HEREDITARILY IRREDUCIBLE MAPS

HUSSAM ABOBAKER AND WLODZIMIERZ J. CHARATONIK

Abstract. A map $f : X \to Y$ from a continuum $X$ onto a continuum $Y$ is said to be hereditarily irreducible, if $f(A) \subsetneq f(B)$ for any subcontinua $A$ and $B$ such that $A \subsetneq B$. We investigate properties of hereditarily irreducible maps between continua. Special attention is given to maps between graphs and maps from the interval.

1. Introduction

The notion of hereditarily irreducible map generalizes the notion of an arcwise increasing maps. In fact the two notions coincide if the domain is an arc. Therefore our investigation of hereditarily irreducible map is an extension of the work done by B. Espinoza and E. Matsuhashi [4]. We generalize some of their theorems to more general settings and we answer a problem from their article.

In section 3 we introduce a new notion of an order of a point and compare it to the order in the classical sense. The two notions agree on graphs, but they differ even for locally connected continua. Then we investigate hereditarily irreducible maps between graphs. Section 4 is devoted to the existence of hereditarily irreducible maps from the arc. Such maps were introduced in [7] under the name of “arcwise increasing maps”; further investigation was done in [4]. We find necessary and sufficient conditions for for existence of such maps. In section 5 and 6 we investigate hereditarily irreducible maps between graphs and onto dendrites, respectively. We devote section 7 to open and hereditarily irreducible maps. We show that such maps must be homeomorphism if the domain is a locally connected continuum, but it does not have to be a homeomorphism if we relax the assumption of locally connectedness of the domain. Finally, in section 8 we give an answer to a question by Espinoza and Matsuhashi in [4].

2010 Mathematics Subject Classification. 54F15, 54C35.

Key words and phrases. continuum, hereditarily irreducible map, order of a point.
2. Preliminaries

In this section we introduce notions that will be used throughout this paper. A space $X$ is called a \textit{continuum} if it is non-empty compact connected metric space. A subset of a continuum $X$ which is itself a continuum is called a \textit{subcontinuum} of $X$. A continuum is \textit{non-degenerate} if it contains more than one point. An \textit{arc} is a homeomorphic image of the closed unit interval $[0, 1]$ and a \textit{simple closed curve} is a homeomorphic image of a circle.

If $X$ is a continuum and $h : [0, 1] \to X$ is a homeomorphism onto its image, we call $h(0)$ and $h(1)$ the \textit{endpoints} of the arc given by $h$. An arc $A$ with endpoints $a, b$ is a \textit{free arc} in $X$ provided that $A \setminus \{a, b\}$ is an open subset of $X$. If $h$ is a homeomorphism on $(0, 1)$ and $h(0) = h(1)$, we call the image of $h$ a \textit{loop} in $X$. A loop is a \textit{free loop} in $X$ if $h([0, 1]) \setminus \{h(0)\}$ is an open subset of $X$. If $A$ is a subset of a continuum $X$, then the \textit{interior} of $A$ in $X$ is denoted by $\text{int}_X(A)$, the \textit{closure} of $A$ in $X$ is denoted by $\text{cl}_X(A)$, and the \textit{cardinality} of the set $A$ is denoted by $\text{card}(A)$.

If $X$ is a continuum with metric $d$, $A \subseteq X$, $x \in X$, and $\delta > 0$, then $\text{diam}(A)$ denotes the diameter of $A$, $d(x, A)$ denotes the infimum of the set $\{d(x, a) : a \in A\}$, and $B_\delta(x)$ denotes the set $\{y \in X : d(x, y) < \delta\}$.

By a \textit{map} we mean a continuous function. If $f : X \to Y$ is a map, and $A \subseteq X$, then $f|A$ denotes the restriction of $f$ to $A$. A continuum $X$ is said to be \textit{dendrite} provided that it is locally connected and contains no simple closed curve. A map $f : X \to Y$ from a continuum $X$ onto a continuum $Y$ is said to be \textit{hereditarily irreducible} map, if $f(A) \subsetneq f(B)$ for any subcontinua $A$ and $B$ such that $A \subsetneq B$. A map $f$ from a continuum $X$ onto a continuum $Y$ is said to be \textit{open} if it maps every open subset of $X$ onto an open subset of $Y$.

3. Order of a point and functions between graphs

In this section we introduce a new concept of an order of a point. It generalizes the concept of order in the classical sense. The two concepts coincide for graphs, but are different even for locally connected continua.

Let us recall the classical definition of an order.

\textbf{Definition 3.1.} Let $X$ be a continuum and $p$ be a point in $X$, and let $\alpha$ be a cardinal number. We say that $\text{ordc}_X(p) \geq \alpha$ if there are arcs $A_\gamma$ for $0 \leq \gamma \leq \alpha$ in $X$ such that $p$ is an endpoint of $A_\gamma$, and $A_\gamma \cap A_\delta = \{p\}$ for $0 \leq \gamma, \delta \leq \alpha$, $\gamma \neq \delta$. Finally, we let $\text{ordc}_X(p) = \alpha$ if $\text{ordc}_X(p) \geq \alpha$ and $\text{ordc}_X(p) \geq \beta$ is not true for any $\beta > \alpha$. The point
$p$ is called an endpoint of $X$ if $\text{ord}_X(p) = 1$ and a ramification point of $X$ if $\text{ord}_X(p) \geq 3$; the set of all endpoints of $X$ is denoted by $E(X)$ and the set of all ramification points of $X$ by $R(X)$.

Our new definition is very similar to the classical one.

**Definition 3.2.** Let $X$ be a continuum and $p$ be a point in $X$, and let $\alpha$ be a cardinal number. We say that $\text{ord}_X(p) \geq \alpha$ if there are arcs $A_\gamma$ for $0 \leq \gamma \leq \alpha$ in $X$ such that $p$ is an endpoint of $A_\gamma$ and $\text{int}(A_\gamma) \cap \text{int}(A_\delta) = \emptyset$ for $0 \leq \gamma, \delta \leq \alpha, \gamma \neq \delta$. Finally, we let $\text{ord}_X(p) = \alpha$ if $\text{ord}_X(p) \geq \alpha$ and $\text{ord}_X(p) \geq \beta$ is not true for any $\beta > \alpha$.

\[ \text{ab} \]
\[ \text{a} \quad \text{b} \]

**Figure 1.** $\text{ord}_D(a) = 1$ and $\text{ord}_D(a) = c$.

Let us show an example where the two notions of order do not coincide.

