Inducible Mappings Between Hyperspaces

by

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Summary. Given a continuum $X$ we denote by $2^X$ and $C(X)$ the hyperspace of all nonempty compact subsets and of all nonempty subcontinua of $X$. For any two continua $X$ and $Y$ and a mapping $f : X \to Y$ let $2^f$ and $C(f)$ stand for the induced mappings between corresponding hyperspaces. A mapping $g$ between the hyperspaces is inducible if there exists a mapping $f$ such that $g = 2^f$ or $g = C(f)$, respectively. Necessary and sufficient conditions are shown under which a given mapping $g$ is inducible.

1. Introduction and preliminaries. All spaces considered in this paper are assumed to be metric. A mapping means a continuous function. A continuum means a compact connected space. Given a continuum $X$ with a metric $d$, we let $2^X$ to denote the hyperspace of all nonempty closed subsets of $X$ equipped with the Hausdorff metric $H$ defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(equivalently: with the Vietoris topology: see e.g. [2, (0.1), p. 1 and (0.12), p. 10]. Further, we denote by $C(X)$ the hyperspace of all subcontinua of $X$, i.e. of all connected elements of $2^X$. The reader is referred to Nadler’s book [2] for needed information on the structure of hyperspaces. In particular, the following fact is well-known (see [2, Theorem (1.13), p. 65]).

FACT 1.1. For each continuum $X$ the hyperspace $C(X)$ is a subcontinuum of the hyperspace $2^X$.

We denote by $F_1(X)$ the hyperspace of singletons. The following proposition is a consequence of definitions.

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PROPOSITION 1.2. For each continuum $X$ the space $F_1(X)$ of singletons is homeomorphic (even isometric) to $X$, and thus it is a subcontinuum of the hyperspace $C(X)$. Consequently,

\[(1.2) \quad X \cong F_1(X) \subset C(X) \subset 2^X.\]

Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we consider mappings (called the induced ones)

\[2f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)\]

defined by

\[2f(A) = f(A) \quad \text{for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \quad \text{for every } A \in C(X).\]

Thus, by Fact 1.1, the following is obvious.

**FACT 1.3.** For every continua $X$ and $Y$ and for each mapping $f : X \to Y$ we have $2f|C(X) = C(f)$.

Proofs of the following facts are straightforward.

**FACT 1.4.** Let a mapping $f : X \to Y$ between continua $X$ and $Y$ be given. Then

\[(1.3) \quad 2f(F_1(X)) \subset F_1(Y); \quad (1.4) \quad C(f)(F_1(X)) \subset F_1(Y).\]

**FACT 1.5.** Let a mapping $f : X \to Y$ between continua $X$ and $Y$ be given, and let $A, B$ be closed subsets of $X$ with $A \subset B$. Then

\[(1.5) \quad 2f(A) \subset 2f(B); \quad (1.6) \quad C(f)(A) \subset C(f)(B) \quad \text{provided that } A \text{ and } B \text{ are connected}.\]

2. Inducible mappings. In connection with the concept of the induced mappings one can ask under what conditions an arbitrary mapping between either the hyperspaces $2^X$ and $2^Y$ or the hyperspaces $C(X)$ and $C(Y)$ is an induced one. An answer to this question is presented below. To formulate a necessary and sufficient condition for a mapping between hyperspaces to be an induced one it is convenient to introduce some auxiliary concepts and notation.

Given two mappings between hyperspaces $g_1, g_2 : 2^X \to 2^Y$ (or $g_1, g_2 : C(X) \to C(Y)$), we will write $g_1 \prec g_2$ provided that $g_1(A) \subset g_2(A)$ for each $A \in 2^X$ (for each $A \in C(X)$, respectively). The following properties of the relation $\prec$ on the set of all mappings between hyperspaces are consequences of the above definition.
**OBSERVATION 2.1.** The relation \(<\) is an order on the set of all mappings between hyperspaces (either \(2^X\) and \(2^Y\), or \(C(X)\) and \(C(Y)\)), that is, the following properties are true for every mappings \(g_1, g_2, g_3\) between corresponding hyperspaces:

1. \(g_1 < g_2\) and \(g_2 < g_3\) implies \(g_1 < g_3\);
2. \(g_1 < g_2\) and \(g_2 < g_1\) implies \(g_1 = g_2\);
3. \(g_1 < g_1\).

