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GATE CONTINUA, ABSOLUTE TERMINAL CONTINUA AND ABSOLUTE RETRACTS

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Abstract. The following classes $\mathcal{K}$ of continua are studied in the paper: (hereditarily decomposable) arc-like, hereditarily irreducible, atriodic, and containing no n-od. If $X, Y, Z, K \in \mathcal{K}$, with $X \cup Y = Z$, $X \cap Y = K$ and $X \neq Z \neq Y$, then K is called a gate continuum in X for $\mathcal{K}$. We characterize gate subcontinua in members of the above classes $\mathcal{K}$. The characterizations are used to construct absolute terminal continua for $\mathcal{K}$, i.e., continua X in $\mathcal{K}$ that are terminal in $Y \in \mathcal{K}$ whenever $X \subset Y$. These results are applied to investigating properties of absolute retracts for $\mathcal{K}$.

1. Introduction

Investigations which are the subject of the present paper have its roots in studies of absolute retracts for various classes of continua (see [21], [9] and [8], for example). Examining properties of absolute retracts for arc-like continua led to distinguish a class of their subcontinua called gate continua. The concept generalizes the one of an end point as defined by Bing in [3, Section 5, p. 660], and its role can be compared to the role of outlet points in [17].

To study gate continua for any pair (X, Y) of continua with $X \subset Y$ we introduce an integer function $\psi(X, Y)$ describing a certain type of ramification of Y at X. The function can be seen as a general analog of the order of a graph at a point. The function ψ is used to characterize gate subcontinua in the members of the following classes of continua: (hereditarily decomposable) arc-like, hereditarily irreducible, atriodic, and containing no n-od.

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The concept of a terminal continuum in the sense of Wallace, see [30], proved to be important and useful, and was extensively studied (see e.g. [16], [25] and [26]). Studying gate continua we have found, for any class $\mathcal{K}$ mentioned above (excluding hereditarily decomposable arc-like continua) nondegenerate continua in $\mathcal{K}$ which are always terminal whenever embedded in a member of $\mathcal{K}$. We call such continua absolute terminal in $\mathcal{K}$.

Next we study absolute retracts for the classes $\mathcal{K}$. We concentrate our attention on finding necessary conditions for a continuum to be an absolute retract for $\mathcal{K}$. The conditions are written in the form of an alternative, however we do not have any examples showing that all parts of the alternative can be realized.

2. Preliminaries

By a *space* we mean a topological space, and a *mapping* means a continuous function. Given a space X and its subspace $Y \subset X$, a mapping $r : X \to Y$ is called a *retraction* if the restriction $r|Y$ is the identity. Then Y is called a *retract* of X. The reader is referred to [4] and [15] for needed information on these concepts.

In this paper the term a *class* of spaces means a topological class of spaces, i.e., for any member M of the class, all homeomorphic copies of M belong to the class. Let $\mathcal{C}$ be a class of *compacta*, i.e., of compact metric spaces. Following [15, p. 80], we say that a space $Y \in \mathcal{C}$ is an *absolute retract for the class* $\mathcal{C}$ (abbreviated AR($\mathcal{C}$)) if for any space $Z \in \mathcal{C}$ such that Y is a subspace of Z, Y is a retract of Z. The concept of an AR space originally had been studied by K. Borsuk, see [4].

By a *continuum* we mean a connected compactum. A class of continua is said to be *hereditary* provided that if a continuum is in the class, then each of its subcontinua also belongs to the class.

A *curve* means a one-dimensional continuum. For a given a continuum X, an *arc component* of X means the union of all arcs A such that $p \in A \subset X$ for some point p of X. A continuum X is said to be *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected. X is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A locally connected dendroid is called a *dendrite*. A *tree* means a graph containing no simple closed curve, or, in other words, a dendrite having finitely many end points (in the classical sense; this means that if an arc contains the point, then the point is an end point of the arc).

A continuum is said to be *decomposable* provided that it can be represented as the union of two its proper subcontinua. Otherwise it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its subcontinua is decomposable (indecomposable, respectively). A hereditarily unicoherent and hereditarily decomposable continuum is called a λ-*dendroid*.

A continuum X is said to be *irreducible* (between points a and b of X) provided that no proper subcontinuum of X contains these points. Points a and b are then called *points of irreducibility of* X. An irreducible continuum X is said to be of *type* λ if each indecomposable subcontinuum of X has empty interior.

A subcontinuum X of a continuum Y is called *terminal in* Y if every subcontinuum of Y that intersects X and $Y \setminus X$ contains X. This concept should not be confused with other ones under the same name, as e.g. in [2, Definition 1.1, p. 7], [13, p. 461], or in [27, 1.54, p. 107].

A continuum X is said to be *arc-like* (*tree-like*) provided that it is the inverse limit of an inverse sequence of arcs (of trees). It is known that each arc-like continuum is irreducible, and that it can be embedded in the plane.

For a given integer $n \geq 3$, a continuum X is called an n-*od* provided that there is a subcontinuum Z of X such that $X \setminus Z$ is the union of n nonempty sets each two of which are mutually separated in X. By a *s*imple n-od we mean a space which is homeomorphic to the cone over over an n-point discrete space. A (simple) 3-od is called a *(simple) triod*. A continuum is said to be *atriodic* provided that it does not contain any triod. It is known that each arc-like continuum is atriodic (see [3, p. 653]).

A mapping $f : X \to Y$ between continua is said to be:
— *monotone* if it has connected point inverses;
— *atomic* provided that $f^{-1}(y)$ is a terminal subcontinuum of X for each point $y \in Y$ (for other equivalent conditions see [24, Proposition 4, p. 536]);
— an ε-*mapping* (for a given $\varepsilon > 0$) provided that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in Y$.

The following notation will be used. Given an inverse sequence $\mathbf{S} = \{X_n, f_n\}$ of compact spaces X_n with bonding mappings $f_n : X_{n+1} \to X_n$, where the set of positive integers $\mathbb{N}$ is taken as the directed set of indices, we denote by $X = \varprojlim \mathbf{S}$ its inverse limit and by $\pi_n : X \to X_n$ the projections. Further, let $f^n_m : X_n \to X_m$ be the bonding mapping $f^n_m = f_m \circ f_{m+1} \circ \cdots \circ f_{n-1}$ of $\mathbf{S}$ for $m < n$, and $f^n_n = \mathrm{id}\,|X_n$. In particular, $f^{n+1}_n = f_n$.

3. Gate continua for certain classes of spaces

In this chapter we introduce and study a concept of a gate continuum for a class of continua. We study gate continua for the following classes of spaces: arc-like λ-dendroids, in abbreviation $\lambda\mathcal{AL}$, i.e., hereditarily decomposable arc-like continua, arc-like continua — $\mathcal{AL}$, hereditarily irreducible continua — $\mathcal{HI}$, atriodic continua — $\mathcal{NT}$, and finally, for a fixed $n \geq 3$, we denote by $\mathcal{N}n$-$\mathcal{OD}$ the class of all continua containing no n-od. Observe that

$$\lambda\mathcal{AL} \subset \mathcal{AL} \subset \mathcal{HI} \subset \mathcal{NT} = \mathcal{N}3\text{-}\mathcal{OD} \subset \mathcal{N}4\text{-}\mathcal{OD} \subset \dots \tag{3.1}$$

Definition 3.2. Let $\mathcal{K}$ be a class of continua and let $X \in \mathcal{K}$. Then a subcontinuum K of X is said to be a *gate continuum in* X *for* $\mathcal{K}$ (or shortly a *gate continuum in* X if it is clear what class $\mathcal{K}$ is considered for), provided there exists a continuum $Y \in \mathcal{K}$ such that $Y \setminus X \neq \emptyset$, $X \cap Y = K$ and $X \cup Y \in \mathcal{K}$.

The following concept has been introduced in [8]. A class $\mathcal{S}$ of nonempty spaces is called *unionable* provided that for all members X, Y of $\mathcal{S}$ if $X \cap Y \in \mathcal{S}$, then $X \cup Y \in \mathcal{S}$. The following classes of spaces are unionable: compact spaces of dimension less than or equal to n, continua, tree-like continua, λ-dendroids, dendroids, dendrites (see [8, Observation 2.2]), while the classes of hereditarily decomposable continua as well as of unicoherent ones are not (see [8, Examples 2.3 and 2.4]).

Observe that if X is any member of a unionable hereditary class $\mathcal{K}$, then any subcontinuum of X is gate in X for $\mathcal{K}$. Indeed, take the union of two copies of X with only points of K in common. Since $\mathcal{K}$ is unionable, the union is in the class $\mathcal{K}$. Thus K is gate in X for $\mathcal{K}$.

However, it is not the case for the classes investigated in this chapter, i.e., for the classes listed in (3.1).

We will see that the concept of a gate continuum is closely related to the following one of an end point. Here by an *end point* of a continuum X we mean a point $p \in X$ such that for any two continua P and Q containing p we have either $P \subset Q$ or $Q \subset P$ (compare [3, Section 5, condition (B), p. 660]). First we collect below several observations, the proofs of which we leave to the reader as an exercise. The proof of (3.3.4) may require the fact that in an arc-like continuum for any $\varepsilon > 0$ we always can start building an ε-chain cover at an end point (see [3, Theorem 12, p. 661 and Theorem 10, p. 659]), while the fact that an end point of a continuum is a point of irreducibility (see [3, Theorem 12, p. 661]) may be needed to prove (3.3.3).

Observation 3.3. *Let p be an end point of a continuum X and pq be an arc such that $pq \cap X = \{p\}$. Then, letting $Y = X \cup pq$ we have the following implications:*
(3.3.1) *if Z is a continuum in Y, then $Z \cap pq$ and $Z \cap X$ are continua;*
(3.3.2) *if X is atriodic, then so is Y;*
(3.3.3) *if X is irreducible, then so is Y;*
(3.3.4) *if X is arc-like, then so is Y;*
(3.3.5) *if X is hereditarily decomposable, then so is Y.*

This observation leads to the following corollary which shows that gate continua generalize the concept of an end point.