**Example 3.3.** Consider an arc $A$ with endpoints $a, b$ and a dense countable subset $Q$ of $A$. At each point of $Q$ erect an arc such that diameters of those arcs converge to 0 (see Figure 1). The union of $A$ and the erected arcs is a dendrite $D$. For the point $a \in D$ we have $\text{ord}_D(a) = 1$ and $\text{ord}_D(a) = c$.

The following lemma plays a crucial role in proving our next Theorems.

**Lemma 3.4.** If $f : X \to Y$ is a hereditarily irreducible map from a locally connected continuum $X$ onto a locally connected continuum $Y$ and $A, B$ are subcontinua of $X$ satisfying $\text{int}_X(A) \cap \text{int}_X(B) = \emptyset$, then $\text{int}_Y(f(A)) \cap \text{int}_Y(f(B)) = \emptyset$.

**Proof.** Suppose on the contrary that $\text{int}_Y(f(A)) \cap \text{int}_Y(f(B)) \neq \emptyset$. Denote by $U$ the set $\text{int}_Y(f(A)) \cap \text{int}_Y(f(B))$. Choose two points $p, q$
in \((A \cap f^{-1}(U)) \setminus B\) such that there is an arc \(A\) with endpoints \(p, q\) contained in \(A \cap f^{-1}(U)\). Let \(R\) be the shortest arc in \(X\) intersecting \(\{p, q\}\) and \(B\). Observe that only one point of \(\{p, q\}\) is in \(R\). Without loss of generality, assume \(q \in R\). Then we have \((R \cup B) \subset (A \cup R \cup B)\) and \(f(R \cup B) = f(A \cup R \cup B)\) contrary to our assumption \(f\) was hereditarily irreducible. \(\square\)

**Theorem 3.5.** If \(X\) is a locally connected continuum and \(\text{ord}_X(x)\) is finite for each \(x \in X\), then every non-degenerate subcontinuum of \(X\) has non-empty interior.

**Proof.** Suppose on the contrary that there is a subcontinuum \(Y\) of \(X\) with empty interior. Let \(A\) be an arc in \(Y\) and \(p, q\) be the endpoints of \(A\). For any \(x \in A\), denote by \(A_x\) the subarc of \(A\) with endpoints \(p\) and \(x\). Then \(\{A_x : x \in A \setminus p\}\) is an uncountable family of arcs with empty interiors, so \(\text{ord}_X(x) = \mathfrak{c}\), a contradiction. \(\square\)

**Remark 3.6.** An analogous statement of Theorem 3.5 is not true if we change the newly defined order \(\text{ord}\) to the classical order \(\text{ord}_c\). Really, the dendrite pictured in Figure 1 has all point of classical order finite, but the arc \(ab\) has empty interior.

**Corollary 3.7.** If \(X\) is a locally connected continuum and \(\text{ord}_X(x)\) is finite for each \(x \in X\), then the union of all free arcs in \(X\) is dense.

The following lemma is needed in the proof of the next theorem.

**Lemma 3.8.** If \(f : X \to Y\) is a map from a locally connected continuum \(X\) onto a locally connected continuum \(Y\) such that \(\text{ord}_Y(y)\) is finite and \(\text{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \text{ord}_X(x)\) for each \(y \in Y\), then for any two subcontinua \(A, B\) of \(X\) satisfying \(\text{int}_X(A) \cap \text{int}_X(B) = \emptyset\), we have \(\text{int}_Y(f(A)) \cap \text{int}_Y(f(B)) = \emptyset\).

**Proof.** Let \(A\) and \(B\) be two subcontinua of \(X\) satisfying \(\text{int}_X(A) \cap \text{int}_X(B) = \emptyset\) and suppose that \(\text{int}_Y(f(A)) \cap \text{int}_Y(f(B)) \neq \emptyset\). By Corollary 3.7 the open set \(\text{int}_Y(f(A)) \cap \text{int}_Y(f(B))\) contains a free arc \(C\) and there is a free arc \(P\) in \(X\) such that \(f(P) \subset \text{int}_Y(C)\). For any \(p \in \text{int}(P)\), we have \(\text{ord}_X(p) = \text{ord}_Y(f(p)) = 2\) and there is a point \(q \in B\) such that \(f(q) = f(p)\), so \(\text{ord}_Y(f(p)) \neq \sum_{x \in f^{-1}(f(p))} \text{ord}_X(x)\), a contradiction. \(\square\)
Theorem 3.9. If \( f : X \to Y \) is a map from a locally connected continuum \( X \) onto a locally connected continuum \( Y \) such that \( \text{ord}_Y(y) \) is finite for each \( y \) in \( Y \) and \( \text{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \text{ord}_X(x) \) for each \( y \) in \( Y \), then \( f \) is a hereditarily irreducible map.

Proof. Let \( f : X \to Y \) be a map from a locally connected continuum \( X \) onto a locally connected continuum \( Y \) such that \( \text{ord}_Y(y) \) is finite for each \( y \) in \( Y \) and \( \text{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \text{ord}_X(x) \) for each \( y \) in \( Y \). Let \( A \) and \( B \) be subcontinua of \( X \) such that \( A \subsetneq B \). Let \( C \) be a non-degenerate continuum in \( B \setminus A \). Since \( \text{ord}_Y(y) \) and \( \text{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \text{ord}_X(x) \) for each \( y \) in \( Y \), then \( f(C) \) is a non-degenerate subcontinuum of \( Y \). It follows from Theorem 3.5 that \( \text{int}_Y(f(C)) \neq \emptyset \). By Corollary 3.7 there is a free arc \( P \) in \( \text{int}_Y(f(C)) \). By Lemma 3.8 we have \( \text{int}_Y(f(A)) \cap \text{int}_Y(f(C)) = \emptyset \). So \( f(p) \notin f(A) \) for any \( p \) in the interior of the free arc \( P \). Thus \( f(A) \neq f(B) \), and hence \( f \) is a hereditarily irreducible map. \( \square \)

Theorem 3.10. If \( f : X \to Y \) is a hereditarily irreducible map from a locally connected \( X \) onto a locally connected continuum \( Y \), then \( \text{ord}_X(x) \leq \text{ord}_Y(f(x)) \) for all \( x \) in \( X \).

