

## INDUCTIVELY OPEN MAPPINGS AND SPACES WITH OPEN COMPONENTS

J. J. Charatonik, W. J. Charatonik, Wrocław, Poland and B. Ricceri,  
Catania, Italy

*Abstract.* An equivalence between inductive openness of a continuous real function and the condition saying that each point of the range is an interior with respect to the image of a connected subset of the domain, proved in [6], is further investigated. It is shown that one implication can be extended to multifunctions, while various attempts to generalize the other one fail. Spaces are characterized on which each (inductively open) continuous real function satisfies the above condition. Several related results are obtained.

A concept of inductive openness of mappings was introduced and investigated by A. V. Arhangel'skii [1] in 1966. During last two decades the concept has been recognized as an important and interesting generalization of openness, with many applications.

Working in this area, the third named author proved ([6], Theorem 2, p. 486) that for continuous locally nonconstant real functions  $f$  on locally connected topological spaces inductive openness of  $f$  is equivalent to the condition that each point of the range lies in the interior of the image of a connected subset of the domain.

Later the same author was able to extend the result by proving it for a class of not necessarily continuous functions: it is just Theorem 9 of [7]. Another extension is possible if an arbitrary linearly ordered space is substituted in place of the real line as the range of the considered function. The present authors will not give any proof of this extension: the reader is kindly requested to verify that each step of the proof of Theorem 2 of [6] is valid also in the new circumstances (i.e. with a linearly ordered range instead of the space of reals). So, we have the following theorem.

**1. THEOREM.** *Let  $X$  and  $Y$  be topological spaces such that  $X$  is locally connected and  $Y$  is linearly ordered, and let a continuous function  $f: X \rightarrow Y$  satisfy the condition  $\text{int} f^{-1}(y) = \emptyset$  for each point  $y \in \text{int} f(X)$ . Then, the following are equivalent:*

---

*Mathematics subject classification (1980):* 54C10, 54C60, 54D05.

*Key words and phrases:* Connected, inductively open, linearly ordered, locally connected, multifunction, open components, real-valued, semicontinuous.

- (i)  $f$  is inductively open on  $X$ ;  
 (ii) for each point  $y \in f(X)$  there is a connected set  $X_y \subset X$  such that  $y \in \text{int}_{f(X)} f(X_y)$ .

In the present paper, by means of a very careful analysis of each assumption made in Theorem 1, we show that the above quoted Theorem 9 of [7] is, substantially, the only nontrivial "reasonable" extension of Theorem 2 of [6] one can do. We get also some positive results. Namely structural characterizations are obtained of two kinds of topological spaces: of these for which each inductively open multifunction has a property that corresponds to (ii) of Theorem 1, and of these Urysohn spaces for which each continuous real function has property (ii).

The term a *space* means a topological space. For a subset  $A$  of a space  $X$  we shall use the symbols  $\text{cl } A$ ,  $\text{int } A$  and  $\text{bd } A$  to denote its closure, interior and boundary respectively. If the interior of  $A$  is considered with respect to a subspace  $S$  of  $X$ , we shall write  $\text{int}_S A$ .

Let  $X$  and  $Y$  be two nonempty sets. A *multifunction* from  $X$  into  $Y$  is a function from  $X$  into the family  $2^Y$  of all nonempty subsets of  $Y$ . Given a multifunction  $F$  from  $X$  into  $Y$  (briefly,  $F: X \rightarrow 2^Y$ ) and nonempty sets  $A \subset X$  and  $B \subset Y$ , we put  $F(A) = \cup \{F(x) : x \in A\}$ ,  $F^+(B) = \{x \in X : F(x) \subset B\}$  and  $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ .

We shall use upper case characters  $F, G, \dots$  to denote multifunctions, while lower case letters  $f, g, \dots$  will mean single-valued functions.

Given two spaces  $X$  and  $Y$ , a multifunction  $F: X \rightarrow 2^Y$  is said to be:

- *upper (lower) semicontinuous*, if for each open set  $V \subset Y$  the set  $F^+(V)$  (the set  $F^-(V)$ ) is open in  $X$ ;
- *continuous*, if it is both upper and lower semicontinuous;
- *open*, if the set  $F(A)$  is open in  $F(X)$  whenever  $A$  is open in  $X$ ;
- *inductively open*, if there exists a subspace  $X^*$  of  $X$  (called the *induced domain* for  $F$ ) such that  $f(X^*) = F(X)$  and that the restricted multifunction  $F|X^*: X^* \rightarrow 2^{F(X)}$  is open.

We start our study with a result which indicates that a possible generalization of Theorem 1 to lower semicontinuous multifunctions with connected values is trivial, because the considered multifunction becomes single-valued. The case of upper semicontinuous multifunctions will be discussed later (see Example 17 below showing that no similar generalization is possible for this case).