Corollary 3.4. *Let $\mathcal{K}$ be any of the classes $\lambda\mathcal{AL}$, $\mathcal{AL}$, $\mathcal{HI}$, $\mathcal{NT}$, and let $X \in \mathcal{K}$. Then for any point $p \in X$ the singleton $\{p\}$ is gate in X for $\mathcal{K}$ if and only if p is an end point of X.*

We say that $X_1 \cup \cdots \cup X_n$ is an *irreducible union* provided that no one of the sets $X_1, \ldots, X_n$ is contained in the union of the others. The next definition, closely related to the one of a gate continuum, will also be extensively employed. For any pair of continua K, X with $\emptyset \neq K \subset X$ define $\psi(K,X) = 0$ if $K = X$, and, if $K \neq X$, let $\psi(K,X)$ be the largest positive integer n such that there are continua $K_1, \ldots, K_n \in C(X)$ satisfying $K \subset K_1 \cap \cdots \cap K_n$ and $K_1 \cup \cdots \cup K_n$ is an irreducible union. If there is no such $n \in \mathbb{N}$ we write $\psi(K,X) = \infty$. One can extend this definition for varying infinite cardinals, but this is not needed here. We also define $\psi(X) = \sup\{\psi(K,X) : K \in C(X)\}$.

The above definition is related to the notion of an n-od by the following characterization of Sorgenfrey.

Theorem 3.5 (Sorgenfrey)**.** *A continuum contains an n-od if and only if it contains n subcontinua $X_1, \ldots, X_n$ such that $X_1 \cup \cdots \cup X_n$ is an irreducible union and that $X_1 \cap \cdots \cap X_n \neq \emptyset$.*

Outline of the proof. The definition of an n-od is equivalent by [29, Theorem 1.2, p. 441] (extended to n-ods in place of triods) to one of an n-od of type 3 as defined in [29, Condition 3, p. 440]. Combining this with [29, Theorem 1.8, p. 443] we get the conclusion. □

By the above characterization we have the following three propositions.

Proposition 3.6. *A nondegenerate continuum X is hereditarily indecomposable if and only if $\psi(X) = 1$.*

Proposition 3.7. *If $X \in \mathcal{HI}$, then $\psi(X) \leq 2$.*

Proposition 3.8. *$X \in \mathcal{N}n$-$\mathcal{OD}$ if and only if $\psi(X) < n$.*

By the definition we note the following properties of the function ψ.

Proposition 3.9. *If $K_1, K_2 \in C(X)$ and $K_1 \subset K_2$, then $\psi(K_1, X) \geq \psi(K_2, X)$.*

Proposition 3.10. *If $K, X \in C(Y)$ and $K \subset X$, then $\psi(K, X) \leq \psi(K, Y)$.*

Now we prove three general properties of the function ψ.

Proposition 3.11. *Let $\{X_n, f_n\}$, where $n \in \mathbb{N}$, be the inverse sequence of continua X_n with arbitrary bonding mappings $f_n : X_{n+1} \to X_n$ having a continuum $X = \varprojlim \{X_n, f_n\}$ as its inverse limit. Let $K_n \in C(X_n)$ be such that $f_n(K_{n+1}) \subset K_n$ and $K = \varprojlim \{K_n, f_n | K_{n+1}\}$. If m is a positive integer such that $\psi(\pi_n(K), X_n) < m$ for each $n \in \mathbb{N}$, then $\psi(K, X) < m$.*

Proof. Suppose that there are continua $L_1, \dots, L_m$ such that $K \subset L_1 \cap \dots \cap L_m$, and that $L_1 \cup \dots \cup L_m$ is an irreducible union. Since the projections $\pi_n : X \to X_n$ are ε-mappings for sufficiently great $n \in \mathbb{N}$, there is an $n_0 \in \mathbb{N}$ such that $(\pi_{n_0}(L_1) \cup K_{n_0}) \cup \dots \cup (\pi_{n_0}(L_m) \cup K_{n_0})$ is an irreducible union, contrary to the assumption. □

Corollary 3.12. *Let m be a positive integer, and let $\{K_n\}$ be a decreasing sequence of subcontinua of X. If $\psi(K_n, X) < m$ for each n, then*

$$\psi(\bigcap \{K_n : n \in \mathbb{N}\}, X) < m.$$

Proof. Indeed, it is enough to take f_n^{n+1} as the identity mapping on X for each $n \in \mathbb{N}$. Then $f_n^{n+1} | K_{n+1}$ is the embedding, so the inverse limit is equal to the intersection. □

Recall that a mapping $f : X \to Y$ between continua is said to be *confluent* provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f. For confluent mappings we have the following result.

Proposition 3.13. *Let $f : X \to Y$ be a confluent mapping from a continuum X onto Y, and let $K \in C(X)$. Then $\psi(K, X) \geq \psi(f(K), Y)$.*

Proof. Put $m = \psi(f(K), Y)$ and let $L_1, \dots, L_m$ be subcontinua of Y such that $f(K) \subset L_1 \cap \dots \cap L_m$ and that $L_1 \cup \dots \cup L_m$ is an irreducible union. For $j \in \{1, \dots, m\}$ let M_j be the component of $f^{-1}(L_j)$ that contains K. Then $K \subset M_1 \cap \dots \cap M_m$ and $M_1 \cup \dots \cup M_m$ is an irreducible union, so $\psi(K, X) \geq m$. □

Since for inverse systems of compact spaces if the bonding mappings are confluent, then the projections are confluent, too (see [7, Corollary 4, p. 5]), as a corollary to Proposition 3.13 we get for confluent mappings a similar result as Proposition 3.11, but with the opposite inequalities.

Proposition 3.14. *Let $\{X_n, f_n\}$, where $n \in \mathbb{N}$, be the inverse sequence of continua X_n with confluent bonding mappings $f_n : X_{n+1} \to X_n$ having a continuum $X = \varprojlim \{X_n, f_n\}$ as its inverse limit, and let $\pi_n : X \to X_n$ be the projections. Let $K \in C(X)$, and let m be a positive integer. If there is an $n_0 \in \mathbb{N}$ such that $\psi(\pi_{n_0}(K), X_{n_0}) \geq m$, then $\psi(K, X) \geq m$.*

Proposition 3.15. *If X, Y, K are continua such that $X \cap Y = K$, then*

$$\psi(K, X \cup Y) = \psi(K, X) + \psi(K, Y).$$

Proof. Let $K_1, \dots, K_n$ be continua in $X \cup Y$ such that $K \subset K_1 \cap \dots \cap K_n$ and let $K_1 \cup \dots \cup K_n$ be an irreducible union. Then the sets $L_i = K_i \cap X$ and $M_i = K_i \cap Y$ are continua and the number of all L_i's (of all M_i's) having a point not belonging to any other L_j (other M_j) is at most $\psi(K, X)$ (at most $\psi(K, Y)$, respectively). Thus it follows that $\psi(K, X \cup Y) \leq \psi(K, X) + \psi(K, Y)$.

Now, if $P_1, \dots, P_m \in C(X)$, $Q_1, \dots, Q_l \in C(Y)$, $K \subset P_1 \cap \dots \cap P_m \cap Q_1 \cap \dots \cap Q_l$ and the unions $P_1 \cup \dots \cup P_m$ and $Q_1 \cup \dots \cup Q_l$ are irreducible ones, then $P_1 \cup \dots \cup P_m \cup Q_1 \cup \dots \cup Q_l$ is an irreducible union, too. This implies $\psi(K, X \cup Y) \geq \psi(K, X) + \psi(K, Y)$. □

Proposition 3.16. *If X, Y, K are continua such that $X \cap Y = K$ and*

(3.16.1) *each continuum in $X \cup Y$ intersecting both $X \setminus Y$ and $Y \setminus X$ contains K,*

then

$$\psi(X \cup Y) = \max\{\psi(X), \psi(Y), \psi(K, X) + \psi(K, Y)\}.$$

Proof. The inequality $\max\{\psi(X), \psi(Y), \psi(K, X) + \psi(K, Y)\} \leq \psi(X \cup Y)$ is a consequence of Propositions 3.10 and 3.15.

To prove the other inequality, let $K_1 \cup \dots \cup K_n$ be the irreducible union of continua K_i, where $i \in \{1, \dots, n\}$, in $X \cup Y$ with $K_1 \cap \dots \cap K_n \neq \emptyset$ such that n is maximal. Define $L_i = K_i \cap X$ and $M_i = K_i \cap Y$ for $i \in \{1, \dots, n\}$. Observe that, by assumption (3.16.1), both L_i and M_i are connected for each $i \in \{1, \dots, n\}$.

Consider the following "cutting off" procedure. If some L_i containing K (some M_i containing K) is a subset of the union of some other L_i's (some other M_i's), then we replace K_i by M_i (by L_i, respectively). Observe that making such a replacement we again obtain the irreducible union of n continua in $X \cup Y$ with the nonempty intersection. Proceeding such replacements finitely many times we obtain the irreducible union of n continua with the nonempty intersection such that each further modification of this kind can change nothing. Without loss of generality we may assume that $K_1 \cup \cdots \cup K_n$ is the irreducible union such that the above cutting off procedure can change nothing.

One can observe that the following cases cover all of the possibilities.

CASE 1. There exists K_r such that $K_r \cap (X \setminus Y) \neq \emptyset \neq K_r \cap (Y \setminus X)$. Then, because of the cut off procedure, $(K_1 \cup L_r) \cup \cdots \cup (K_{r-1} \cup L_r) \cup M_r \cup (K_{r+1} \cup L_r) \cup \cdots \cup (K_n \cup L_r)$ is an irreducible union, and thus $\psi(X \cup Y) = n \leq \psi(K, X \cup Y) = \psi(K, X) + \psi(K, Y)$ according to Proposition 3.15.

CASE 2. $K_1 \cup \cdots \cup K_n \subset X$. Then $\psi(X \cup Y) = n \leq \psi(X)$.

CASE 3. $K_1 \cup \cdots \cup K_n \subset Y$. Then $\psi(X \cup Y) = n \leq \psi(Y)$.

CASE 4. For each $i \in \{1, \ldots, n\}$ either $K_i \subset X$ or $K_i \subset Y$, and neither Case 2 nor Case 3 holds. Then $K_1 \cap \cdots \cap K_n \cap K \neq \emptyset$, and thus, for each $i \in \{1, \ldots, n\}$, the set $K_i \cup K$ is connected. Then $(K_1 \cup K) \cup \cdots \cup (K_n \cup K)$ is an irreducible union, whence $\psi(X \cup Y) = n \leq \psi(K, X \cup Y) = \psi(K, X) + \psi(K, Y)$. The proof is complete. □

Although no class $\mathcal{N}n$-$\mathcal{OD}$ is unionable, we have the following kind of "weak unionability" for these classes. Its proof is an immediate consequence of Proposition 3.16.

Corollary 3.17. *Let X, Y, K be continua in the class $\mathcal{N}n$-$\mathcal{OD}$ for some integer $n \geq 3$ such that $X \cap Y = K$ and condition* (3.16.1) *is satisfied. Then $X \cup Y \in \mathcal{N}n$-$\mathcal{OD}$ if and only if $\psi(K, X) + \psi(K, Y) < n$.*

Recall the following result (see [1, Theorem, p. 35]). If H is a locally compact, non-compact, metric space, then each continuum is a remainder of H in some compactification of H.