Proof. Let \( x \in X \), and let \( \text{ord}_X(x) = \alpha \). By our assumption, there are arcs \( A_\gamma \) for \( 0 \leq \gamma \leq \alpha \) in \( X \) such that \( x \in E(A_\gamma) \) for each \( \gamma \), and \( \text{int}(A_\gamma) \cap \text{int}(A_\delta) = \emptyset \) for \( 0 \leq \gamma, \delta \leq \alpha, \gamma \neq \delta \). We have, \( f(A_\gamma) \) is a locally connected continuum for all \( \gamma \), therefore, for each \( \gamma \) there is an arc \( B_\gamma \) in \( f(A_\gamma) \) such that \( f(x) \in E(B_\gamma) \). By Lemma 3.4, \( \text{int}(B_\gamma) \cap \text{int}(B_\delta) = \emptyset \) for \( \gamma \neq \delta \). Thus, \( \text{ord}_Y(f(x)) \geq \alpha \), and hence, \( \text{ord}_X(x) \leq \text{ord}_Y(f(x)) \). \( \square \)

Theorem 3.11. If \( f : X \to Y \) is a hereditarily irreducible map from a locally continuum \( X \) onto a continuum \( Y \) and \( y \) is a point in \( Y \), then

\[
\text{ord}_Y(y) \geq \sum_{x \in f^{-1}(y)} \text{ord}_X(x).
\]

Proof. Let \( \alpha \) be a cardinal number, and let \( \sum_{x \in f^{-1}(y)} \text{ord}_X(x) = \alpha \). Then, there are arcs \( A_\gamma \) for \( 0 \leq \gamma \leq \alpha \) in \( X \) such that one of the endpoints of each \( A_\gamma \) is in \( f^{-1}(y) \) and \( \text{int}_X(A_\gamma) \cap \text{int}_X(A_\delta) = \emptyset \) for \( \gamma \neq \delta \). Since \( f(A_\gamma) \) is a locally connected continuum, then there is an arc \( B_\gamma \) in \( f(A_\gamma) \) such that \( y \in E(B_\gamma) \). By Lemma 3.4, we have \( \text{int}_Y(f(A_\gamma)) \cap \text{int}_Y(f(A_\delta)) = \emptyset \) for \( \gamma \neq \delta \). Since \( B_\gamma \subseteq f(A_\gamma) \), then \( \text{int}_Y(B_\gamma) \cap \text{int}_Y(B_\delta) = \emptyset \) for \( \gamma \neq \delta \). Therefore, \( \text{ord}_Y(y) \geq \alpha \), and hence, \( \text{ord}_Y(y) \geq \sum_{x \in f^{-1}(y)} \text{ord}_X(x) \). \( \square \)
Let us notice that for graphs the two notions of order coincide.

**Observation 3.12.** For every graph $G$ and a point $p \in G$ we have $\operatorname{ord}_G(p) = \operatorname{ord}_G(p)$.

Our next Theorem shows that Theorem 3.11 can be strengthen if the domain and the range of our functions are graphs.

**Theorem 3.13.** If $f : X \to Y$ is a hereditarily irreducible map from a graph $X$ onto a graph $Y$ and $y$ is a point in $Y$, then $\operatorname{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x)$.

**Proof.** Suppose there is a hereditarily irreducible map $f$ from a graph $X$ onto a graph $Y$, and let $y$ be a point in $Y$. By Theorem 3.11, we have $\operatorname{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x)$. We will show that $\operatorname{ord}_Y(y) \leq \sum_{t \in f^{-1}(y)} \operatorname{ord}_X(t)$. If $\operatorname{ord}_Y(y) = n$ for some positive integer $n$, then there are $n$ arcs $A_1, A_2, ..., A_n$ in $Y$ such that $y \in E(A_i)$ for all $i$, $i = 1, 2, ..., n$, and $\operatorname{int}_Y(A_i) \cap \operatorname{int}_Y(A_j) = \emptyset$ for $i \neq j$. Since $Y$ is a graph, we may assume that $A_i \cap A_j = \{y\}$, $i \neq j$, and $\operatorname{ord}_Y(z) = 2$ for all $z \in A_i \setminus \{y\}$. For each $i$, there is a non-degenerate component $C_i$ of $f^{-1}(A_i)$ such that $f(C_i) \subseteq A_i$ and $y \in C_i$. By Theorem 3.10, $\operatorname{ord}_X(x) \leq 2$ for all $x \in C_i \setminus f^{-1}(y)$. Since $f$ is a hereditarily irreducible, then $C_i$ contains no simple closed curve, therefore, $C_i$ is an arc in $X$. Since $A_i \cap A_j = \{y\}$, then $\operatorname{int}_X(C_i) \cap \operatorname{int}_X(C_j) = \emptyset$ for $i \neq j$. For each $i$, we have $E(C_i) \cap f^{-1}(y) \neq \emptyset$ since $f$ is a hereditarily irreducible map. It follows that $\sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x) \geq n$. Thus, $\operatorname{ord}_Y(y) \leq \sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x)$, and hence, $\operatorname{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x)$. \qed

**Example 3.14.** The assumption that $X$ and $Y$ are graphs in Theorem 3.13 is essential, in general the equality does not hold. In fact there is a hereditarily irreducible map $f$ from $[0,1]$ onto a locally connected continuum $X$ pictured in Figure 2 such that $f^{-1}(f(1)) = 1$ and $\operatorname{ord}_X(f(1)) = 3 > 1 = \operatorname{ord}_{[0,1]}(1)$. The map is sketched in Figure 3.