**2. THEOREM.** *Let a multifunction  $F: X \rightarrow 2^Y$  from a space  $X$  to a linearly ordered space  $Y$  satisfy the following conditions:*

- (1) *for each point  $x \in X$  the value  $F(x)$  is connected,*
- (2)  *$F$  is lower semicontinuous,*
- (3) *the set  $\{y \in F(X) : \text{int } F^-(y) = \emptyset\}$  is dense in  $F(X)$ . Then  $F$  is single-valued.*

*Proof.* Suppose, on the contrary, that there is a point  $x_0 \in X$  such that  $F(x_0)$  is a nondegenerate connected interval from  $p$  to  $q$ . By connectivity of  $F(x_0)$  we can choose two points  $y_1, y_2 \in ]p, q[$  with  $y_1 < y_2$ . By lower semicontinuity of  $F$ , the set  $U = F^{-1}(]p, y_1[) \cap F^{-1}(]y_2, q[)$  is an open neighborhood of  $x_0$ , and for each point  $x \in U$  we have  $F(x) \cap ]p, y_1[ \neq \emptyset \neq F(x) \cap ]y_2, q[$ . Hence, for each point  $y \in ]y_1, y_2[$ , connectivity of the values of  $F$  implies that  $y \in F(x)$  for each  $x \in U$ , thus  $U \subset \text{int } F^{-1}(y)$ , contrary to (3).

**3. COROLLARY.** *Let  $X$  and  $Y$  be spaces and  $F: X \rightarrow 2^Y$  be a multifunction such that:*

1°  $X$  is locally vconnected,

2°  $Y$  is linearly ordered,

3° for each point  $x \in X$  the value  $F(x)$  is connected,

4°  $F$  is lower semicontinuous,

5° condition  $\text{int } F^{-1}(y) = \emptyset$  holds for each point  $y \in \text{int } F(X)$ . Then the following conditions are equivalent:

(I)  $F$  is inductively open on  $X$ .

(II) for each point  $y \in F(X)$  there exists a connected set  $X_y \subset X$  such that  $y \in \text{int}_{F(X)} F(X_y)$ .

*Proof.* Really, by Theorem 2, the function  $F$  is single-valued, and therefore the equivalence (I)  $\Leftrightarrow$  (II) holds by Theorem 1.

In Corollary 3 the equivalence (I)  $\Leftrightarrow$  (II) is shown under assumptions 1° – 5°. However, as it can easily be observed from the proof of Theorem 1 given in [6], Theorem 2, p. 486, these assumptions are employed to show the implication (II)  $\Rightarrow$  (I) only; no one of them is really needed in proving (I)  $\Rightarrow$  (II). But this does not mean that (I) implies (II) for an arbitrary space  $X$ . Namely we have to assume that each point of  $X$  has a connected neighborhood. This conditions is obviously equivalent to the fact that each component of  $X$  is open.

Recall that a space in which each point has a linearly ordered (local) base is called a *lob space*. Note that each metrizable space is a lob space. The reader can consult Section 1 of [5], p. 368–373 for more information on the lob spaces. Further, recall that an ordinal number  $\alpha$  is equal to the set of all ordinal numbers less than  $\alpha$ , and thus it can also be understood as a topological space, the topology on  $\alpha$  being always taken as the order topology.

A cardinal number is understood as the smallest ordinal number of a given cardinality. We denote by  $\omega$  and  $\omega_1$  the smallest countable and the smallest uncountable ordinals, respectively. Further,  $\chi(x, X)$  means the *character of the point  $x$*  in the space  $X$ , i.e., the smallest cardinal number  $\alpha$  such that  $x$  has a local base of cardinality  $\alpha$ . And  $\chi(X)$  means the character of the space  $X$ , i.e., the supremum of all  $\chi(x, X)$  over all  $x \in X$ .

We have the following result from which the implication (I) $\Rightarrow$ (II) of Theorem 1 and of Corollary 3 immediately follows.

4. THEOREM. *The following conditions are equivalent for a space  $X$ :*

- (4) *each point of  $X$  has a connected neighborhood,*
  - (5) *for every space  $Y$  each inductively open multifunction  $F: X \rightarrow 2^Y$  has property (II),*
- and each of them implies*
- (6) *each continuous inductively open function  $f: X \rightarrow \chi(X) + 1$  has property (II).*

*Moreover, if  $X$  is a lob space, then all these conditions are equivalent.*

*Proof.* (4) $\Rightarrow$ (5). Fix a point  $y \in F(X)$ , take  $x \in X^* \cap F^{-1}(y)$  (where  $X^*$  is an induced domain for  $F$ ), and denote by  $X_y$  a connected neighborhood of  $x$  in  $X$ . Thus  $X^* \cap \text{int } X_y$  is an open subset of  $X^*$  containing  $x$ , and so the set  $F(X^* \cap \text{int } X_y)$  is open in  $F(X)$  and contains  $y$ . Hence

$$y \in \text{int}_{F(X)} F(X^* \cap X_y) \subset \text{int}_{F(X)} F(X_y).$$

To see the implication (5) $\Rightarrow$ (4) it is enough to take  $F: X \rightarrow 2^X$  defined by  $F(x) = \{x\}$ . The implication (5) $\Rightarrow$ (6) is obvious.