Let \(X\) and \(Y\) be continua. A mapping between hyperspaces, \(g : 2^X \to 2^Y\) (or \(g : C(X) \to C(Y)\)), is said to be **inducible** provided that there exists a mapping \(f : X \to Y\) such that \(g = 2^f\) (or \(g = C(f)\), respectively). We have the following characterization of inducible mappings.

**THEOREM 2.2.** Let continua \(X\) and \(Y\) be given. A mapping between hyperspaces, \(g : 2^X \to 2^Y\) (or \(g : C(X) \to C(Y)\)), is inducible if and only if each of the following three conditions is satisfied:

1. \(g(F_1(X)) \subseteq F_1(Y)\);
2. \(A \subseteq B\) implies \(g(A) \subseteq g(B)\) for every \(A, B \in 2^X\) (for every \(A, B \in C(X)\), respectively);
3. \(g\) is minimal with respect to the order \(<\), i.e., if a mapping \(g_0 : 2^X \to 2^Y\) (or \(g_0 : C(X) \to C(Y)\)) satisfies (2.5), and \(g_0 < g\), then \(g = g_0\).

**Proof.** Since the proof is the same for both considered cases, i.e., for mappings between the hyperspaces \(2^X\) and \(2^Y\) as well as between \(C(X)\) and \(C(Y)\), we present a proof for the first case only.

Assume \(g\) is inducible, i.e., \(g = 2^f\) for some \(f : X \to Y\). Then (2.4) and (2.5) follow from Facts 1.4 and 1.5 correspondingly. Let \(g_0\) satisfy (2.5), and let \(g_0 < 2^f\). Then for each \(x \in X\) we have \(g_0(\{x\}) \subseteq 2^f(\{x\}) = \{f(x)\}\), and thus

\((2.7)\) \(g_0(\{x\}) = \{f(x)\}\).

For each \(A \in 2^X\) and for each \(x \in A\) we have \(\{x\} \subseteq A\), whence \(g_0(\{x\}) \subseteq g_0(A)\) by (2.5). Taking the union over all points \(x \in A\) and using (2.7) we get

\[\bigcup \{g_0(\{x\}) : x \in A\} = \bigcup \{\{f(x)\} : x \in A\} = f(A) \subseteq g_0(A).\]

Since the last inclusion holds for each \(A \in 2^X\), we conclude that \(2^f < g_0\). This implies \(2^f = g_0\) by (2.2), thus (2.6) follows. If a mapping \(g\) satisfies conditions (2.4)–(2.6), then one can define \(f : X \to Y\) putting \(f(x)\) to be the only point in the set \(g(\{x\})\). Thus for each \(A \in 2^X\) we have \(2^f(A) = f(A) = \bigcup \{f(x) : x \in A\} = \bigcup \{g(\{x\}) : x \in A\} \subseteq g(A)\) by (2.5). Thus \(2^f < g\), and by (2.6) we get \(2^f = g\). The proof is finished.
3. Examples. The following examples show that conditions (2.4), (2.5) and (2.6) of Theorem 2.2 are independent in the sense that no one of them is implied by the two others.

Example 3.1. There are mappings \( g : [0,1] \to [0,1] \) and \( h : C([0,1]) \to C([0,1]) \) that satisfy (2.4) and (2.5) but not (2.6).

Proof. Denote by \( \text{diam} \, A \) the diameter of a nonempty closed subset \( A \) of the closed unit interval \([0,1]\) of reals, and by \( N(A,r) \) the closed ball of radius \( r \) about the set \( A \), i.e., \( N(A,r) = \{ x \in [0,1] : \text{there is a point } y \in A \text{ such that } |x - y| \leq r \} \). Define \( g : [0,1] \to [0,1] \) by \( g(A) = N(A, \text{diam} \, A) \). Then \( g \) satisfies (2.4) and (2.5). To see (2.6) does not hold define \( g_1 : [0,1] \to [0,1] \) as the identity mapping, and note that \( g_1 < g \) and \( g_1 \neq g \). Finally put \( h = g|C([0,1]) \).