**Corollary 3.15.** If $f : X \to Y$ is a map from a graph $X$ onto a graph $Y$, then $f$ is hereditarily irreducible if only if $\operatorname{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \operatorname{ord}_X(x)$.

**Proof.** It follows from Theorem 3.5 and Theorem 3.13. \qed

**Corollary 3.16.** If $f : X \to Y$ is a hereditarily irreducible map between graphs $X$ and $Y$, then the number of points of odd order of $Y$ is less than or equal to the number of points of odd order of $X$.

**Proof.** By Theorem 3.13 each point of odd order in $Y$ has at least one point in its preimage of odd order. \qed
Problem 3.17. Is the conclusion of Corollary 3.16 true for any hereditarily irreducible map between locally connected continua $X$ and $Y$?

Our next result is an immediate consequence of Theorem 3.13

Corollary 3.18. If $X$ is a graph, $f : X \to X$ is hereditarily irreducible map, then $f$ is a homeomorphism.

Proof. Let $X$ be a graph, and let $f : X \to X$ be a hereditarily irreducible map. Note that $E(X)$ and $R(X)$ are finite sets and $f$ is onto, therefore, it follows from Theorem 3.13 that $f$ maps endpoints to endpoints and ramification points to ramification points, and $f$ is one-to-one on $E(X)$ and $R(X)$. Since $f$ maps endpoints to endpoints, then, using Theorem 3.11, we can conclude that $f$ maps points of order 2 to points of order 2 and $f$ is one-to-one on $X \setminus (E(X) \cup R(X))$. Thus, $f$ is a homeomorphism. \qed
The following example shows Corollary 3.18 doesn’t hold if we remove the assumption that \( X \) is a graph.

**Example 3.19.** Let \( X = \bigcup_{i=1}^{\infty}(C_i \cup L_i) \), as shown in Figure 4, where \( C_i \) is a circle, \( \text{diam}(C_i) \to 0 \) as \( i \to 0 \) and \( L_i \) is an arc, \( \text{diam}(L_i) \to 0 \) as \( i \to 0 \). Let \( f \) be a map from \( X \) onto \( X \) define as following. Let \( f(L_1) = C_1 \), image of the endpoints of \( L_1 \) under \( f \) is equal to \( p \), and \( f(C_i) = C_{i+1}, \ i \geq 1, \ f(L_i) = L_{i+1}, \ i \geq 2 \). Then, \( f \) is hereditarily irreducible map but \( f \) is not one-to-one.

![Figure 4. The continuum \( X \).](image)

In some cases the converse to Corollary 3.16 is also true. This is the famous Euler’s theorem about Königsberg bridges, see e.g. [4, Theorem 3.6, p. 79].

**Theorem 3.20.** If \( G \) is a graph, then there is a hereditarily irreducible map from \([0, 1]\) onto \( G \) if and only if \( G \) has at most two points of odd order.

Similarly, we can characterize hereditarily irreducible images of the simple closed curve.

**Theorem 3.21.** A graph \( G \) is an image of \( S^1 \) under a hereditarily irreducible map if and only if each point of \( G \) has even order.

*Proof.* The proof is very similar to the proof of Theorem 3.20. \( \square \)
Generalizations of Theorems 3.20 and 3.21 to arbitrary locally connected continua are not true. This can be seen by the following example.

**Example 3.22.** The continuum $X$ pictured in Figure 5 is locally connected, every subcontinuum of $X$ has non-empty interior, $X$ has only two points of odd order, but there is no hereditarily irreducible map from $[0,1]$ onto $X$.

![Figure 5. The continuum $X$ has only two points of odd order.](image)

The reason why Example 3.22 works is the existence of points of infinite orders. Therefore the following problems seems interesting.

**Problem 3.23.** Suppose $X$ is a locally connected continuum in which the union of free arcs is dense, all points of $X$ are of finite order, and $X$ has at most two points of odd order. Does there exist a hereditarily irreducible map from $[0,1]$ onto $X$?

Characterizations of hereditarily irreducible maps of other graphs are much less obvious. Our next example shows that the invariants in Theorem 3.10 and Corollary 3.16 are not enough to characterize hereditarily irreducible images of a simple triod.

**Example 3.24.** There is a graph $G$ with the following properties:

1. $G$ has one ramification point of odd order;
2. $G$ has three endpoints;
3. $G$ is not an image of a simple triod under a hereditarily irreducible map.

*Proof.* Let $G$ be the graph in Figure 6, and let $r$ be the only point of order three. Suppose there is a hereditarily irreducible map from a simple triod $T$ onto $G$. Then the three endpoints of $T$ have to go to the three endpoints of $G$ and, consequently, by Theorem 3.13 the image of
the vertex of $T$ is $r$. Let $p$ be a point of the interior of the free arc left to $r$. Then there are at least three points of order two in the preimage of $p$. This contradicts Theorem 3.13. □

Figure 6. The graph $G$ with one ramification point of odd order.

The next theorem gives another characterization of hereditarily irreducible maps between graphs.

**Theorem 3.25.** If $f : X \to Y$ is a surjective map between graphs, then the following conditions are equivalent:

1. $f$ is a hereditarily irreducible map;
2. $\text{card}(f^{-1}(y)) < \aleph_0$ for any $y \in Y$ and $\text{card} \left( \{ y \in Y : f^{-1}(y) \text{ is non-degenerate} \} \right) < \aleph_0$.

**Proof.** To show that $(1) \implies (2)$, let $y \in Y$. Since $Y$ is a graph, then $\text{ord}_Y(y) < \aleph_0$. By Theorem 3.13, we have $\text{ord}_Y(y) = \sum_{x \in f^{-1}(y)} \text{ord}_X(x)$, therefore, $\text{card}(f^{-1}(y)) < \aleph_0$.

