Self-Homeomorphic Star Figures

WLODZIMIERZ J. CHARATONIK University of Wroclaw, Wroclaw, Poland, and McNeese State University, Lake Charles, Louisiana

ANNE DILKS DYE and JAMES F. REED McNeese State University, Lake Charles, Louisiana

1. Introduction

The work done in sections 1, 2, and 3 of this paper was included [2]. Since part of the talk given at the Lafayette Conference on Continua Theory and Dynamical Systems was taken from that paper, we include for completeness a summary of the theorems and examples contained in it. Proofs and details are in the original paper. In Section 4 we include some new work which was in the talk in Lafayette, but not in the original paper.

Some basic definitions, which will be used throughout the paper, are necessary. By a continuum, we mean a compact connected metric space. A neighborhood of a point x in a topological space X is a set N satisfying $x \in \text{int}(N)$. A point $x \in X$ is called a cut point of X [4, III, p. 41] if $X \setminus \{x\}$ is not connected. A point $x \in X$ is called a local cut point of X provided there is a connected neighborhood N of x, such that x is a cut point of N. Note that a local cut point is a point of local connectedness of the space X. We will, however, mainly use the notion of a local cut-point in locally connected spaces anyway. For connected sets, this definition of a cut point is equivalent to the definition of a separating point in [4, III, 8, p. 58], and the definition of a local cut point is equivalent to that of a local separating point in [4, III, 9, p. 61]. For a given space X the set of local cut points of X is denoted by LC(X).

2. Definitions and Basic Properties

In this section we recall definitions of four types of self-homeomorphic spaces and we discuss interrelations between them and some of their basic properties.

2.1 Definition: A topological space X is called self-homeomorphic if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X.

2.2 Definition: A topological space X is called strongly self-homeomorphic if for any open set $U \subseteq X$ there is a set V with nonempty interior such that $V \subseteq U$, and V is homeomorphic to X.

2.3 Definition: A topological space X is called pointwise self-homeomorphic at a point $x \in X$

if for any neighborhood U of x, there is a set V such that $x \varepsilon V \subseteq U$ and V is homeomorphic to X. The space X is called pointwise self-homeomorphic if it is pointwise self-homeomorphic at each point.

2.4 Definition: A topological space X is called strongly pointwise self-homeomorphic at a point $x \varepsilon X$ if for any neighborhood U of x, there is a neighborhood V of x such that $V \subseteq U$ and V is homeomorphic to X. The space X is called strongly pointwise self-homeomorphic if it is strongly pointwise self-homeomorphic at each of its points.

Recall that metric spaces X and Y are called *similar* if there is a surjection $f: X \to Y$ such that there is a positive constant c satisfying $d(f(x), f(y)) = c \cdot d(x, y)$. Such a map is called a similarity. If we replace the condition that V is homeomorphic to X in each of the above definitions with the condition that V is similar to X, we get definitions of self-similar spaces. Thus every (strongly, pointwise or strongly pointwise) self-similar space is (strongly, pointwise or strongly pointwise) self-homeomorphic, but not conversely. Of course we could consider any class of mappings besides homeomorphisms or similarities. For example, if we use affine transformations we get analogous definitions of self-affine spaces. However, homeomorphisms and similarities seem to be the most interesting ones.

An arc, and *n*-dimensional cube and a one point union of two *n*-dimensional cubes with the identification point of the boundary of both of the *n*-dimensional cubes are examples of strongly pointwise self-homeomorphic continua.

We note some basic properties of self-homeomorphic spaces.

2.5 Theorem: The following diagram of implications applies to the above definitions.

$$\begin{array}{ccc} 2.2 & \rightarrow & 2.1 \\ \uparrow & & \uparrow \\ 2.4 & \rightarrow & 2.3 \end{array}$$

In [2], one can find examples showing that none of the implications in the above diagram can be reversed, and that there is no relationship between 2.2 and 2.3.

3. Methods of Constructing Self-homeomorphic spaces.

In this section we recall a method of constructing self-homeomorphic spaces. The method comes from [1, 3.7, pp. 80-85] and its details and proofs of the theorems can be found in [2]. Let us recall some common notations. For a given compact metric space X, a nonempty subset A of X, and a positive number r, we put $N(A, r) = \{x \varepsilon X\colon \text{there exists } a \varepsilon A \text{ with } d(a, x) < r\}$.

