because the function  $f$  defined by  $f(x, y) = \sqrt{x^2 + y^2}$  satisfies (a)–(d)).

## References

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