Now we will show that $\text{card}(\{ y \in Y : f^{-1}(y) \text{ is non-degenerate} \}) < \aleph_0$. Let $y$ be any point of $Y$ such that $f^{-1}(y)$ is non-degenerate. It follows from Theorem 3.13 that $\text{ord}_Y(y) \geq 2$ and $y \in Y \setminus E(Y)$. Since $Y$ is a graph then $\text{card}(R(Y)) < \aleph_0$. Therefore, $\text{card}(\{ y \in R(Y) : f^{-1}(y) \text{ is non-degenerate} \}) < \aleph_0$. If $y \notin R(Y)$, then it clear that $f^{-1}(y) \subseteq E(X)$. Since $X$ is a graph, then $\text{card}(E(X)) < \aleph_0$, and therefore, $\text{card}(\{ y \in Y \setminus (E(Y) \cup R(Y)) : f^{-1}(y) \text{ is non-degenerate} \}) < \aleph_0$. So $\text{card}(\{ y \in Y : f^{-1}(y) \text{ is non-degenerate} \}) < \aleph_0$.

To show that $(2) \implies (1)$, let $A$ and $B$ be two subcontinua of $X$ such that $A \subsetneq B$. Since $\text{card}(B \setminus A) > \aleph_0$, $\text{card}(f^{-1}(f(a))) < \aleph_0$ for any
Theorem 3.26. If $f$ is a hereditarily irreducible map from a locally connected continuum $X$ onto a graph $Y$, then $X$ is a graph.

Proof. Let $f$ be a hereditarily irreducible map from a locally continuum $X$ onto a graph $Y$. By Theorem 3.10, $\text{ord}_X(x) \leq \text{ord}_Y(f(x))$ for each $x \in X$. Since $Y$ is a graph, then $\text{ord}_Y(y) < \aleph_0$ for every $y \in Y$. Therefore, $\text{ord}_X(x) < \aleph_0$ for each $x \in X$. So to show that $X$ is a graph, it is enough to show that $\text{card}(R(X)) < \aleph_0$. It follows from Theorem 3.11 that $\text{card}(R(X)) < \aleph_0$. Thus $X$ is a graph. \qed

4. HEREDITARILY IRREDUCIBLE MAPS FROM AN ARC

In this section we give two characterizations of hereditarily irreducible images of an arc. In the case of the image is a graph we have a full characterization by Theorem 3.20. Also if the image is a continuum that does not contain free arc, we always have such mapping by [4, Theorem 4.21, p. 87].

Theorem 4.1. Suppose $X$ and $Y$ are locally connected continua such that $Y \subseteq X$. If $Y$ contains all free arcs of $X$ and there is a hereditarily irreducible map from $[0, 1]$ onto $Y$, then there is a hereditarily irreducible map from $[0, 1]$ onto $X$.

Proof. Let $X$ be a locally connected continuum, $d$ a convex metric on $X$, and let $Y$ be a locally connected subcontinuum of $X$ containing all free arcs of $X$. Suppose there is a hereditarily irreducible map $f$ from $[0, 1]$ onto $Y$. Let $\{A_\lambda : \lambda \in \Lambda\}$ be the set of interiors of all maximal free arcs in $X$; here the set $\Lambda$ is either finite or countable infinite. Put $U = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda)$, and let $T$ be a component of $[0, 1] \setminus U$. Define $F_T = \{x \in X : d(x, f(T)) = d(x, Y)\}$. We will show that $F_T$ is a connected without free arcs. To see that let $x \in F_T$, and let $y$ be a point in $f(T)$ such that $d(x, y) = d(x, f(T)) = d(x, Y)$. Let $A$ be a convex arc joining $x$ and $y$. Then for every $z \in A$ we have $d(z, Y) = d(z, f(T)) = d(z, y) \leq d(x, y)$, so $z \in F_T$. This shows $F_T$ is arcwise connected.

To see that $F_T$ contains no free arc, it is enough to notice that $T \subset [0, 1] \setminus U$ and $f(U)$ contains all free arcs in $X$. Observe that if
$F_T \cap (X \setminus Y) \neq \emptyset$, then $\text{int}(F_T) \cap (X \setminus Y) \neq \emptyset$. Moreover, if $T$ and $T'$ are two different components of $[0,1] \setminus U$, then $\text{int}(F_T) \cap \text{int}(F_{T'}) = \emptyset$. Note also that if $F_T$ is non-degenerate for some component $T$ of $[0,1] \setminus U$, then $\text{int}(F_T) \neq \emptyset$. Therefore there is at most countable set $\mathcal{J}$ such that $F_{T_j}$ is non-degenerate for each $j \in \mathcal{J}$. Let $\alpha : [0,1] \to [0,1]$ be a monotone map such that if $F_{T_j}$ is non-degenerate, then $\alpha^{-1}(T_j)$ is non-degenerate. For each $j \in \mathcal{J}$, let $D_j$ be a dense countable subset of $\alpha^{-1}(T_j)$. By Theorem 4.21 [4], there is a hereditarily irreducible map $g_i : \alpha^{-1}(T_j) \to F_{T_j}$ that is one-to-one on $D_j$. Define $g : [0,1] \to X$ by

$$g(t) = \begin{cases} g_i(t) & \text{if } t \in \alpha^{-1}(T_i) \text{ for some } i \in \mathcal{J}; \\ f(\alpha(t)) & \text{otherwise}. \end{cases}$$

Observe that $g$ is a hereditarily irreducible map from $[0,1]$ onto $X$, as required.

As an illustration of an application of Theorem 4.1 let us have the following example.

**Example 4.2.** There is a hereditarily irreducible map from $[0,1]$ onto the continuum $X$ pictured in left part of Figure 7.

![Figure 7](image)

**Figure 7.** The continuum $X$ (left) and the graph $G$ (right).

**Proof.** Note that the graph $G$ pictured in the right part of Figure 7 has only two point of odd order, so it is a hereditarily irreducible image of $[0,1]$ by Theorem 3.20. Note also that $G$ contains all free arcs of $X$, therefore the existence of the required map follows from Theorem 4.1. \qed
Definition 4.3. Let $X$ be a locally connected continuum. We define an equivalence relation $\sim$ on $X$ by letting $x \sim y$ if only if there is an arc with empty interior that contains both $x$ and $y$ or $x = y$.