For two nonempty compact subsets A and B of X we define the Hausdorff distance dist(A, B)

$= \inf\{r > 0 : A \subseteq N(B, r)$ and $B \subseteq N(A, r)\}$, and we denote by 2^X the hyperspace of all nonempty compact subsets of X equipped with the Hausdorff distance.

3.1 Definition: Let (X, d) be a metric space. A map f: X $\rightarrow$ X is called contractive if there is a constant $0 \leq s < 1$ such that $d(f(x), f(y)) \leq s \cdot d(x, y)$ for every x, y ε X. Any such number s is called a contractivity factor for f.

For any given compact metric space X and a set of contractive mappings $\{f_i: i \in I\}$ having common contractivity factor $s < 1$, let $F: 2^X \rightarrow 2^X$ be defined by $F(A) = cl(\cup\{f_i(A): i \in I\})$. Then F is contractive with contractivity factor s (see [2, Theorem 3.3]) and therefore according to the Banach Contraction Mapping Theorem (see e.g.[1, Theorem 1, p. 76]), there is exactly one fixed set under F, i.e. a set $A \in 2^X$ such that $F(A) = A$. Such a set A can be obtained as the limit Lim $F^n(B)$ for any $B \in 2^X$. Then we will write $A = \mathcal{F}(X, \{f_i : i \in I\})$.

Now we present some conditions on mappings f_i under which the set $\mathcal{F}(X, \{f_i : i \in I\})$ is self-homeomorphic.

3.2 Theorem. If the mappings $f_i : X \rightarrow X$ for $i \in I$ are embeddings, then $\mathcal{F}(X, \{f_i : i \in I\})$ is self-homeomorphic.

3.3 Theorem. If the mappings $f_i : X \rightarrow X$ for $i \in I$ are embeddings and satisfy int $f_i(X) \neq \emptyset$ and int $f_i(X) \cap$ int $f_j(X) = \emptyset$ for $i,j \in I$ with $i \neq j$, then $\mathcal{F}(X, \{f_i : i \in I\})$ is strongly self-homeomorphic.

3.4 Theorem. If the mappings $f_i : X \rightarrow X$ for $i \in I$ are embeddings and the set I is finite, then $\mathcal{F}(X, \{f_i : i \in I\})$ is pointwise self-homeomorphic.

3.5 Theorem. Under the assumptions of Theorem 3.4 the set is either a locally connected continuum, or is not connected.

4. A Family of Self-homeomorphic Plane Continua

In this section we will define a countable family of pointwise self-homeomorphic and strongly self-homeomorphic plane continua. Each member S_n, for $n \geq 2$, of this family will be located in a regular polygonal disk with vertices at $v_k = (1+a_n)\exp \frac{2k\pi i}{n}$, where a_n is a scale factor defined in Theorem 4.7. Also, each S_n will be defined as the only fixed set under some natural mapping. We will also investigate topological properties of S_n's.

Denote by P_n, for $n > 2$, the polygonal disk with its vertices $v_0, v_1, \ldots v_{n-1}$, as defined above. For $k \in \{0, 1, 2, 3, \ldots, n-1\}$, let f_k be the contractive mapping from P_n into P_n with contractivity factor a_n chosen in such a way that

(1) $f_i(P_n) \cap f_{i+1}(P_n) \neq \emptyset$ and (2) int $f_i(P_n) \cap$ int $f_j(P_n) = \emptyset$ for $i \neq j$

and such that f_k maps P_n onto a similar copy, of P_n centered at v_k as defined above. See Figure 1. We note that the image of P_n is "scaled down" by a factor of a_n (as defined in Theorem 4.7). Condition (2) above, along with Theorem 3.3 give that the fixed set under this set of mappings is strongly self-homeomorphic. Condition (1) above gives that $\bigcup_{k=0}^{n-1} f_k(P_n) = F(P_n)$ is connected. By the remarks following definition 3.1, $S_n = \mathcal{F}(P_n, \{f_i : i \in \{0, 1, \ldots, n-1\}\}) = \lim_{m\to\infty} F^m(P_n)$ and hence S_n is connected. Therefore, by Theorems 3.4 and 3.5 we have the following.

4.1 Proposition. For any $n>2$, the set S_n is a strongly self-homeomorphic, pointwiese self-homeomorphic, locally connected continuum.

Observe that S_2 is the segment with vertices $v_0 = 1$ and $v_1 = -1$, S_3 is the Sierpiński Triangle (see Figure 2) and S_4 is the solid square with vertices ± 1, $\pm i$. Observe also that $a_2 = a_3 = a_4 = \frac{1}{2}$. Pictures of S_3, S_5, S_6, and S_8 are in Figure 2.