Let a continuum X be given. Then, by the above quoted result, X is a remainder of some compactification $Y = H \cup X$ of a copy H of the half line $[0, \infty)$, i.e., $\operatorname{cl} H \setminus H = X$ and Y is compact. Denote by $\mathcal{R}(X)$ the class of all such compactifications Y. Observe the following statement.

Statement 3.18. *For any $Y \in \mathcal{R}(X)$ the continuum X is terminal in Y, and the quotient space Y/X is an arc.*

Proposition 3.19. *If $\mathcal{K}$ is any of the classes of continua listed in* (3.1)*, then for each continuum $X \in \mathcal{K}$ we have $\mathcal{R}(X) \subset \mathcal{K}$.*

PROOF. First we show the result for $\mathcal{K} = \mathcal{AL}$. Let $Y \in \mathcal{R}(X)$ for some $X \in \mathcal{AL}$. Then the quotient mapping $Y \to Y/X$ is atomic with arc-like fibers, and Y/X

is an arc, according to Statement 3.18. By [24, Proposition 11 (iii), p. 537] it follows that Y is in $\mathcal{AL}$.

If $\mathcal{K} = \lambda\mathcal{AL}$, the result is a consequence of the previous case; and it is obvious for other classes $\mathcal{K}$. □

In the following theorem we characterize gate continua for classes $\mathcal{N}n$-$\mathcal{OD}$, where $n \in \{3, 4, \dots\}$.

Theorem 3.20. *Let $n \in \{3, 4, \dots\}$ and $X \in \mathcal{N}n$-$\mathcal{OD}$. Then for any continuum $K \subset X$ the following conditions are equivalent:*

(3.20.1) *K is gate in X for $\mathcal{N}n$-$\mathcal{OD}$;*
(3.20.2) *there exists a continuum Z such that $X \cup Z \in \mathcal{N}n$-$\mathcal{OD}$, $Z \setminus X \neq \emptyset$, and K is a component of $X \cap Z$;*
(3.20.3) *$\psi(K, X) < n - 1$;*
(3.20.4) *for each $Y \in \mathcal{R}(K)$ such that $X \cap Y = K$ we have $X \cup Y \in \mathcal{N}n$-$\mathcal{OD}$.*

Proof. Implications (3.20.1) ⇒ (3.20.2) ⇒ (3.20.3) and (3.20.4) ⇒ (3.20.1) are obvious. To see (3.20.3) ⇒ (3.20.4) note that $\psi(K, Y) = 1$ for any $Y \in \mathcal{R}(K)$, and apply Corollary 3.17. □

Our next theorem is a consequence (and a generalization) of a result of Bing, [3, Theorem 12, p. 661]. Indeed, it suffices to shrink the continuum P below to a point and apply the result of Bing to the space X/P.

Theorem 3.21. *For each pair P, X of continua with $P \subset X$ we have $\psi(P, X) = 1$ if and only if any continuum X' such that $P \subset X' \subset X$ is irreducible with respect to containing P and some point in X'.*

Corollary 3.22. *If a continuum X is irreducible and $\psi(P, X) = 1$ for some $P \in C(X)$, then P contains a point of irreducibility of X.*

Proof. Let $x \in X \setminus P$ be a point such that any continuum in X containing P and x equals X (see Theorem 3.21), and let a continuum Q be irreducible between some point $p \in P$ and x. Then $P \cup Q = X$. If $Q = X$, the conclusion holds. If $P \setminus Q \neq \emptyset$, then X is irreducible between some points $p' \in P \setminus Q$ and $q \in Q \setminus P$. □

The next result is a variant of the "weak unionability" for the class $\mathcal{HI}$ of all hereditarily irreducible continua.

Theorem 3.23. *Let $X, Y, K \in \mathcal{HI}$ be such continua that $X \cap Y = K$ and*

(3.16.1) *each continuum in $X \cup Y$ intersecting both $X \setminus Y$ and $Y \setminus X$ contains K.*

Then

$$X \cup Y \in \mathcal{HI} \quad \textit{if and only if} \quad \psi(K,X) + \psi(K,Y) \leq 2.$$

Proof. Since all hereditarily irreducible continua are atriodic, the former condition implies the latter one by Corollary 3.17.

Assume $\psi(K,X) + \psi(K,Y) \leq 2$. The only nontrivial case to be proved is $\psi(K,X) = \psi(K,Y) = 1$. So, assume that these equalities hold. Let Z be any continuum in $X \cup Y$. If either $Z \subset X$ or $Z \subset Y$, then Z is irreducible by the assumption. Otherwise Z intersects both $X \setminus Y$ and $Y \setminus X$, and thus $K \subset Z$. We have $\psi(K, Z \cap X) = \psi(K, Z \cap Y) = 1$ by Proposition 3.10. So $Z \cap X$ is irreducible between some points $x \in X \setminus Y$ and $p_1 \in K$, and $Z \cap Y$ is irreducible between some points $y \in Y \setminus X$ and $p_2 \in K$ by Corollary 3.22. Take any continuum in Z irreducible between x and y, and observe that this continuum equals Z. Hence $X \cup Y \in \mathcal{HI}$. □

Now we present a characterization of gate continua for the class $\mathcal{HI}$. The characterization is similar to that of the classes $\mathcal{N}n$-$\mathcal{OD}$. Observe that conditions (3.24.1) and (3.24.2) of Theorem 3.27 below are essentially different, because there are non-hereditarily unicoherent hereditarily irreducible continua. Indeed, let X and Y be two disjoint hereditarily indecomposable continua such that X is irreducible between points $x_1, x_2 \in X$ and Y is irreducible between points $y_1, y_2 \in Y$. Identifying the pairs $\{x_1, y_1\}, \{x_2, y_2\}$ in the union $X \cup Y$ and taking singletons for other fibers, we obtain an upper semicontinuous decomposition of $X \cup Y$. Then the quotient space of this decomposition is not unicoherent, but it belongs to $\mathcal{HI}$.

Theorem 3.24. *For each continuum $X \in \mathcal{HI}$ and for each $K \in C(X)$ the following conditions are equivalent:*

(3.24.1) *K is a gate continuum in X for $\mathcal{HI}$;*
(3.24.2) *there exists a continuum Z such that $X \cup Z \in \mathcal{HI}$, $Z \setminus X \neq \emptyset$, and K is a component of $X \cap Z$;*
(3.24.3) *$\psi(K,X) \leq 1$;*
(3.24.4) *for each $Y \in \mathcal{R}(K)$ such that $X \cap Y = K$ we have $X \cup Y \in \mathcal{HI}$.*

Proof. Implications (3.24.1) ⇒ (3.24.2) ⇒ (3.24.3) and (3.24.4) ⇒ (3.24.1) are obvious. To prove (3.24.3) ⇒ (3.24.4) apply Theorem 3.23. □

Now we formulate a theorem concerning conditions under which the union of two arc-like continua is arc-like.

Theorem 3.25. *Let a class $\mathcal{K}$ be either $\lambda\mathcal{AL}$ or $\mathcal{AL}$, and let continua $X, Y, K \in \mathcal{K}$ be such that $X \cap Y = K$, $X \setminus Y \neq \emptyset \neq Y \setminus X$, and*

(3.16.1) *each continuum in $X \cup Y$ intersecting both $X \setminus Y$ and $Y \setminus X$ contains K.*

Then the following conditions are equivalent:

(3.25.1) *$X \cup Y$ belongs to $\mathcal{K}$;*
(3.25.2) *$\psi(K, X) = \psi(K, Y) = 1$;*
(3.25.3) *$X \cup Y$ is atriodic.*

Proof. Assume (3.25.1). The numbers $\psi(K, X)$ and $\psi(K, Y)$ must be positive by their definitions and by the assumptions of the theorem. None of them can excess 1 by Propositions 3.15 and 3.8 (for $n = 3$; the continuum $X \cup Y$ is in the class $\mathcal{K}$ whose elements are atriodic). This implies (3.25.2).

Assume (3.25.2). Applying Theorem 3.23 we infer that $X \cup Y$ is atriodic, so (3.25.3) holds.

Assume (3.25.3). A result of J. B. Fugate [13, Theorem 1, p. 466] says that the union of two arc-like continua with the nonempty intersection is arc-like if and only if it is atriodic and unicoherent. Therefore $X \cup Y$ is arc-like. In the case of $\mathcal{K} = \lambda\mathcal{AL}$ one can easily observe that $X \cup Y$ is hereditarily decomposable. Thus (3.25.1) holds. This finishes the proof. □

The next theorem is a generalization of a theorem of Bing, [3, Theorem 10, p. 659]. Its proof can exactly be rewritten from that of Bing just changing the meaning of p being a singleton in [3] to being the continuum P, replacing the phrase "$f_1 - \overline{f}_2$ contains p" with "$f_1 - \overline{f}_2$ intersects P" in the 13-th line from bottom on p. 659 of [3], and noting that the condition $\psi(P, M) = 1$ implies an analog of the condition (A) in [3, p. 659] by Theorem 3.25.

Theorem 3.26. *If D is a chain cover of a continuum M and P is a continuum in M satisfying $\psi(P, M) = 1$, then there is a chain cover $E = (e_1, \dots, e_n)$ of M such that E is a refinement of D and $e_1 \setminus e_2$ intersects P.*

Further, we obtain a result which, among others, contains a generalization of two theorems of Bing, [3, Theorems 12 and 13, p. 661].

Theorem 3.27. *For any nonempty subcontinuum K of an arc-like continuum X the following six conditions are equivalent:*

(3.27.1) *K is a gate continuum in X for the classes $\mathcal{AL}$, $\mathcal{HI}$ and $\mathcal{NT}$; moreover, K is a gate continuum for the class $\lambda\mathcal{AL}$ provided that $X \in \lambda\mathcal{AL}$;*
(3.27.2) *$\psi(K, X) \leq 1$;*
(3.27.3) *for each $Y \in \mathcal{R}(K)$ such that $X \cap Y = K$ the union $X \cup Y$ is arc-like;*

(3.27.4) *each continuum X' such that $K \subset X' \subset X$ is irreducible with respect to containing K and some point in X;*

(3.27.5) *for each $\varepsilon > 0$ there exists an ε-chain cover $(C_1, \ldots, C_n)$ of X such that $(C_1 \setminus \operatorname{cl} C_2) \cap K \neq \emptyset$;*

(3.27.6) *the image of K is an end point of the quotient continuum X/K.*

Proof. First we show that (3.27.1) is equivalent to (3.27.2). For $\mathcal{K} = \mathcal{AL}$ or $\mathcal{K} = \lambda\mathcal{AL}$ take as Y a compactification of the real half line $[0, \infty)$ having K as the remainder (see the above quoted result of [1, Theorem, p. 35]) and use the equivalence between (3.25.1) and (3.25.2). For $\mathcal{K} = \mathcal{HI}$ apply the equivalence between (3.24.1) and (3.24.3). And for $\mathcal{K} = \mathcal{NT}$ use the equivalence between (3.20.1) and (3.20.3) for $n = 3$.