Example 3.2. There are mappings \( g : [0,1] \to [0,1] \) and \( h : C([0,1]) \to C([0,1]) \) that satisfy (2.4) and (2.6) but not (2.5).

Proof. It is enough to define \( g \) by \( g(A) = \{ \min A \} \) for \( A \in [0,1] \). Note that if \( g_1 < g \) then \( g_1 = g \), and that there is no \( g_1 < g \) satisfying (2.5); thus (2.6) is satisfied vacuously. Similarly we define \( h(A) = \{ \min A \} \) for \( A \in C([0,1]) \).

Example 3.3. Let \( S \) denote the unit circle in the complex plane. Then there is a mapping \( g : S \to S \) that satisfies (2.5) and (2.6) but not (2.4).

Proof. Define \( g(A) = \{ z \in S : z^2 \in A \} \). Then (2.5) is satisfied, while (2.4) is not. To see (2.6) holds, assume that there is a mapping \( g_1 : S \to S \) with \( g_1 < g \) and \( g_1 \neq g \). Let \( A_0 \in S \) be such that \( g(A_0) \setminus g_1(A_0) \neq \emptyset \) and let \( z_0 \) be a point of this difference. Then \( g_1(\{z_0^2\}) \subset g(\{z_0^2\}) = \{z_0, -z_0\} \). On the other hand \( z_0 \notin g_1(A_0) \) and thus by (2.5) we get

\[
(3.1) \quad g_1(\{z_0^2\}) = \{-z_0\}.
\]

Observe that

\[
S = \{ z \in S : g_1(\{z^2\}) \text{ is a one-point set} \} \cup \{ z \in S : g_1(\{z^2\}) = \{z, -z\} \}.
\]

Continuity of \( g_1 \) implies that both members of this union are closed and, since they are disjoint, one of them must be empty. But it follows from (3.1) that the former is not empty, so we conclude that \( g_1(\{z^2\}) \) is a one-point set for all \( z \in S \). Therefore

\[
S = \{ z \in S : g_1(\{z^2\}) = \{z\} \} \cup \{ z \in S : g_1(\{z^2\}) = \{-z\} \}.
\]

Arguing as previously we see that one member of this union is empty, and again by (3.1) we infer that the latter one is not empty. Thus we have

\[
g_1(\{z^2\}) = \{-z\} \quad \text{for all } z \in S.
\]

Taking in particular \( z = 1 \) and \( z = -1 \)
we get $g_1({1}) = g_1({1^2}) = \{-1\}$ and $g_1({1}) = g_1({(-1)^2}) = \{1\}$, a contradiction.

Recall that in [1, p. 173] a continuum named an *arc of pseudo-arcs* has been described. It is known that the continuum is irreducible and it has a continuous decomposition into pseudo-arcs such that the decomposition space is an arc.

**Example 3.4.** Let $X = [0,1]$ and let $Y$ be an arc of pseudo-arcs. Then there is a mapping $g : C(X) \to C(Y)$ that satisfies (2.5) and (2.6) but not (2.4).

**Proof.** Let $f : Y \to X$ be the natural projection (i.e., the quotient mapping). Thus $f$ is open and monotone. Define $g : C(X) \to C(Y)$ by $g(A) = f^{-1}(A)$ for each $A \in C(X)$. Then $g$ is continuous by openness of $f$, (2.4) does not hold, but (2.5) does. We will argue for (2.6). Let $g_1 : C(X) \to C(Y)$ with $g_1 \prec g$ be given. Then $Z = \bigcup\{g_1({t}) : t \in [0,1]\}$ is a subcontinuum of $Y$ as the union of a subcontinuum of $C(Y)$ (see [2, (1.49), p. 102]). Moreover, $Z$ intersects $g(\{0\})$ and $g(\{1\})$, and since $Y$ is irreducible between any two points one of which is in $g(\{0\})$ and the other is in $g(\{1\})$, we conclude that $Z = Y$, and thus $g_1({t}) = g({t})$ for each $t \in [0,1]$. Now for each $A \in C(X)$ we have

$$g(A) \supset g_1(A) \supset \bigcup\{g_1({t}) : t \in A\} = \bigcup\{g({t}) : t \in A\} = g(A).$$

Thus $g_1(A) = g(A)$, and the proof is finished.

**REFERENCES**