Now we will prove that S_m is homeomorphic to S_n if and only if m and n are multiples of 4 with m, n > 4. We start with a lemma.

4.2 Lemma. For each $n \geq 2$ the intersection $f_i(P_n) \cap f_{i+1}(P_n)$ is a common edge of the polygons $f_i(P_n)$ and $f_{i+1}(P_n)$ if n is a multiple of 4, and it is a common vertex of these polygons otherwise.

Proof. Because of the definition of S_n and the convexity of P_n, the intersection $f_i(P_n) \cap f_{i+1}(P_n)$ is either an edge or a vertex of both $f_i(P_n)$ and $f_{i+1}(P_n)$. If it is an edge, then it is perpendicular to $f_i(v_i v_{i+1})$. Thus $f_i(P_n)$, and hence P_n has two perpendicular edges. This can happen only in the case when n is a multiple of 4.

4.3 Theorem. If $n > 4$ is a multiple of 4, then S_n is homeomorphic to the Sierpiński Universal Plane Curve.

Proof. For any such n, by 4.2, $f_i(P_n) \cap f_{i+1}(P_n)$ is a common edge. In the limit, $\lim F^n(P_n)$, this creates a Cantor set of points and hence the continuum S_n will contain no local cut points. According to the Whyburn Characterization Theorem (see [5, Theorem 3, p. 322]) every locally connected, one-dimensional, plane continuum with no local cut points is homeomorphic to the Sierpiński Universal Plane Curve.

Now we are going to prove that if one of the numbers m or n is not a multiple of 4, then S_m is not homoeomophic to S_n. To this aim we recall the notion of the structure of the set of local cut-points of a continuum as defined in [2, Definition 5.1].