Next observe that if $K = X$, then the equivalence of (3.27.2) and (3.27.3) follows from [24, Proposition 11 (iii), p. 537], while the other equivalences are trivial.

So, assume that $K \neq X$ or, equivalently, that $\psi(K, X) > 0$. Then (3.27.2) implies (3.27.3) again by [24, Proposition 11 (iii), p. 537] and by Theorem 3.25, while the opposite implication holds just by Theorem 3.25. The equivalence of (3.27.2) and (3.27.4) is established in Theorem 3.21. The implication from (3.27.2) to (3.27.5) is a consequence of Theorem 3.26.

Assume (3.27.5). We will show (3.27.6). Given an $\varepsilon > 0$, let $(C_1, \ldots, C_n)$ be the ε-chain cover of X guaranteed by (3.27.5). Take the chain cover $D = (q(C_1 \cup \cdots \cup C_k), q(C_{k+1}), \ldots, q(C_n))$ of X/K, where $q : X \to X/K$ is the quotient mapping, and $C_i \cap K \neq \emptyset$ if and only if $i \in \{1, \ldots, k\}$. In this way an ε-chain $(C'_1, \ldots, C'_m)$ of X/K with $q(K) \subset C'_1$ is obtained for any $\varepsilon > 0$. Thus $q(K)$ is an end point of X/K in view of [3, Theorem 13, p. 661]. Therefore (3.27.5) implies (3.27.6).

The implication from (3.27.6) to (3.27.2) can immediately be observed from the definitions. The proof is then complete. □

Corollary 3.28. *Let $\mathcal{K}$ be any of the classes of continua listed in* (3.1)*, let $X \in \mathcal{K}$ and $K \in C(X)$. Then the following three conditions are equivalent.*

(3.28.1) *K is a gate continuum in X for $\mathcal{K}$;*

(3.28.2) *for every continuum $Y \in \mathcal{R}(K)$ if $X \cap Y = K$ then $X \cup Y \in \mathcal{K}$;*

(3.28.3) *$\psi(K, X) < 2$ if $\mathcal{K} \in \{\lambda\mathcal{AL}, \mathcal{AL}, \mathcal{HI}, \mathcal{NT}\}$, and $\psi(K, X) < n - 1$ if $\mathcal{K} = \mathcal{N}n\text{-}\mathcal{OD}$.*

Proof. If $\mathcal{K}$ is either $\lambda\mathcal{AL}$ or $\mathcal{AL}$, the conclusion follows from Theorem 3.27. For $\mathcal{K} = \mathcal{HI}$ it follows from Theorem 3.24. Finally if $\mathcal{K} = \mathcal{N}n\text{-}\mathcal{OD}$, apply Theorem 3.20. □

Corollary 3.29. *Let $\mathcal{K}$ be any of the classes of continua listed in* (3.1)*, and let $X \in \mathcal{K}$. If $K \in C(X)$ is a gate continuum in X for $\mathcal{K}$, then*

(3.29.1) *each continuum L such that $K \subset L \subset X$ is gate in X for $\mathcal{K}$;*

(3.29.2) *there exists a minimal continuum $K_0 \subset K$ which is gate in X for $\mathcal{K}$.*

Proof. To see (3.29.1) use Corollary 3.28 and Proposition 3.9. To see (3.29.2) use Corollary 3.12 and apply the Kuratowski-Zorn Lemma. □

4. Absolute terminal continua

Accept the following definition.

Definition 4.1. A continuum X in a class $\mathcal{K}$ of continua is an *absolute terminal continuum for $\mathcal{K}$* provided that for each $Y \in \mathcal{K}$ such that $X \subset Y$ the continuum X is terminal in Y.

The above concept should not be confused with one of the absolutely terminal continuum as defined in [2, Definition 4.1, p. 34] and studied in [6], which will not be used in the present paper.

As a consequence of the definitions we have the following.

Proposition 4.2. *Let $\mathcal{K}$ be an arbitrary class of continua. If X is an absolute terminal continuum for $\mathcal{K}$, then X contains no proper gate continuum for $\mathcal{K}$.*

Our next result is related to the opposite implication.

Proposition 4.3. *Let $\mathcal{K}$ be a hereditary class of hereditarily unicoherent continua. If a member X of $\mathcal{K}$ contains no proper gate continuum for $\mathcal{K}$, then X is an absolute terminal continuum for $\mathcal{K}$.*

Proof. Indeed, if not, then there is a continuum $Y \in \mathcal{K}$ containing X such that X is not terminal in Y. Thus there exists a continuum $Z \subset Y$ such that $X \cap Z \neq \emptyset$ and $X \setminus Z \neq \emptyset \neq Z \setminus X$. Then $K = X \cap Z$ is a gate continuum in X for $\mathcal{K}$. □

As a consequence of Corollary 3.28 and Propositions 4.2 and 4.3 we get the following corollary.

Corollary 4.4. *If $\mathcal{K}$ is one of the classes of continua listed in* (3.1)*, then the following three conditions are equivalent for a continuum $X \in \mathcal{K}$:*

(4.4.1) *X is an absolute terminal continuum for $\mathcal{K}$;*

(4.4.2) *X contains no proper gate continuum in $\mathcal{K}$;*
(4.4.3) *for each $K \in C(X)\setminus\{X\}$ we have $\psi(K,X) = 2$ if $\mathcal{K} \in \{\lambda\mathcal{AL}, \mathcal{AL}, \mathcal{HI}, \mathcal{NT}\}$ and $\psi(K,X) = n-1$ if $\mathcal{K} = \mathcal{N}n\text{-}\mathcal{OD}$.*

Now we will show examples of absolute terminal continua for classes listed in (3.1). We start with classes $\mathcal{AL}$, $\mathcal{HI}$ and $\mathcal{NT} = \mathcal{N}3\text{-}\mathcal{OD}$. Recall that Bing in [3, Example 7, p. 662] has constructed an arc-like continuum with no end point. However, his example (the union of two Knaster simplest indecomposable continua with the end points identified only) contains proper gate continua (for the classes $\mathcal{AL}$, $\mathcal{HI}$ and $\mathcal{NT}$). Here we extend this result by showing that there are continua having no proper gate subcontinua for these classes.

Example 4.5. *There is an arc-like continuum X containing no proper gate subcontinuum for the classes $\mathcal{AL}$, $\mathcal{HI}$ and $\mathcal{NT}$.*

Proof. By Corollary 4.4 and the equivalence of conditions (3.27.1) and (3.27.2) in Theorem 3.24 it is enough to construct an arc-like continuum X such that for each proper subcontinuum K of X we have $\psi(K,X) = 2$.

Let $f : [0,1] \to [0,1]$ be a (piecewise linear) mapping satisfying

$$f(0) = \tfrac{1}{2},\ \ f(\tfrac{1}{5}) = 1,\ \ f(\tfrac{2}{5}) = 0,\ \ f(\tfrac{3}{5}) = 1,\ \ f(\tfrac{4}{5}) = 0,\ \ f(1) = \tfrac{1}{2},$$

and being linear on each of the intervals $[\frac{i}{5}, \frac{i+1}{5}]$ for $i \in \{0,\dots,4\}$. Define $X_n = [0,1]$ and $f_n = f : X_{n+1} \to X_n$ for each $n \in \mathbb{N}$, and finally

$$X = \varprojlim \{X_n, f_n\}.$$

To show that for each proper subcontinuum K of X we have $\psi(K,X) = 2$ it suffices to prove that each proper subcontinuum K of X is an arc, and that there exists an arc L in X containing K such that no end point of K is an end point of L. So, let K be a proper subcontinuum of X. Denote by $\pi_n : X \to X_n$ the natural projection, and consider two cases.

Case 1. $(\frac{1}{2}, \frac{1}{2}, \dots) \in K$.
Let n be a positive integer such that $\pi_n(K)$ is a proper subinterval of $[0,1]$. Then either 0 or 1 is not in $\pi_n(K)$. By the symmetry of the graph of f we can assume that $0 \notin \pi_n(K)$. Since $\frac{1}{2} \in \pi_{n+1}(K)$, we see that $\pi_{n+1}(K) \subset (\frac{2}{5}, \frac{4}{5})$. Denote by L_1 an interval such that

$$\pi_{n+1}(K) \subset \operatorname{int} L_1 \subset (\tfrac{2}{5}, \tfrac{4}{5}). \tag{4.5.1}$$

For each positive integer i let L_i denote the component of $(f^{n+i}_{n+1})^{-1}(L_1)$ containing $\pi_{n+i}(K)$. Then $f^{n+i}_{n+1}|L_i$ is a homeomorphism by (4.5.1), and therefore

$L = \varprojlim\{L_i, f_{n+i}|L_{i+1}\}$ is an arc as the inverse limit of arcs with homeomorphisms as bonding mappings. It satisfies the required conditions.

CASE 2. $(\frac{1}{2}, \frac{1}{2}, \dots) \notin K$.

Let n be a positive integer such that $\frac{1}{2} \notin \pi_n(K)$. Then 0, $\frac{1}{2}$ and 1 are not in $\pi_{n+i}(K)$ for $i \in \mathbb{N}$. Let L_1 be a subcontinuum of $X_{n+1} = [0,1]$ such that none of the points 0, $\frac{1}{2}$, 1 is in L_1. As previously, define L_i as the component of $(f_{n+1}^{n+i})^{-1}(L_1)$ containing $\pi_{n+i}(K)$. Then again $f_{n+1}^{n+i}|L_i$ is a homeomorphism, so $L = \varprojlim\{L_i, f_{n+i}|L_{i+1}\}$ is the required arc. The proof is finished. □

One can show, using [28, Theorem 2.7, p. 21] that the continuum X described in Example 4.5 is indecomposable. In general, we have the following theorem.