Theorem 4.4. Let $X$ be a locally connected continuum. If there is a hereditarily irreducible map from $[0, 1]$ onto $X$, then there is a hereditarily irreducible map from $[0, 1]$ onto $X/\sim$.

Proof. Suppose there is a hereditarily irreducible map $f$ from $[0, 1]$ onto $X$. We will construct a hereditarily irreducible map from $[0, 1]$ onto $X/\sim$. For any $s, t \in [0, 1]$, we define $s \sim t$ if $f(s) \sim f(t)$ in $X/\sim$ for any $x \in [s, t]$. Note that $\sim$ is a monotone relation on $[0, 1]$, so $[0, 1]/\sim$ is homeomorphic to $[0, 1]$. Define $g : [0, 1]/\sim \to X/\sim$ by $g(t) = [f(t)]\sim$. To show that $g$ is hereditarily irreducible map, let $A$ be a subcontinuum of $[0, 1]/\sim$ and $B$ be a proper subcontinuum of $A$, and let $q, q'$ be the natural quotient maps on $X$ and $[0, 1]$ respectively. Since $B$ is proper subcontinuum of $A$, then $q'^{-1}(B) \subset q^{-1}(A)$. Since $q'$ is a monotone map, the sets $\sim_1, q'^{-1}(A)$ and $q^{-1}(B)$ are continua. Also, $f(q'^{-1}(B)) \subset f(q^{-1}(A))$ since $f$ is hereditarily irreducible. Since $B \neq A$, then there is $a \in A$ such that $f(a) \sim f(b)$ for all $b \in B$. Therefore, $g(B) = q(f(q'^{-1}(B))) \subset g(A) = q(f(q^{-1}(A)))$, and thus $g$ is a hereditarily irreducible map. 

As an illustration of an application of Theorem 4.4 let us have the following example.

Example 4.5. There is no hereditarily irreducible map from $[0, 1]$ onto the continuum $X$ pictured in left part of Figure 8.

Proof. The continuum $X/\sim$ pictured in the right part of Figure 8 is just a simple triod, so it contains four points of odd order, therefore, by Theorem 3.20, there is no hereditarily irreducible map from $[0, 1]$ onto $X/\sim$. Consequently, by Theorem 4.4 there is no hereditarily irreducible map from $[0, 1]$ onto $X$.

We do not know if the converse Theorem 4.4 is true.

Problem 4.6. Suppose $X$ is a locally connected continuum and there is a hereditarily irreducible map from $[0, 1]$ onto $X/\sim$. Can we prove that there is hereditarily irreducible map from $[0, 1]$ onto $X$?

To show our next Theorem we need to recall the definition and properties of a dendrite called $D_3$. This is a dendrite with a dense set of ramification points, each point of classical order 3. It is known, (see e.g. [1, (6), p. 490]), that the above properties characterize $D_3$, i.e. any two dendrites with a dense set of ramification points, each of classical
order 3, are homeomorphic. Note that every point of \( D_3 \) has infinite order in the new definition. Here we will use the fact that any dendrite with a dense set of ramification points contains \( D_3 \).

**Theorem 4.7.** Let \( f : [0,1] \to X \) be a hereditarily irreducible map from \([0,1]\) onto a locally connected continuum \( X \). If \( X \) is not arc, then \( X \) contains either a simple closed curve or a copy of \( D_3 \).

**Proof.** If \( X \) contains a simple closed curve, we are done. Otherwise we will show that \( X \) contains a copy of \( D_3 \). On the contrary, suppose that \( X \) contains no copy of \( D_3 \). Since \( X \) is not an arc, there are two elements \( t_1 \) and \( t_2 \) in the unit interval \([0,1]\) such that \( t_1 < t_2 \) and \( f(t_1) = f(t_2) \). Since \( f([t_1,t_2]) \) is a locally connected continuum and \( X \) contains no copy of \( D_3 \), then \( f([t_1,t_2]) \) contains a free arc \( A \). Let \( a, b \) be two distinct elements in the interior of \( A \). By Theorem 3.10, \( f \) is one-to-one on \( f^{-1}(ab) \). Choose \( t_3, t_4 \) in \([t_1,t_2]\) such that \( f(t_3) = a \) and \( f(t_4) = b \). By symmetry, we may assume that \( t_3 < t_4 \). Since \( X \) is locally connected containing no simple closed curve, then it is a dendrite, and therefore, \( f([t_1,t_2]) \setminus \text{int}(ab) \) has two components. If \( f(t_1) \) and \( a \) are in the same component, then there is an element \( s \in [t_4,t_2] \) such that \( f(s) = a \); since \( f(t_1) = f(t_2) \) and for the same reason if \( f(t_1) \) and \( b \) are in the same component, then there is an element \( t \in [t_1,t_3] \) such that \( f(t) = b \), but this contradicts the facts that \( f^{-1}(a) = \{t_3\} \) and \( f^{-1}(b) = \{t_4\} \). Thus, \( X \) contains a copy of \( D_3 \).

As a corollary we get the following result.
Corollary 4.8. If there is a hereditarily irreducible map from $S^1$ onto a continuum $X$, then $X$ contains either a simple closed curve or a copy of $D_3$.

5. Hereditarily irreducible images of graphs

In [4] the authors proved that for every locally connected continuum $X$ without free arcs there is a hereditarily irreducible map from $[0, 1]$ onto $X$. The goal of this section is to generalize the result to have any graph in the domain, not just $[0, 1]$.

We have to start with the definitions of necessary symbols. Let $X$ and $Y$ be continua, $\bar{x} = \{x_i\}_{i=1}^n \subset X$, $\bar{y} = \{y_i\}_{i=1}^n \subset Y$ be any subsets with $n$ elements, not necessarily different, and let $\mathcal{C}(X,Y)$ denote the set of all maps from $X$ to $Y$. We define the following sets

1. $S(X,Y) = \{f \in \mathcal{C}(X,Y) : f \text{ is a surjective map}\}$.
2. $\mathcal{C}(X,Y,\bar{x},\bar{y}) = \{f \in \mathcal{C}(X,Y) : f(x_i) = y_i \text{ for all } i \leq n\}$.
3. $S(X,Y,\bar{x},\bar{y}) = S(X,Y) \cap \mathcal{C}(X,Y,\bar{x},\bar{y})$.
4. $A_F(X,Y) = \{f \in \mathcal{C}(X,Y) : f^{-1}(f(x)) = \{x\} \text{ for each } x \in F\}$

Let us recall two important results from [4].