4.4 Definition. We say that two continua X and Y have the same structure of the sets of local

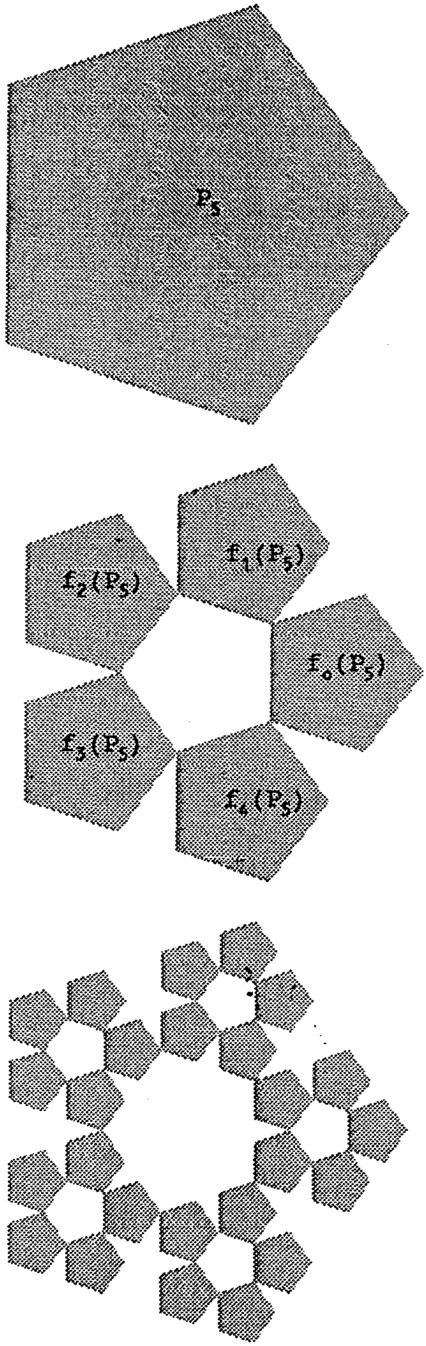

Figure 1

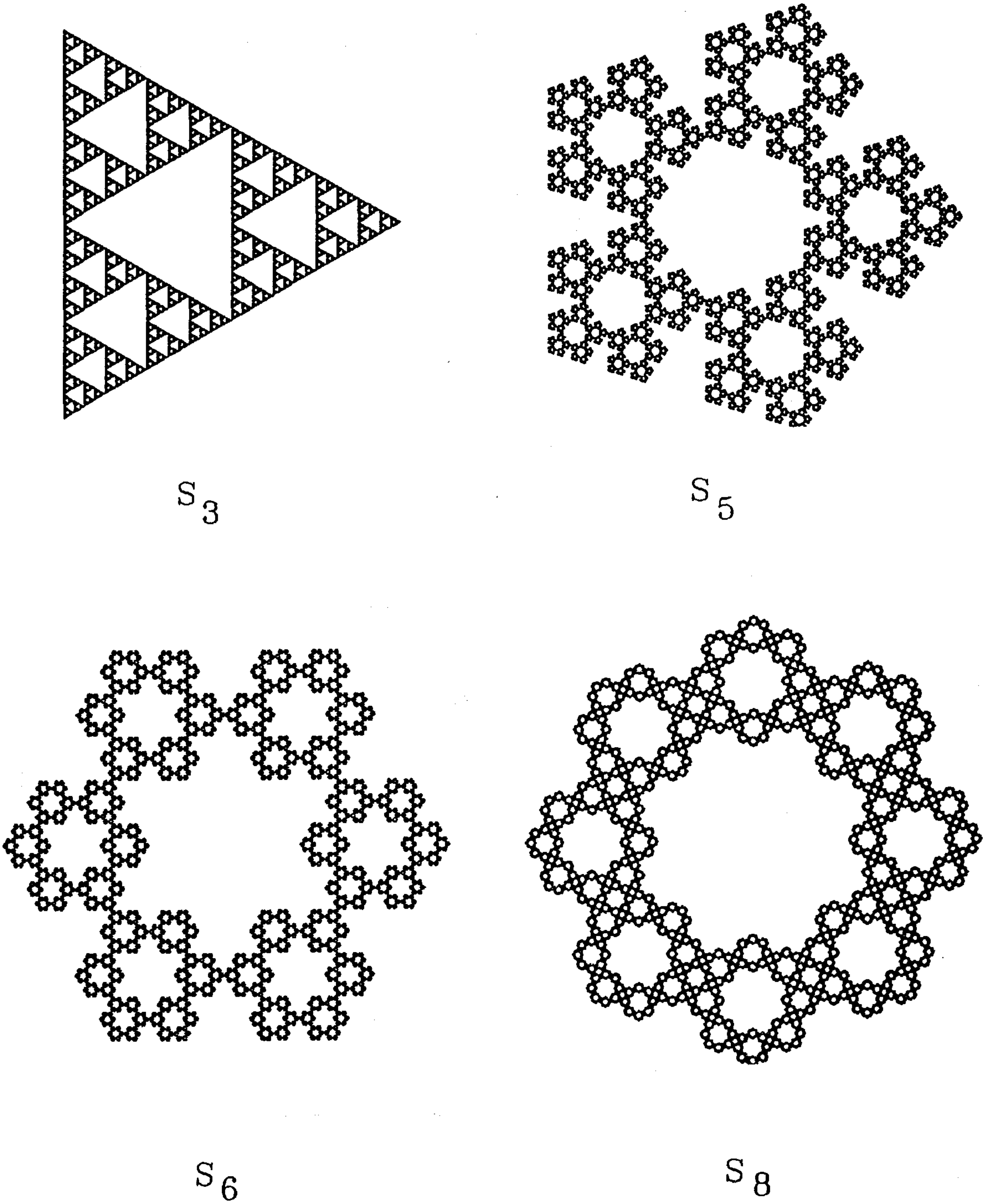

Figure 2

cut points if there is a homeomorphism α: LC(X) $\rightarrow$ LC(Y) such that a set A$\subseteq$ LC(X) disconnects X if and only if α(A) disconnects Y and the number of components of X\A is the same as the number of components of Y\α(A).

In general the structure of the set of local cut points does not distinguish self-homeomorphic continua. In [2, Example 5.2] an example is shown of two nonhomeomorphic strongly self-homeomorphic and pointwise self-homeomorphic plane curves U and H having the same structure of the sets of local cut points. However, we will show that for the family $\{S_n : n \varepsilon \text{ N with } n \geq 2 \text{ and } n \neq 4\}$ the structures of local cut points determines homeomorphic members of the family.

4.5 Proposition. Let m, n $\geq$ 2 be two distinct natural numbers. If one of them is not a multiple of 4, then S_m and S_n do not have the same structure of the sets of local cut points.

Proof. Observe that S_2 is the only member of the family $\{S_n : n \varepsilon \text{ N with } n\geq 2\}$ having (global) cut points, and that S_n has no local cut points if and only if n is a multiple of 4.