Theorem 4.6. *Let $\mathcal{K} \subset \mathcal{HI}$ be a hereditary class of continua. Then every absolute terminal continuum for the class $\mathcal{K}$ is indecomposable.*

PROOF. Assume on the contrary an absolute terminal continuum X for $\mathcal{K}$ is the union of two its proper subcontinua A and B. We will show that $\psi(A, X) = 1$. Indeed, if not, there are continua X_1 and X_2 such that $A \subset X_1 \cap X_2$ and $X_1 \cup X_2$ is an irreducible union. Since every hereditarily irreducible continuum is hereditarily unicoherent, $A \cup (X_1 \cap B) \cup (X_2 \cap B)$ is an irreducible union, so it is a triod, and therefore it is not an irreducible continuum. □

By Example 4.5 we see that there are absolute terminal continua for the classes $\mathcal{AL}$ and $\mathcal{NT}$. Note that solenoids are also absolute terminal continua for the class $\mathcal{NT}$ of all atriodic continua. Now we will use solenoids to construct absolute terminal continua for each of the classes $\mathcal{N}n$-$\mathcal{OD}$, where $n \geq 3$.

Example 4.7. *For each integer $n \geq 3$ there is an absolute terminal continuum in the class $\mathcal{N}n$-$\mathcal{OD}$.*

PROOF. By the equivalence between conditions (4.4.1) and (4.4.3) in Theorem 4.4 absolute terminal continua for the class $\mathcal{N}n$-$\mathcal{OD}$ are precisely continua X satisfying $\psi(K, X) = n - 1$ for each proper subcontinuum K of X. For $n = 3$ a circle, any solenoid, or the continuum described in Example 4.5 are such continua. So assume $n \geq 4$.

Let T be a simple $(n-2)$-od with the center v, and let a point $p \notin T$ be given. Denote by L a ray (i.e., a homeomorphic copy of $[0, \infty)$) emanating from p and approximating T so that $L \cap T = \emptyset$ and $\operatorname{cl} L = L \cup T$ (see the above quoted result of [1]). Put $M = L \cup T$.

In a solenoid S we delete an open set U homeomorphic to a Cantor bundle $(0,1) \times C$, where C stands for the Cantor ternary set. Next we replace $\operatorname{cl} U$ in

S by $M \times C$ identifying points $(p, c) \in M \times C$ with $(0, c) \in \operatorname{cl} U$ and identifying $(v, c) \in M \times C$ with $(1, c) \in \operatorname{cl} U$. The obtained space is the needed continuum X. Then each proper subcontinuum K of X is a subcontinuum of the finite union $M_1 \cup \cdots \cup M_j$ of some $j \in \mathbb{N}$ copies of M such that the image of v in M_i is identified with the image of p in M_{i+1} for each $i \in \{1, \ldots, j-1\}$. The reader can verify that $\psi(K, X) = n - 1$. □

Corollary 4.8. *There exist absolute terminal continua for the classes $\mathcal{AL}$, $\mathcal{HI}$, and $\mathcal{N}n$-$\mathcal{OD}$ for each $n \in \{3, 4, \ldots\}$.*

Note further that for the class of all hereditarily indecomposable continua any element of the class is an absolute terminal continuum for this class. The class $\lambda\mathcal{AL}$ cannot be joined to ones listed in Corollary 4.8. The next corollary, which is a consequence of Theorem 4.6, shows this.

Corollary 4.9. *No nondegenerate continuum is absolute terminal for the class $\lambda\mathcal{AL}$.*

5. Absolute retracts for some classes of continua

Now we will investigate properties of absolute retracts for the classes $\mathcal{K}$ of continua considered in the previous sections and listed in (3.1). In forthcoming paper the authors showed the following result.

Theorem 5.1. *The inverse limit of an inverse sequence of trees with confluent bonding mappings is an absolute retract for the class of hereditarily unicoherent continua.*

This implies that so called Knaster type continua (see [5, p. 219, and Corollary 6.2, p. 229]; compare [12]) are absolute retracts for arc-like continua. Namely we have the following corollary.

Corollary 5.2. *The inverse limit of an inverse sequence of arcs with confluent (equivalently: with open) bonding mappings is in* $\mathrm{AR}(\mathcal{AL})$.

Instead of the arc approximation property (proved for absolute retracts for the classes studied in [8]) which implies density of all arc components of the continuum, we can only show that, in general, absolute retracts for the classes in (3.1) have at least one arc component dense.

Theorem 5.3. *Let a class $\mathcal{K}$ of continua be such that*

$$\mathcal{K} \cap \mathcal{R}(X) \neq \emptyset \quad \text{for each} \quad X \in \mathcal{K}. \tag{5.3.1}$$

Then each member of $\mathrm{AR}(\mathcal{K})$ *has a dense arc component.*

Proof. Let $X \in \mathrm{AR}(\mathcal{K}) \subset \mathcal{K}$, and take any $Y \in \mathcal{K} \cap \mathcal{R}(X)$. Thus, by the assumption, there is a retraction $r : Y \to X$. Then $r(Y \setminus X)$ is a dense arcwise connected subset of X contained in the required arc component. □

Corollary 5.4. *Let $\mathcal{K}$ be any class of continua listed in* (3.1)*. Then each member of* $\mathrm{AR}(\mathcal{K})$ *has a dense arc component.*

In the following part of this section we investigate members of $\mathrm{AR}(\lambda\mathcal{AL})$, $\mathrm{AR}(\mathcal{AL})$, $\mathrm{AR}(\mathcal{HI})$, and irreducible continua in $\mathrm{AR}(\mathcal{N}n\text{-}\mathcal{OD})$ for $n \geq 3$. Thus all considered AR's are irreducible in this part.

In what follows let X^* denote a continuum which is irreducible between some distinct points a_1 and a_2 and has a dense arc component. Recall that a *closed domain* of a space means a subset of the space which equals the closure of its interior. Denote by $\boldsymbol{\mathcal{D}}_1$ (by $\boldsymbol{\mathcal{D}}_2$) the family of all closed connected domains of X^* that contain the point a_1 (the point a_2, respectively) and are different from X^*, and note that these families are monotone in the sense that they are linearly ordered by inclusion. The next statement is a consequence of [19, §48, VII, Theorem 2, p. 215].

Statement 5.5. *The following implications hold for a continuum X^*:*

(a) *If* $\operatorname{card} \boldsymbol{\mathcal{D}}_1 = 0$*, then the continuum X^* is indecomposable.*
(b) *If* $\operatorname{card} \boldsymbol{\mathcal{D}}_1 = 1$*, then $X^* = P^* \cup Q^*$, where P^* and Q^* are indecomposable subcontinua, $a_1 \in P^* \setminus Q^*$ and $a_2 \in Q^* \setminus P^*$.*

A subarc pq of a continuum is said to be *free* provided that $pq \setminus \{p, q\}$ is an open subset of the continuum.

Lemma 5.6. *If $D, D' \in \boldsymbol{\mathcal{D}}_1$ with $D \subsetneq D'$, then there is a free arc p^1p^2 in X^* such that $D' = D \cup p^1p^2$. For each arc $p^1q \subset p^1p^2$ the continuum $D \cup p^1q$ is an element of $\boldsymbol{\mathcal{D}}_1$.*

Proof. Note that $F = \operatorname{cl}(X^* \setminus D')$ is in $\boldsymbol{\mathcal{D}}_2$ by [19, §48, III, Theorem 5, p. 196]. Since X^* contains a dense arc component, there is an arc xy in X^* such that $x \in \operatorname{int} D$ and $y \in \operatorname{int} F$. The arc xy contains an arc p^1p^2 with $p^1 \in D$ and $p^2 \in F$ which joins D and F irreducibly. Since X^* is irreducible, the arc p^1p^2 satisfies the conclusion. □

The next statement is an immediate consequence of the second part of Lemma 5.6.

Statement 5.7. *If* $\operatorname{card} \boldsymbol{\mathcal{D}}_1 > 1$*, then* $\operatorname{card} \boldsymbol{\mathcal{D}}_1 = \mathfrak{c}$ *(and* $\operatorname{card} \boldsymbol{\mathcal{D}}_2 = \mathfrak{c}$*).*

Define

$$A_i^* = \bigcap\{D : D \in \boldsymbol{\mathcal{D}}_i\} \text{ for } i \in \{1,2\}, \tag{5.8}$$

and observe that A_1^* and A_2^* are disjoint subcontinua of X^*.

Lemma 5.9. *If* $\operatorname{card}\boldsymbol{\mathcal{D}}_1 > 1$, *then for* $i \in \{1,2\}$ *there exists a decreasing sequence* $\{D_n^i\}$ *of elements of* $\boldsymbol{\mathcal{D}}_i$ *such that* $D_n^i \neq A_i^*$ *for each* $n \in \mathbb{N}$ *and* $A_i^* = \bigcap\{D_n^i : n \in \mathbb{N}\}$.

Proof. If A_i^* belongs to $\boldsymbol{\mathcal{D}}_i$, then the conclusion follows from Lemma 5.6. If not, it is a consequence of the definition of A_i^* and of monotonicity of the family $\boldsymbol{\mathcal{D}}_i$. □

Lemma 5.10. *If* $\operatorname{card}\boldsymbol{\mathcal{D}}_1 > 1$, *then the continuum* X^* *has the form of the union*

$$X^* = A_1^* \cup L^* \cup A_2^*,$$

where L^* *is an open subset of* X^* *homeomorphic to the real line* $\mathbb{R}$, *and the sets* A_1^*, L^* *and* A_2^* *are mutually disjoint.*

Proof. According to Lemma 5.9 there are, for $i \in \{1,2\}$, sequences of subcontinua $\{D_n^i\}$ of X^* such that

$$D_n^i \in \boldsymbol{\mathcal{D}}_i, \quad D_{n+1}^i \subset D_n^i, \quad D_n^1 \cap D_n^2 = \emptyset \quad \text{for each } n \in \mathbb{N}$$

and

$$A_i^* = \bigcap\{D_n^i \in \boldsymbol{\mathcal{D}}_i : n \in \mathbb{N}\}.$$

Then $\operatorname{cl}(X^* \setminus D_n^2) \in \boldsymbol{\mathcal{D}}_1$, thus $\operatorname{cl}(X^* \setminus D_n^2) = D_n^1 \cup p_n^1 p_n^2$ as in Lemma 5.6, and for each $n \in \mathbb{N}$ we have

$$X^* = D_n^1 \cup p_n^1 p_n^2 \cup D_n^2, \quad D_n^i \cap p_n^1 p_n^2 = \{p_n^i\}, \quad \text{for } i \in \{1,2\}.$$

Then $L^* = \bigcup\{p_n^1 p_n^2 : n \in \mathbb{N}\}$ is homeomorphic to $\mathbb{R}$, and the conclusion follows. □

The next theorem summarizes the above information on the structure of the continuum X^*. It is a consequence of Statements 5.5 and 5.7 and Lemma 5.10.