Theorem 5.1. [4, Theorem 4.13, p.85]. Let $X$ be a 1-dimensional continuum, $Y$ a non-degenerate locally connected continuum without free arcs, $F$ a 0-dimensional closed subset of $X$, $\bar{x} \subset X$, and $\bar{y} \subset Y$. If $F \cap \bar{x} = \emptyset$, then $S(X,Y,\bar{x},\bar{y}) \cap A_F(X,Y)$ is a dense $G_δ$-subset of $S(X,Y,\bar{x},\bar{y})$.

Corollary 5.2. [4] Let $X$ be a 1-dimensional continuum, $Y$ a non-degenerate locally connected continuum without free arcs, $T$ a 0-dimensional $F_σ$-subset of $X$. Then $S(X,Y) \cap A_T(X,Y)$ is a dense $G_δ$-subset of $S(X,Y)$.

Theorem 5.3. Let $X$ be a graph, and let $Y$ be a non-degenerate locally connected continuum without free arcs, then there is a hereditarily irreducible map from $X$ onto $Y$.

Proof. Let $X$ be a graph, and let $Y$ a non-degenerate locally connected continuum without free arcs. Let $T$ be a countable dense subset of $X$. By Corollary 5.2, $S(X,Y) \cap A_T(X,Y)$ is a dense $G_δ$-subset of $S(X,Y)$. Let $f \in S(X,Y) \cap A_T(X,Y)$. We will show that $f$ is a hereditarily irreducible map. Let $A, B$ be two subcontinua of $X$ such that $A \subset B$. Since $T$ is dense subset of $X$, then $T \cap (B \setminus A) \neq \emptyset$. Let $t \in T \cap (B \setminus A)$, then $f^{-1}(f(t)) = \{t\}$. Therefore, $f(A) \subset f(B)$, and hence, $f$ is a hereditarily irreducible map. □
6. Hereditarily irreducible maps onto dendrites

**Theorem 6.1.** Let $D$ be a dendrite containing no copy of $D_3$. If there is a hereditarily irreducible map $f$ from a locally connected continuum $X$ onto $D$, then $f$ is a homeomorphism.

*Proof.* Let $x_1, x_2$ be two distinct elements of $X$, and let $A$ be an arc in $X$ irreducible between $\{x_1, x_2\}$. Since $D$ is a dendrite contains no copy of $D_3$, then $f(A)$ contains no simple closed curve and no copy of $D_3$. By Proposition 4.7, $f$ is a homeomorphism on $A$. So, $f(x_1) \neq f(x_2)$, and therefore, $f$ is one-to-one on $X$. Since $f$ is one-to-one map from a compact space $X$ onto a Hausdorff space $D$, then $f$ is a homeomorphism. □

**Corollary 6.2.** If $f : X \to Y$ is a hereditarily irreducible map between locally connected continua, and $X$ is not homeomorphic to $Y$, then $Y$ contains either a simple closed curve or a copy of $D_3$.

The following example shows that the assumption of local connectedness in Corollary 6.2 is essential.

**Example 6.3.** Let $f$ be a hereditarily irreducible map from a continuum $X$ onto a continuum $Y$ pictured in Figure 9. $Y$ is not homeomorphic to $X$ and contains no simple closed curve nor a copy of $D_3$.

![Figure 9. The map $f$ from $X$ onto $Y$.](image)
Theorem 6.4. A dendrite $D$ is an image of $[0, 1]$ under hereditarily irreducible map if only if there is an arc containing all free arcs of $D$.

Proof. First, suppose that there is an arc containing all free arcs of $D$. Then the existence of a hereditarily irreducible map from $[0, 1]$ onto $D$ follows from Theorem 4.1.

Second, suppose $f : [0, 1] \rightarrow D$ is a hereditarily irreducible map. We will show that the arc $f(0)f(1)$ contains all free arcs of $D$. Suppose the contrary, let $a, b$ be two distinct interior points of a free arc in $D$. Let $t_a, t_b$ be two points in $[0, 1]$ such that $f(t_a) = a$ and $f(t_b) = b$. The difference $D \setminus ab$ has two components, say $C_1, C_2$. Because $a$ and $b$ are not in the arc $f(0)f(1)$, the arc $f(0)f(1)$ is in one of these components. We may assume $f(0)f(1) \subset C_1$. Let $s$ be a point such that $f(s) \in C_2$; then there is a point $s' \in (s, 1)$ such that $f(s') = a$. Thus $f(t_a) = f(s') = a$, contrary to Theorem 3.11. □

Theorem 6.5. For a dendrite $D$ the following conditions are equivalent.

a) $R(D)$ is dense in $D$.

b) $D$ contains no free arc.

c) $E(D)$ is dense in $D$.

d) For any graph $G$ there is a hereditarily irreducible map from $G$ onto $D$.

e) There is a hereditarily irreducible map $f$ from $S^1$ onto $D$.

f) There is a graph $G$ with no cut points and a hereditarily irreducible map from $G$ onto $D$.