Fix a number n$\geq$ 3 which is not a multiple of 4, and let p ε S_n be any local cut point. We will show that there are exactly n$-$1 points $q_1, q_2, \ldots q_{n-1}$ of S_n such that each of the sets $\{p, q_j\}$ for j ε $\{1,\ldots,n-1\}$ disconnects S_n. Note that this property depends on the structure of the sets of local cut points only, and it distinguishes S_n's if n is not a multiple of 4.

Indeed, if p is a local cut point of S_n, then there exists a natural number k such that p is the only point of the intersection

$$f_{i_k}(f_{i_{k-1}}(\ldots f_{i_1}(P_n)\ldots)) \cap f_{(i_k+1)\text{mod } n}(f_{i_{k-1}}(\ldots f_{i_1}(P_n)\ldots)),$$

for some $i_1, \ldots i_k$ ε $\{0,\ldots,n-1\}$. For each i ε $\{0,\ldots,n-1\}\backslash\{i_k\}$ the intersection

$$f_i(f_{i_{k-1}}(\ldots f_{i_1}(P_n)\ldots)) \cap f_{(i+1)\text{mod } n}(f_{i_{k-1}}(\ldots f_{i_1}(P_n)\ldots))$$

is a one point set. Denote the points of the intersections by $q_1, \ldots, q_{n-1}$ and note that each two element set $\{p, q_i\}$ for j ε $\{1,\ldots,n-1\}$ disconnects S_n as required. This finishes the proof.

The following corollary is a consequence of Theorem 4.3 and Proposition 4.5.

4.6 Corollary. Two continua S_m and S_n for m, n $\geq$ 2 are homeomorphic if and only if either m = n or m, n > 4 and both m and n are multiples of 4.

We end the paper by calculating the scale a_n of the similarities $f_0, f_1, \ldots, f_{n-1}$ needed to construct the continuum S_n, as in the definitions of S_n (see the beginnning of this section).

4.7 Theorem. For every n $\geq$ 2 let m be the nearest integer to $\frac{n+2}{4}$ if n is not a multiple of 4 and let $m = \frac{n}{4}$ if n is a multiple of 4. Then the scale a_n is expressed by

(4.7.1) $$a_n = \frac{\sin\frac{\pi}{n}}{2\sin\frac{m\pi}{n}\cos\frac{(m-1)\pi}{n}}.$$

Proof. Let p be the point of intersection $f_0(P_n) \cap f_1(P_n)$. Let m and j be integers such that $p \in f_m(f_0(P_n)) \cap f_j(f_0(P_n))$, i.e.,

(4.7.2) $$p = f_m(f_0(v_m)) \text{ and } p = f_j(f_0(v_j)),$$

where v_k is the k-th vertex of P_n, for $k \in \{1, \ldots, n-1\}$ (see the beginning of this section).

Because of the symmetry of S_n with respect to the line $y = \tan\frac{\pi}{n}$ we have $m+j = n+1$. Observe that among all directions of vectors $\overline{f_0(0)f_0(v_k)}$ the direction of $\overline{f_0(0)f_0(v_m)}$ is the closest to the direction of the vector $\overline{f_0(0)f_1(0)}$. Calculating the directions we conclude that among the numbers $\frac{2\pi k}{n}$ the number $\frac{2\pi m}{n}$ is the closest to $\frac{(n+2)\pi}{2n}$, and hence m is the closest integer to $\frac{n+2}{4}$. Further, by (4.7.2) we have $1 + a_n(v_m - 1) = v_1 + a_n(v_j - v_1)$, and so $a_n(1 - v_1 + v_j - v_m) = 1 - v_1$. Passing to the real parts of both sides we get

$$a_n(1 - \cos\frac{2\pi}{n} + \cos\frac{2j\pi}{n} - \cos\frac{2m\pi}{n}) = 1 - \cos\frac{2\pi}{n}.$$

After some elementary calculations we obtain (4.7.1).

References

1. M. Barnsley, (1988) Fractals Everywhere, Academic Press, Inc., New York, New York.

2. W.J. Charatonik and A. Dilks, *On self-homeomorphic spaces,* Topology Appl. (to appear).

3. K. Kuratowski, (1968) Topology, Vol. II, Academic Press, Inc., New York, New York.

4. G.T. Whyburn, (1942) Analytic Topology, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I.

5. G.T. Whyburn, *Topological Characterizations of the Sierpiński Curve,* Fund. Math. 45 (1958), 320-324.