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Some Remarks on Continuously Homogeneous Continua

by

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Summary. The paper is devoted to continuously homogeneous continua. We consider products, hyperspaces and arc components of such continua.

In this paper we generalize some facts and answer some questions of [4]. After preliminaria we give two counterexamples concerning products. The first is an example of a continuously homogeneous product of two continua one of which has, while the other has not, this property. The second example is a continuously homogeneous countable power X^{∞} of a non-continuously homogeneous continuum X. Both these examples answer Question 1 of [4], p. 346 in the negative.

Next we prove two theorems on hyperspaces of continua. Namely we show that the hyperspace 2^X is continuously homogeneous for an arbitrary continuum X, and the hyperspace $C(X)$ is such for a continuum X containing an open subsets with uncountably many components.

The last part of the paper is devoted to arc components of continuously homogeneous continua. We show that if such a continuum has finitely many arc components, then each of them is dense. This generalizes Propositions 4 and 5 of [4], p. 354.

All spaces considered in this paper are metric and all mappings are continuous. A continuum is a compact, connected space.

The following concept is due to David P. Bellamy. A space X is called continuously homogeneous (shortly c.h.) if for every two points $x, y \in X$ there is a continuous surjection of X onto itself sending x to y.

For a continuum X we denote by $C(X)$ the hyperspace of all subcontinua of X and by 2^X the hyperspace of all compact nonempty subsets of X.

Let I denote the unit interval $[0, 1]$. The cone $X \times I / X \times \{0\}$ over a space X is denoted by Cone (X) and we write $(x, t) \in \text{Cone}(X)$ for $x \in X$ and $t \in I$ understanding $(x, 0)$ as the vertex of the cone for every $x \in X$.

Recall a useful

THEOREM A ([4], Proposition 2, p. 346). *If there exist surjections $f: X \to Y$ and $g: Y \to X$, then X is continuously homogeneous if and only if Y is.*

1. Products. Krupski has shown in Proposition 1 of [4] that if all factor spaces are c.h. then their product is c.h., too, and he has asked whether the converse is true ([4], Question 1, p. 346). We answer this question in the negative by showing two counterexamples.

EXAMPLE 1. There are two continua X and Y such that X is and Y is not continuously homogeneous and their product is continuously homogeneous.

Indeed, let X be the pseudo-arc and Y be the sinusoidal curve (i.e. $\mathrm{cl}\,\{(x, \sin(1/x)): 0 < x \leqslant 1\}$). Then X is even homogeneous and hence c.h. and Y is not c.h. by Corollary 1 of [4], p. 354. To see $X \times Y$ is c.h. it is enough, by Theorem A, to define surjections $f: X \times Y \to X$ and $g: X \to X \times Y$. We can take the natural projection as the mapping f. To define the mapping g let us recall that there are surjections $g_1: X \to X \times X$ (see [2], Theorem 3.5, p. 387 and [3], Theorem 3.2, p. 293) and $g_2: X \to Y$ (see [2], Theorem 4.1, p. 389). So we can put $g = (\mathrm{id} \times g_2) \circ g_1$, where id denote the identity map on X.

In the rest part of this section we use the concept of smoothness for dendroids and fans. The reader can find the needed definitions in [1].

EXAMPLE 2. There exists a smooth fan F such that F is not continuously homogeneous while its countable power F^∞ is.

Really, let F be the non-c.h. smooth fan described in [4], Example 1, p. 348. To see F^∞ is c.h. we prove two propositions.

PROPOSITION 1. *Let a space X be given. If there exists a surjection $f: \mathrm{Cone}(X) \to X$, then the countable power X^∞ is continuously homogeneous.*

Proof. We use Theorem A. Observe that the existence of a surjection $f: \mathrm{Cone}(X) \to X$ implies connectedness of X, and hence X can be mapped onto the interval I (if X is degenerate then the proof is obvious). Shrinking the set $X \times \{0\} \subset X \times I$ to a point we conclude that there exists a surjection $g: X \times X \to \mathrm{Cone}(X)$. So the power g^∞ is a surjection of X^∞ onto $(\mathrm{Cone}(X))^\infty$. Similarly the power f^∞ maps $(\mathrm{Cone}(X))^\infty$ onto X^∞. To complete the proof observe that $(\mathrm{Cone}(X))^\infty$ is c.h. since $\mathrm{Cone}(X)$ is ([4], Proposition 3, p. 346) and the product of c.h. spaces is c.h. ([4], Proposition 1, p. 346).