Proof. The equivalence of conditions (a), (b), and (c) are shown in [2, Theorem 4.6, p. 10]. The implication (c) ⇒ (d) follows from Theorem 5.3. The implications (d) ⇒ (e) and (e) ⇒ (f) are trivial. So we only need to show (e) ⇒ (b). Suppose on the contrary that $f : S^1 \rightarrow D$ is hereditarily irreducible map and $D$ contains a free arc. Let $c$ be an interior point of that arc, then $D \setminus \{c\}$ has two components. Let us choose $t_0, t_1 \in S^1$ such that $f(t_0)$ and $f(t_1)$ are in different components of $D \setminus \{c\}$. Let $A$ and $B$ be arcs in $S^1$ such that $A \cap B = \{t_0, t_1\}$. Since $f(A)$ and $f(B)$ contain the point $c$, so there are points $s_1 \in A$ and $s_2 \in B$ such that $f(s_1) = f(s_2) = c$. Then $\text{ord}_{S^1}(s_1) = \text{ord}_{S^1}(s_2) = 2$, while $\text{ord}_D(c) = 2 < \text{ord}_{S^1}(s_1) + \text{ord}_{S^1}(s_2)$, contrary to Theorem 3.11. □

7. Mappings

Theorem 7.1. Let $f$ be a hereditarily irreducible map from a locally connected continuum $X$ onto a locally connected continuum $Y$, then $f$ is an open map if only if $f$ is a homeomorphism.
Proof. Suppose, on the contrary, that $f$ is not a homeomorphism. Then by our assumption, there are $x_1$ and $x_2$ in $X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$. Let $Z$ be a continuum neighborhood of $y$ in $Y$ such that $x_1$ and $x_2$ lying in different components of $f^{-1}(Z)$. Let $C_1$ and $C_2$ be components of $f^{-1}(Z)$ that contain $x_1$ and $x_2$ respectively. Since $f$ is open map, then $f$ maps $C_1$ and $C_2$ onto $Z$. Let $A$ be an arc in $X$ irreducible between $C_1$ and $C_2$, then $C_1 \cup A$ and $C_1 \cup C_2 \cup A$ are subcontinua of $X$, $C_1 \cup A \subsetneq C_1 \cup C_2 \cup A$, and $f(C_1 \cup A) = f(C_1 \cup C_2 \cup A)$ but this contradicts our assumption that $f$ is a hereditarily irreducible map. Thus, $f$ is a homeomorphism.

The assumption that $X$ and $Y$ are locally connected continuum in Theorem 7.1 is essential. The following example shows that without the assumption of connectedness Theorem 7.1 does not hold.

Example 7.2. Let $S^1$ be the unit circle $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$. Define $f, g : S^1 \to S^1$ by $f(z) = z^3$ and $g(z) = z^2$. The inverse limit space $\varprojlim \{ S^1, f \}$ with one bonding mapping is called a triadic solenoid $\Sigma_3$. The commutative diagram below induces a map $g^* : \Sigma_3 \to \Sigma_3$ (See [3], p. 101).

\[
\begin{array}{cccccc}
S^1 & f & S^1 & f & S^1 & \cdots \\
g\downarrow & g\downarrow & g\downarrow & g\downarrow & g\downarrow & \\
S^1 & f & S^1 & f & S^1 & \cdots
\end{array}
\]

We will show that the map $g^*$ is hereditarily irreducible and open, but not a homeomorphism. To see it is not a homeomorphism, let us observe that $g^*((1,1,1,...)) = g^*((-1,-1,-1,...)) = (1,1,1,...) \in \Sigma_3$.

To show that $g^*$ is hereditarily irreducible, let $A$ and $B$ be two continua in $\Sigma_3$ such that $A$ is a proper subcontinuum of $B$. Let $\pi_n : \Sigma_3 \to S^1$ be the projection onto the $n$-th factor, i.e $\pi_n((x_1,x_2,...)) = x_n$. Since $A \neq B$, there is an index $n$ such that $\pi_n(A) \neq \pi_n(B)$. Since $\pi_n(A)$ and $\pi_n(B)$ are continua, thus arcs in $S^1$, the length of $\pi_n(A)$ is less than $2\pi$. Observe that the length of $\pi_{n+1}(A)$ is three times less than the length of $\pi_n(A)$, so it is less than $2\pi/3$. Thus $\pi_{n+1}(A) \subsetneq \pi_{n+1}(B)$ and the length of $\pi_{n+1}(A)$ is less than $2\pi/3$. Applying the function $g$, we see that $g(\pi_{n+1}(A))$ has length less than $4\pi/3$ and that $g(\pi_{n+1}(A)) \subsetneq g(\pi_{n+1}(B))$. This implies that $g^*(A) \subsetneq g^*(B)$ as required. To see that $g^*$ is open, observe that the map $g$ is open and that the diagram above is exact. Then the conclusion follows from Theorem 4 in [9].
HEREDITARILY IRREDUCIBLE MAPS 19

8. Answers

In [4, Theorem 4.21.3, Question 6.4, p. 92] B. Espinoza and E. Matsuhashi proved that for any locally connected continuum $X$ contains no free arcs and for any two points $p, q \in X$, there is a hereditarily irreducible map $f$ from $[0, 1]$ onto $X$ such that $f(0) = p$ and $f(1) = q$ and posed a question: Is the converse of this Theorem true? We answered their question in the positive.

Theorem 8.1. If $X$ is a locally connected continuum and for any two points $p, q \in X$ there is a hereditarily irreducible map $f$ from $[0, 1]$ onto $X$ such that $f(0) = p$ and $f(1) = q$, then $X$ contains no free arc.

Proof. Suppose on the contrary that $X$ contains a free arc $A$. Choose two distinct points $p, q$ in the interior of $A$ and let $f$ be a hereditarily irreducible map from $[0, 1]$ onto $X$ such that $f(0) = p$ and $f(1) = q$. Choose two distinct points $a, b$ in the interior of $A$ such that $p \in \text{int}(ab)$ and $q \notin ab$. Let $t_1, t_2 \in [0, 1]$ such that $f(t_1) = a$ and $f(t_2) = b$. Note that $p$ or $q$ is in the interior of $f([t_1, t_2])$. Without loss of generality, assume $p \in \text{int} f([t_1, t_2])$, so there is a point $t_3$ in the open interval $(t_1, t_2)$ such that $f(t_3) = p$. Then $\text{ord}_X(p) = 2 < \text{ord}_{[0,1]}(t_3) + \text{ord}_{[0,1]}(0)$, but this contradicts Theorem 3.11. Thus $X$ contains no free arc. □

References

DEPARTMENT OF MATHEMATICS AND STATISTICS, MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY, 400 W 12TH ST, ROLLA MO 65409-0020

E-mail address: haq3f@mail.mst.edu
E-mail address: wjcharat@mst.edu