Theorem 5.11. *The continuum* X^* *satisfies exactly one of the following three conditions:*

(A) X^* *is indecomposable;*
(B) X^* *is the union of two indecomposable continua* P^* *and* Q^* *such that* $P^* \setminus Q^* \neq \emptyset \neq Q^* \setminus P^*$;

(C) $X^* = A_1^* \cup L^* \cup A_2^*$, *where* A_1^*, L^* *and* A_2^* *are mutually disjoint,* L^* *is an open subset of* X^* *homeomorphic to the real line* $\mathbb{R}$, *and* $a_i \in A_i^* = \bigcap\{D : D \in \boldsymbol{\mathcal{D}}_i\}$ *for* $i \in \{1, 2\}$.

For $i \in \{1, 2\}$ denote by $\boldsymbol{\mathcal{D}}_i^*$ the family of all closed connected domains in X^* which, if nonempty, contain the point a_i. Thus $\boldsymbol{\mathcal{D}}_i^* = \boldsymbol{\mathcal{D}}_i \cup \{\emptyset, X^*\}$. The next proposition describes the structure of the continuum X^* if the case (C) of Theorem 5.11 holds.

Proposition 5.12. *If* $\operatorname{card} \boldsymbol{\mathcal{D}}_1 > 1$ *and if, for either* $i = 1$ *or* $i = 2$ *(or both), the continuum* A_i^* *is nondegenerate, then exactly one of the following two conditions holds:*

(5.12.1) A_i^* *is a terminal subcontinuum of* X^**;*

(5.12.2) $A_i^* \in \boldsymbol{\mathcal{D}}_i$, A_i^* *is indecomposable, and* $\operatorname{bd} A_i^*$ *is a singleton.*

Proof. Denote by L the dense arc component of X^*. If A_i^* is not terminal in X^*, then $\operatorname{int} A_i^* \neq \emptyset$, and thus $A_i^* \cap L \neq \emptyset$. Then $\operatorname{bd} A_i^* = \{p_i\}$, where p_i is the first point in the arc joining L^* and $\operatorname{int} A_i^*$ which belongs to A_i^*. Thus, if $j \in \{1, 2\}$ and $j \neq i$, then $A_j^* \cup L^* \cup \{p_i\} \in \boldsymbol{\mathcal{D}}_j$. This implies that $A_i^* = \operatorname{cl}(X^* \setminus (A_j^* \cup L^* \cup \{p_i\})) \in \boldsymbol{\mathcal{D}}_i$ according to [19, §48, III, Theorem 5, p. 196]. Therefore A_i^* is the minimal element in $\boldsymbol{\mathcal{D}}_i$ by the definition of A_i^*. So, the sets $\emptyset$ and A_i^* constitute a jump in the family $\boldsymbol{\mathcal{D}}_i^*$, whence A_i^* is indecomposable by [19, §48, VII, Theorem 2, p. 215]. □

Corollary 5.13. *If a continuum* X^* *belongs to the class* $\lambda\mathcal{AL}$, *then condition* (C) *of Theorem* 5.11 *holds.*

By the definition of gate continua we have the following corollary.

Corollary 5.14. *Let* $\mathcal{K} \in \{\mathcal{AL}, \mathcal{HI}, \mathcal{N}n\text{-}\mathcal{OD}\}$ *for* $n \geq 3$. *If* $X^* \in \mathcal{K}$ *satisfies condition* (B) *of Theorem* 5.11, *then each component of* $P^* \cap Q^*$ *is a gate continuum in both* P^* *and* Q^* *for* $\mathcal{K}$.

To prove further properties of members of $\mathrm{AR}(\mathcal{K})$ for $\mathcal{K}$ in (3.1) we will use two auxiliary constructions. They are presented below.

Let there be given a continuum X with two points $u, v \in X$, and a compact zero-dimensional set Z with a point $z_0 \in Z$. The following construction of a set $S(X, u, v; Z, z_0)$ will be employed.

Construction 5.15. Attach two arcs I_u and I_v to X in such a way that u is an end point of I_u, v is an end point of I_v,

$$I_u \cap (X \cup I_v) = \{u\} \quad \text{and} \quad I_v \cap (X \cup I_u) = \{v\}.$$

Denote by u_0 (by v_0) the other end point of I_u (of I_v, respectively).

If z_0 is an isolated point of Z, then take an arc J_0 disjoint with $I_u \cup X \cup I_v$, define $W = (I_u \cup X \cup I_v) \cup J_0$, and let

$$S(X, u, v; Z, z_0) = ((I_u \cup X \cup I_v) \times \{z_0\}) \cup (J_0 \times (Z \setminus \{z_0\})) \subset W \times Z.$$

We say that a sequence of compact sets A_n converges homeomorphically to a set A provided that there is a sequence of homeomorphisms $h_n : A \to A_n$ that converges to the identity on A.

If z_0 is an accumulation point of Z, let $\{W_n\}$ be a basis of open and closed neighborhoods of z_0 in Z such that $W_1 = Z$ and $W_{n+1} \subsetneq W_n$ for each $n \in \mathbb{N}$. Next, take a sequence of mutually disjoint arcs J_n each of which is the irreducible union of three subarcs, $J_n = u_n^0 u_n \cup u_n v_n \cup v_n v_n^0$, where the points u_n^0, u_n, v_w^0, v_n are mutually distinct, and assume, moreover, that

(5.15.1) the sequence of arcs $u_n^0 u_n$ converges homeomorphically to the arc I_u with $\lim u_n^0 = u_0$ and $\lim u_n = u$;

(5.15.2) the sequence of arcs $v_n^0 v_n$ converges homeomorphically to the arc I_v with $\lim v_n^0 = v_0$ and $\lim v_n = v$;

(5.15.3) $\operatorname{Lim} u_n v_n = X$.

Thus, in particular, we have $\operatorname{Lim} J_n = I_u \cup X \cup I_v$. Define $W = (I_u \cup X \cup I_v) \cup \bigcup\{J_n : n \in \mathbb{N}\}$, and let

$$S(X, u, v; Z, z_0) = ((I_u \cup X \cup I_v) \times \{z_0\}) \cup \bigcup\{J_n \times (W_n \setminus W_{n+1})\} \subset W \times Z.$$

Observe that in both cases the set $S(X, u, v; Z, z_0)$ is compact, the decomposition of it into its components is continuous with the quotient space homeomorphic to Z, and all its components but $(I_u \cup X \cup I_v) \times \{z_0\}$ are arcs. Note further that the space $S(X, u, v; Z, z_0)/(X \times \{z_0\})$ is homeomorphic to $[0, 1] \times Z$. The construction is finished.

In the case when the continuum X is arc-like, the points u and v are opposite end points of X (i.e., for each $\varepsilon > 0$ there is an ε-chain from u to v covering X, see [3, p. 661]), and again Z is a compact zero-dimensional set with a point $z_0 \in Z$ distinguished, we consider the following special construction of a set $T(X, u, v; Z, z_0)$.

Construction 5.16. Let $\{W_n\}$ be a basis of open and closed neighborhoods of z_0 in Z such that $W_{n+1} \subset W_n$ for each $n \in \mathbb{N}$. Next, for each $n \in \mathbb{N}$, put $I_n^0 = [\frac{1}{3}, \frac{2}{3}]$, and let mappings $f_n^0 : I_{n+1}^0 \to I_n^0$ be such that $f_n^0(\frac{1}{3}) = \frac{1}{3}$, $f_n^0(\frac{2}{3}) = \frac{2}{3}$, and that there exists a homeomorphism $h_0 : X \to \varprojlim \{I_n^0, f_n^0\}$ satisfying $h_0(u) = (\frac{1}{3}, \frac{1}{3}, \dots)$ and $h_0(v) = (\frac{2}{3}, \frac{2}{3}, \dots)$.

For each $n \in \mathbb{N}$ let $G_n = [0,1] \times Z$, and define a mapping $f_n : G_{n+1} \to G_n$ by

$$f_n((t,z)) = \begin{cases} (t,z) & \text{if either } \quad t \notin [\frac{1}{3}, \frac{2}{3}] \quad \text{or} \quad z \notin W_n, \\ (f_n^0(t), z) & \text{if } \; t \in [\frac{1}{3}, \frac{2}{3}] \quad \text{and} \quad z \in W_n. \end{cases}$$

We usually will identify the continuum X and the inverse limit $\varprojlim \{I_n^0, f_n^0\}$ by the homeomorphism h_0. Therefore the set

$$T(X,u,v;Z,z_0) = \varprojlim \{G_n, f_n\}$$

contains X. Moreover, the decomposition of this set into its components is continuous, has the quotient space homeomorphic to Z, all components but that which contains X are arcs, and the space $T(X,u,v;Z,z_0)/X$ is homeomorphic to $[0,1] \times Z$. The construction is complete.

It was shown in Proposition 5.12 that if condition (C) of Theorem 5.11 is satisfied, then we have two possibilities for A_i^*: either it is a terminal subcontinuum of X^* or it is indecomposable and has one-point boundary. In the next two propositions we will show further properties of A_i^* in these both cases under an additional assumption that X^* is an absolute retract for any of the classes $\mathcal{AL}$, $\mathcal{HI}$ and $\mathcal{N}n$-$\mathcal{OD}$ for $n \geq 3$.

In the sequential results, viz. Propositions 5.17 and 5.18, Corollary 5.19 and Theorem 5.20 the assumption that the irreducible continuum X^* has a dense arc component can be omitted, since it follows from the assumption that X^* is in $\mathrm{AR}(\mathcal{K})$ by Corollary 5.4.

Proposition 5.17. *Let the continuum X^* satisfying condition* (C) *of Theorem* 5.11 *belong to* $\mathrm{AR}(\mathcal{K})$*, where $\mathcal{K} \in \{\mathcal{AL}, \mathcal{HI}, \mathcal{N}n\text{-}\mathcal{OD}\}$ for $n \geq 3$. If A_i^* is terminal in X^*, then A_i^* is absolute terminal continuum for $\mathcal{K}$.*

Proof. By Corollary 4.4 it is enough to show that A_i^* contains no proper gate continuum for $\mathcal{K}$. Assume on the contrary that there is a proper gate continuum K in A_i^* for $\mathcal{K}$. Let L_1 be a ray approximating K such that $L_1 \cap X^* = \emptyset$. Then $L_1 \cup X^* \in \mathcal{K}$ by Corollary 4.4. Note that, for $j \neq i$, we have $\psi(A_j^*, X^*) = \psi(A_j^*, L_1 \cup X^*) = 1$, so again by Corollary 4.4, if we approximate A_j^* by a ray L_2 so that $L_2 \cap (L_1 \cup X^*) = \emptyset$, then $X = L_1 \cup X^* \cup L_2 \in \mathcal{K}$.