PROPOSITION 2. *Each smooth dendroid X is an image of its cone.*

Proof. Each smooth dendroid admits a radially convex metric d with respect to the initial point p ([1], Theorem 10, p. 310). Sending a point $(x, t) \in \mathrm{Cone}(X)$ to the only point y belonging to the arc px and satisfying $d(y, p)/d(x, p) = t$ we obtain a continuous surjection of $\mathrm{Cone}(X)$ onto X.

Note that the assumption of Proposition 1 is necessary for X^{∞} to be c.h. In fact, if X is the sinusoidal curve, then X^{∞} has a dense arc component and a nowhere dense arc component. Then no point of the dense arc component may be mapped by a surjection onto a point of the nowhere dense arc component.

QUESTION 1. Are there two continua X and Y such that X and Y both are not, while $X \times Y$ is continuoúsly homogeneous? Is it possible to find such an example with $X = Y$?

2. Hyperspaces. We shall prove two theorems.

THEOREM 1. *For each continuum X its hyperspace 2^X is continuously homogeneous.*

Proof. Consider two cases. If X is locally connected, then 2^X is a Hilbert cube ([5], (1.97), p. 137), so it is homogeneous even. If X is not locally connected, then there are surjections mapping 2^X onto the Cantor fan ([5], (1.39), p. 94) and mapping the Cantor fan onto 2^X ([5], (1.33), p. 81). To complete the proof observe that the Cantor fan is c.h. as the cone over the Cantor set ([4], Proposition 3, p. 346) and use Theorem A.

THEOREM 2. *If a continuum X is locally connected or if it contains an open subset with uncounably many components, then $C(X)$ is continuously homogeneous.*

Proof. If X is locally connected, then $C(X)$ is locally connected, too ([5], (1.92), p. 134), so it is c.h. ([4], Theorem 1, p. 347). If X has an open subset with uncountably many components, then there are surjections from $C(X)$ onto the Cantor fan ([5], (1.45), p. 98) and from the Cantor fan onto $C(X)$ ([5], (1.33), p. 81). So the conclusion follows by Theorem A as in the final part of he proof of Theorem 1.

QUESTION 2. Is the hyperspace $C(X)$ continuously homogeneous for every continuum X?

3. Arc components. In this paragraph we prove a theorem which generalizes Proposition 4 and 5 of [4], p. 354.

THEOREM 3. *Let a continuum X be continuously homogeneous and non-arcwise connected with finitely many arc components. Then each of them is dense and boundary.*

Proof. We prove that each arc component is dense. Let $K_1, \ldots, K_n$ be arc components of X. Assume on the contrary that there exists a point $x \in X$ which belongs to closures of exactly m arc components with $m < n$. Since every surjection of X onto itself permutes arc components (and so their closures) and since X is c.h., every point of X belongs to closures of exactly m arc components. Then the family of sets of the form $\overline{K_{\sigma(1)}} \cap \ldots \cap \overline{K_{\sigma(m)}}$, where σ is a one-to-one function from the set $\{1, \ldots, m\}$ into the set $\{1, \ldots, n\}$, is a partition of the continuum X onto finitely many closed subsets of X. This contradics the connectedness of X. So each arc component is dense; thus each one has the dense complement, so it is boundary.

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В. Я. Харатоник, З. Гарнцарек, **Некоторые замечания о непрерывно однородных континуумах**

Настоящая работа посвящена непрерывно однородным континуумам. Рассматриваются произведения, гиперпространства и дуговые компоненты этих континуумов.