Denote by u and v the end points of L_1 and L_2, respectively. Let D be the simplest Brouwer-Janiszewski-Knaster indecomposable continuum, and denote by C_0 the composant of D that contains the (only) end point of D. Further, let U stand for an open subset of D which is homeomorphic to $(0,1) \times Z$, where Z denotes the standard Cantor ternary set, and let $h : (0,1) \times Z \to U$ be the homeomorphism. Fix a point $z_0 \in Z$ such that $h((\frac{1}{2}, z_0)) \in D \setminus C_0$. Replace $\mathrm{cl}\, U$

in D either by $S(X,u,v;Z,z_0)$ if $\mathcal{K} \in \{\mathcal{HI}, \mathcal{N}n\text{-}\mathcal{OD}\}$ (where $n \geq 3$), according to Construction 5.15, or by $T(X,u,v;Z,z_0)$ if $\mathcal{K} = \mathcal{AL}$, according to Construction 5.16, to obtain a continuum Y. We can see Y as D with one arc replaced by $L_1 \cup X^* \cup L_2$. Note that in both cases $Y \in \mathcal{K}$.

We will show that there is no retraction from Y onto X^*. Suppose that there is a retraction $r : Y \to X^*$. Let L' be a ray that contains L_1. Then $r(L')$ is an arcwise connected set, and $\operatorname{cl} L' = L' \cup K$, so $r(L' \cup K) \subset A_i^*$. Since $L' \cup K$ can be taken so large that it is close to Y, we have $r(Y) \subset A_i^*$. In particular $r(X^*) \subset A_i^*$, so r is not a retraction. This contradiction finishes the proof. □

Proposition 5.18. *Let the continuum X^* satisfying condition* (C) *of Theorem* 5.11 *belong to* $\mathrm{AR}(\mathcal{K})$, *where* $\mathcal{K} \in \{\mathcal{AL}, \mathcal{HI}, \mathcal{NT}\}$. *If* $\operatorname{bd} A_i^*$ *is a singleton* $\{p\}$, *then* $\{p\}$ *is the only minimal gate continuum in* A_i^* *for* $\mathcal{K}$.

PROOF. Suppose that there is a gate continuum K in A_i^* for $\mathcal{K}$ such that $p \notin K$. Let L_1 be a ray approximating K such that $L_1 \cap X^* = \emptyset$. Then $L_1 \cup X^* \in \mathcal{K}$ by Corollary 4.4. Since each element of $\mathcal{K}$ does not contain triods, we see that A^* is irreducible with respect to containing K and $\{p\}$, because otherwise $L_1 \cup X^*$ is a triod with its center being the irreducible continuum containing K and $\{p\}$. By Proposition 5.12 the continuum A_i^* is indecomposable. Hence irreducibility of A_i^* with respect to containing K and $\{p\}$ means that K and $\{p\}$ are contained in different composants of A_i^*.

Note that, for $j \neq i$, we have $\psi(A_j^*, X^*) = \psi(A_j^*, L_1 \cup X^*) = 1$. The rest of the proof runs exactly as the corresponding part of the proof of Proposition 5.17, and leads to a contradiction. The argument is complete. □

As a consequence of Propositions 5.17 and 5.18 we have the following corollary.

Corollary 5.19. *Let the continuum X^* satisfying condition* (C) *of Theorem* 5.11 *belong to* $\mathrm{AR}(\mathcal{K})$, *where* $\mathcal{K} \in \{\mathcal{AL}, \mathcal{HI}, \mathcal{NT}\}$. *Then (for* $i \in \{1,2\}$*)* A_i^* *is indecomposable, and either*

(5.19.1) A_i^* *is an absolute terminal continuum for* $\mathcal{K}$, *or*

(5.19.2) $\operatorname{bd} A_i^*$ *is a singleton, and this singleton is the only minimal gate continuum in* A_i^* *for* $\mathcal{K}$.

In the following theorem we use notation of Theorem 5.11.

Theorem 5.20. *Let the continuum X^* satisfy condition* (B) *of Theorem* 5.11 *and belong to* $\mathrm{AR}(\mathcal{K})$, *where* $\mathcal{K} \in \{\mathcal{AL}, \mathcal{HI}, \mathcal{NT}\}$.

(5.20.1) *If $P^* \cap Q^*$ is connected, then P^* and Q^* are the only minimal gate continua in X^* for* $\mathcal{K}$.

(5.20.2) *If $P^* \cap Q^*$ is not connected, then X^* is an absolute terminal continuum for $\mathcal{K}$.*

PROOF. Let G be a gate continuum in X^* for $\mathcal{K}$ and suppose that G contains neither P^* nor Q^*. Let L_1 be a ray approximating G such that $L_1 \cap X^* = \emptyset$. Then $L_1 \cup X^* \in \mathcal{K}$ (by Theorem 3.27 for $\mathcal{K} = \mathcal{AL}$, Theorem 3.24 for $\mathcal{K} = \mathcal{HI}$ and Theorem 3.20 for $\mathcal{K} = \mathcal{NT}$).

CLAIM 1. $G \cap P^* \cap Q^* = \emptyset$.

Suppose on the contrary that $G \cap P^* \cap Q^* \neq \emptyset$. Then $L_1 \cup X^* = (\operatorname{cl} L_1) \cup P^* \cup Q^*$ is an irreducible union with $(\operatorname{cl} L_1) \cap P^* \cap Q^* \neq \emptyset$. Thus it contains a triod by Theorem 3.5, contrary to the atriodicity of the elements of $\mathcal{K}$.

Consequently, by Claim 1, either $G \subset P^*$ or $G \subset Q^*$. Assume $G \subset P^*$.

CLAIM 2. The composant C of P^* containing G is disjoint with Q^*.

In fact, if not, then $L_1 \cup X^* = (\operatorname{cl} L_1 \cup K) \cup P^* \cup Q^*$, where K is the irreducible continuum in Q^* containing G and some point of $C \cap P^* \cap Q^*$. This last union is an irreducible one, and $(\operatorname{cl} L_1 \cup K) \cap P^* \cap Q^* \neq \emptyset$. Applying again Theorem 3.5 we see that $L_1 \cup X^*$ contains a triod contrary to the atriodicity of the elements of $\mathcal{K}$.

To see (5.20.1) assume that $P^* \cap Q^*$ is connected and note that $\psi(Q^*, L_1 \cup X^*) = \psi(Q^*, X^*) = \psi(P^* \cap Q^*, P^*) = 1$ by the atriodicity of X^* and of $X^* \cup L_1$, and by Claims 1 and 2. Consequently, Q^* is a gate subcontinuum in $L_1 \cup X^*$ for $\mathcal{K}$, so if L_2 is a ray approximating Q^* disjoint with $L_1 \cup X^*$, then the continuum $X = L_1 \cup X^* \cup L_2$ is in $\mathcal{K}$. Again the rest of the proof runs exactly as the part of the proof of Proposition 5.17 starting from the second paragraph. Thus the argument for (5.20.1) is complete.

To show (5.20.2) observe that, by the atriodicity of X^* the intersection $P^* \cap Q^*$ has exactly two different components C_1 and C_2. If C_1 and C_2 were contained in the same composant of P^* or of Q^*, then the continuum irreducible with respect to containing C_1 and C_2 would be the center of a triod in X^*, a contradiction.

To prove that there is no proper gate subcontinuum in X^* for $\mathcal{K}$ it is enough to show, by Corollary 3.28, that $\psi(H, X^*) = 2$ for any proper subcontinuum H of X^*.

If $H \subset P^*$ then $\psi(H, X^*) \geq \psi(P^*, X^*) = 2$ by Proposition 3.9. So, let $H \cap P^* \neq \emptyset \neq H \cap Q^*$. If $P^* \setminus H \neq \emptyset \neq Q^* \setminus H$, then $(P^* \cup H) \cup (Q^* \cup H)$ is an irreducible union containing H, whence $\psi(H, X^*) = 2$. If $P^* \subset H$ (for $Q^* \subset H$ the argument is the same), then $H = P^* \cup M_1 \cup M_2$, where M_1 and M_2 are continua contained in the composants containing the continua C_1 and C_2 respectively, therefore again $\psi(H, X^*) = 2$. The proof is complete. □

Let X be a metric space with a metric d. For a mapping $f : A \to B$, where A and B are subspaces of X, we define $d(f) = \sup\{d(x, f(x)) : x \in A\}$. A compactum X is called an *approximative absolute neighborhood retract* (written AANR) provided that whenever X is embedded in a compactum (or, equivalently, in the Hilbert cube) Y, for each $\varepsilon > 0$ there are a neighborhood U of the embedded copy X' of X in Y and a mapping $f : U \to X'$ such that $d(f|X') < \varepsilon$. The reader is referred to [14, p. 9] and [11, Sections 1 and 2, p. 117-119] for a discussion of variants of this concept. Our definition agrees with the one given in [11, Definition 2.3, p. 118].

A continuum X is said to be an *absolute terminal retract* provided that if X is embedded in a continuum Y in such a way that the embedded copy, X', is a terminal subcontinuum of Y, then X' is a retract of Y (see [10, Definition 4.1]).

In [10] relations were shown between absolute retracts for the classes of tree-like continua ($\mathcal{TL}$), λ-dendroids ($\lambda\mathcal{D}$), dendroids ($\mathcal{D}$), arc-like continua ($\mathcal{AL}$) and arc-like λ-dendroids ($\lambda\mathcal{AL}$), as well as between AANR-continua and absolute terminal retracts. Namely we have the following two theorems, [10, Theorems 3.3 and 4.3]. The latter one is the main result of [10].

Theorem 5.21. *Let $\mathcal{K}$ be any of the following classes of continua: $\mathcal{TL}$, $\lambda\mathcal{D}$, $\mathcal{D}$, $\mathcal{AL}$ and $\lambda\mathcal{AL}$. Then each member of* AR($\mathcal{K}$) *is an* AAR.

Theorem 5.22. *A continuum X is an* AANR *if and only if X is an absolute terminal retract.*

Now we will investigate AANR (equivalently AAR) continua for the class $\mathcal{AL}$. We will show that, for this class, they are exactly those continua X which are retracts of the members of $\mathcal{R}(X)$.

Theorem 5.23. *For each arc-like continuum X the following four conditions are equivalent:*

(5.23.1) *X is an absolute terminal retract;*
(5.23.2) *X is an* AANR*;*
(5.23.3) *X is a retract of any continuum $Y \in \mathcal{R}(X)$;*
(5.23.4) *for each $\varepsilon > 0$ there is a mapping $f : X \to A$ onto an arc $A \subset X$ with $d(f) < \varepsilon$.*

Proof. The equivalence of conditions (5.23.1) and (5.23.2) is Theorem 5.22. We will show implications (5.23.1) $\Longrightarrow$ (5.23.3) $\Longrightarrow$ (5.23.4) $\Longrightarrow$ (5.23.2).

Assume (5.23.1). Since each member Y of $\mathcal{R}(X)$ contains X as a terminal subcontinuum, it follows that X is a retract of each continuum $Y \in \mathcal{R}(X)$, so (5.23.3) holds.

Assume now (5.23.3). We will show (5.23.4). Since X is arc-like, it can be represented as $X = \varprojlim \{A_n, f_n\}$, where $A_n = [0,1]$ and $f_n : A_{n+1} \to A_n$ are surjective bonding mappings. Let $X_0 = \{0\} \times \varprojlim \{A_n, f_n\} \subset \mathbb{Q} = [0,1] \times A_1 \times A_2 \times \dots$ and note that $\mathbb{Q}$ is the Hilbert cube. Let

$$A'_n = \{(\tfrac{1}{n}, x_1, \dots, x_n, 0, 0, \dots) : (x_1, \dots, x_n, x_{n+1}, \dots) \in X\}.$$

Then $X_0 \cap X_n = \emptyset = X_m \cap X_n$ for $m \neq n$. The formula

$$h_n(x_n) = (\tfrac{1}{n}, f_1^n(x_n), f_2^n(x_n), \dots, f_n^n(x_n), 0, 0, \dots)$$

defines homeomorphisms $h_n : A_n \to A'_n$, so the sets A'_n are arcs. The mappings $\alpha_n : X_0 \to A'_n$ defined by

$$\alpha_n((0, x_1, x_2, \dots)) = (\tfrac{1}{n}, x_1, \dots, x_n, 0, 0, \dots)$$

(which correspond to the projections $\pi_n : X \to A_n$) approximate the identity $\mathrm{id}\,|X_0$.

For each $n \in \mathbb{N}$ let p_n and q_n stand for the end points of the arc A'_n. Denote by B_n an arc in the considered Hilbert cube $\mathbb{Q}$ such that:

(5.23.5) the end points of B_n are q_n and p_{n+1};
(5.23.6) $B_n \cap (X_0 \cup \bigcup\{A'_m : m \in \mathbb{N}\}) = \{q_n, p_{n+1}\}$;
(5.23.7) $\mathrm{Lim}\, B_n = X_0$.

Then the union $Y = X_0 \cup \bigcup\{A'_n \cup B_n : n \in \mathbb{N}\}$ is a member of $\mathcal{R}(X_0)$ with X_0 homeomorphic to X. Thus there exists a retraction $r : Y \to X_0$. The compositions $g_n = r \circ \alpha_n$ converge to the identity $\mathrm{id}\,|X_0$, and we have $g_n(X_0) = r(A'_n)$. So $g_n(X_0)$ are locally connected subcontinua of X_0. Since X_0 is arc-like, these subcontinua are arcs. Therefore the property described in (5.23.4) is satisfied for X_0 (which is homeomorphic to X). Hence (5.23.4) holds.

To see that (5.23.4) implies (5.23.2) it is enough to show that the condition defining AANR is satisfied for an embedding of the continuum X in the Hilbert cube $\mathbb{Q}$. So, assume that $X \subset \mathbb{Q}$. Take any $\varepsilon > 0$ and let $f : X \to A$ with $d(f) < \varepsilon$ be a mapping as in (5.23.4). Since A is an absolute retract, there exists an extension $f^* : \mathbb{Q} \to A$ of f on $\mathbb{Q}$ (see [15, Theorem 3.2, p. 84]). Then, for sufficiently small neighborhood U of the continuum X in $\mathbb{Q}$ the restriction $f^*|U : U \to A$ satisfies (similarly to f) the condition $d(f^*|U) < \varepsilon$. Therefore (5.23.2) holds. This finishes the proof. $\square$

As consequences of the implication (5.23.3) $\Longrightarrow$ (5.23.4) and of the opposite one of Theorem 5.23 we obtain the following two corollaries, respectively.

Corollary 5.24. *Let X be an arc-like continuum and let $Y \in \mathcal{R}(X)$. If Y is a retract of each member of $\mathcal{R}(Y)$, then there exists a sequence of mappings $g_n : Y \to Y \setminus X$ such that $\lim d(g_n) = 0$.*

Indeed, take mappings $g_n : Y \to A_n$ with $\lim d(g_n) = 0$, where A_n are arcs in Y guaranteed by (5.23.4). Since $Y \setminus X$ is a dense open arc component in Y, almost all arcs A_n are subsets of $Y \setminus X$.

Corollary 5.25. *The $\sin(1/x)$-curve S is a retract of each continuum $X \in \mathcal{R}(S)$.*

In the next example we will see that there are compactifications of the ray with an arc as the remainder which, unlike the $\sin(1/x)$-curve, are not AAR's.

Example 5.26. *There is a compactification Y of the half line $[0,\infty)$ with an arc I as the remainder such that Y is not a retract of some member of $\mathcal{R}(Y)$. In particular, Y is not a member of $\mathrm{AR}(\lambda\mathcal{AL})$.*

Proof. In the plane $\mathbb{R}^2$ let $I = \{0\} \times [-2, 2]$. For each $n \in \mathbb{N}$ define

$$a_n = \langle \tfrac{1}{6n}, -2\rangle, \qquad b_n = \langle \tfrac{1}{6n+1}, 1\rangle, \qquad c_n = \langle \tfrac{1}{6n+2}, -1\rangle,$$
$$d_n = \langle \tfrac{1}{6n+3}, 2\rangle, \qquad e_n = \langle \tfrac{1}{6n+4}, -1\rangle, \qquad f_n = \langle \tfrac{1}{6n+5}, 1\rangle,$$

and let B_n be the broken line with consecutive vertices $a_n, b_n, c_n, d_n, e_n, f_n, a_{n+1}$. Then $Y = I \cup \bigcup\{B_n : n \in \mathbb{N}\}$ is the needed continuum. Note that Y does not satisfy the conclusion of Corollary 5.24, and therefore Y is not a retract of some member of $\mathcal{R}(Y)$. By Theorem 5.21 and the implication (5.23.2) $\Longrightarrow$ (5.23.3) Y does not belong to $\mathrm{AR}(\lambda\mathcal{AL})$. □

The next theorem provides a sufficient condition for some continua X (not necessarily arc-like nor one-dimensional even) to be retracts of all continua in $\mathcal{R}(X)$. This theorem, presented here in connection with Theorem 5.23, is a sample of more general results that will be presented in another paper.

Theorem 5.27. *Let a continuum X contain a point $p \in X$ and a sequence of locally connected continua A_n such that there is a sequence of mappings $f_n : X \to A_n$ satisfying $f_n(p) = p$ for each $n \in \mathbb{N}$ and $\lim d(f_n) = 0$. Then X is a retract of each continuum in $\mathcal{R}(X)$.*

Proof. Let $Y \in \mathcal{R}(X)$ and $L = Y \setminus X$. Choose an increasing (with respect to the natural order on L) sequence of points $p_n \in L$ converging to p. Let $P = \{p, p_1, p_2, \dots\}$. Consider the quotient space $Z = Y/P$ and let $q : Y \to Z$ be the quotient mapping. Since the restriction $q|X$ is a homeomorphism, we can identify X and $q(X)$ under this homeomorphism.

We will use the following assertion which is a particular case of [19, §53, IV, Theorem 1′, p. 347].

(5.27.1) If a locally connected continuum A is embedded in a continuum B so that $\dim(B \setminus A) \leq 1$, then A is a retract of B.

It follows from (5.27.1) that for each $n \in \mathbb{N}$ the mapping f_n can be extended to a mapping $g_n : Z \to A_n$. Then there are neighborhoods U_n of X in Z such that $d(g_n|U_n) < \frac{1}{n}$.

For $m, n \in \mathbb{N}$ let $p_m p_n \subset L$ be the arc having p_m and p_n is its end points, $L_{m,n} = q(p_m p_n)$ and $L_n = \bigcup\{L_{m,n} : m > n\}$. Choose an increasing sequence of positive integers n_k such that $L_{n_k} \subset U_k$. Finally define a mapping $g : Z \to X$ by

$$g(x) = \begin{cases} x & \text{if } x \in X, \\ g_{n_k}(x) & \text{if } x \in L_{n_k, n_{k+1}}. \end{cases}$$

It follows from the properties of the mappings g_n that the mapping g is well defined, continuous and that it is a retraction. Thus the composition $g \circ q : Y \to X$ is the needed retraction. The proof is complete. □

6. Problems

We close the paper stating some problems and questions concerning the subject.

Question 6.1. Let B be the simplest Brouwer-Janiszewski-Knaster indecomposable continuum with the only one end point p (see e.g. [19, §48, V, Example 1 and Fig. 4, p. 204-205]). Are the following continua members of AR($\mathcal{AL}$)?

(6.1.1) the one point union of two copies of B with the end points identified;

(6.1.2) the one point union of B and of an arc ab with the end points p and a identified;

(6.1.3) the union of two copies B_1 and B_2 of B and of an arc ab such that $B_1 \cap ab = \{p_1\} = \{a\}$ and $B_2 \cap ab = \{p_2\} = \{b\}$.

Up to now the only known examples of AR($\mathcal{AL}$) are those presented in Corollary 5.2, and they are absolute retracts for hereditarily unicoherent continua. Thus the following question is of some interest.

Problem 6.2. Find a member of AR($\mathcal{AL}$) which is not an absolute retract for the class of hereditarily unicoherent continua.

In Chapter 5 we described several conditions such that the members of AR($\mathcal{AL}$), AR($\mathcal{HI}$) and AR($\mathcal{NT}$) have to satisfy at least one of them. However, for most of these conditions we have no example satisfying any of them. Thus the following questions arise.

Questions 6.3. Does there exist a member X^* of $\mathrm{AR}(\mathcal{AL})$ that satisfies:

(i) condition (B) of Theorem 5.11?
(ii) condition (C) of Theorem 5.11 with either A_1^* or A_2^* nondegenerate?

The authors would also like to know an answer to the following questions.

Question 6.4. Is each subcontinuum of a member X of $\mathrm{AR}(\lambda\mathcal{AL})$ a retract of X?

Question 6.5. Does each subcontinuum of a member of $\mathrm{AR}(\lambda\mathcal{AL})$ also belong to $\mathrm{AR}(\lambda\mathcal{AL})$?

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