Now we show that (6) implies (4) provided  $X$  is a lob space. Assume  $x'$  is a point of  $X$  such that no neighborhood of  $x'$  is connected. We construct a continuous inductively open function  $f: X \rightarrow \chi(X) + 1$  without property (II). To this aim we define a limit ordinal  $\mu$  and two transfinite sequences  $\{X(\alpha): \alpha < \mu\}$  and  $\{U(\alpha): \alpha < \mu\}$  of open subsets of  $X$  such that

$$(a) \quad x' \in X(\beta) \subset X(\alpha) \text{ for } \alpha < \beta < \mu,$$

and

$$(b) \quad U(\alpha) \cap U(\beta) = \emptyset \text{ if } \alpha \neq \beta.$$

Put  $X(0) = X$  and  $U(0) = \emptyset$ . If  $X(\alpha)$  is defined as an open set containing  $x'$ , it is not connected by the assumption, and therefore it is the union of two nonempty disjoint open subsets  $X(\alpha + 1)$  and  $U(\alpha + 1)$  with  $x' \in X(\alpha + 1)$ . Assume now that, given a limit ordinal  $\lambda$ , the open sets  $X(\alpha)$  satisfying (a) are defined for all  $\alpha < \lambda$ . Then we put  $X(\lambda) = \bigcap \{X(\alpha): \alpha < \lambda\}$  and  $U(\lambda) = \emptyset$  provided that the considered intersection is open; otherwise we define  $\mu = \lambda$  and we stop the procedure. Therefore the set  $Z = \bigcap \{X(\alpha): \alpha < \mu\}$  is not open, and we have  $X = Z \cup \bigcup \{U(\alpha): \alpha < \mu\}$ .

Define now a mapping  $g: X \rightarrow \mu + 1$  by  $g(x) = \mu$  if  $x \in Z$  and  $g(x) = \alpha$  if  $x \in U(\alpha) \subset X \setminus Z$  for  $\alpha < \mu$ . Observe that, since  $U(\lambda) = \emptyset$  for each limit ordinal  $\lambda$ , the ordinal  $\mu$  is the only nonisolated point of  $g(X)$ , and that  $g$  is continuous.

Since  $Z$  is not open, there exists a point  $z \in Z \cap \text{cl}(\bigcup\{U(\alpha) : \alpha < \mu\})$ . The point  $z$  has a linearly ordered base indexed by  $\chi(z, X)$ , and thus there is a transfinite sequence of points  $\{x_\alpha \in X \setminus Z : \alpha < \chi(z, X)\}$  such that  $x_\alpha \rightarrow z$ . Define a function  $h : g(X) \rightarrow g(\{x_\alpha : \alpha < \chi(z, X)\} \cup \{\mu\})$  by  $h(\mu) = \mu$  and, for  $\alpha < \mu$ , putting  $h(\alpha)$  to be the smallest ordinal  $\gamma$  such that  $\alpha \leq \gamma$  and  $\gamma = g(x_\beta)$  for some  $\beta < \chi(z, X)$ . Observe that  $h$  is continuous and that  $hg(X)$  is a set of ordinals which is isomorphic with a subset of  $\chi(z, X) + 1 \subset \chi(X) + 1$ . Let  $i$  denote any isomorphism of  $hg(X)$  into  $\chi(X) + 1$  and let  $f = ihg : X \rightarrow \chi(X) + 1$ . Then  $f$  is continuous, and it is inductively open because  $X^* = \{x_\alpha : \alpha < \chi(z, X)\} \cup \{z\}$  can be taken as an induced domain for  $f$ . Letting  $y = i(\mu)$  we see that each connected set  $X_y \subset X$  such that  $y \in f(X_y)$  has to be contained in  $Z$  and therefore  $f(X_y) = \{y\}$ , so condition (II) does not hold. The proof is complete.

Recal that each metrizable space  $X$  is first countable, i.e.,  $\chi(X) = \omega$  (see [4], p. 311), whence the inclusion  $\chi(X) + 1 \subset \mathbb{R}$  follows (where  $\mathbb{R}$  denotes the space of reals). Further, each metrizable space is a lob one. Therefore we get the following corollary to Theorem 4.

5. COROLLARY. *For each metrizable space  $X$  conditions (4), (5), (7) each continuous inductively open real function  $f$  on  $X$  has property (II), and (8) each continuous inductively open function  $f : X \rightarrow \omega + 1$  has property (II), are equivalent.*

The following example (see Statement 6 below) shows that the assumption that  $X$  is a lob space is essential in the implication (6)  $\Rightarrow$  (4) of Theorem 4. Before we describe the example, we recall some auxiliary concepts.

Denote by  $\beta\omega$  the Čech-Stone compactification of the space  $\omega$  of all nonnegative integers. Remind that for every subset  $A \subset \omega$  its closure in  $\beta\omega$  is an open and closed subset of  $\beta\omega$ , and that for any two disjoint subsets  $A$  and  $B$  of  $\omega$  we have  $\text{cl } A \cap \text{cl } B = \emptyset$  (see e.g. [4], Exercise 3.6 A, p. 233).

6. STATEMENT. *There exists a space  $X$  such that each continuous inductively open function  $f : X \rightarrow Y$  of  $X$  into a linearly ordered space  $Y$  has property (II), while there is a point of  $X$  without any connected neighbourhood.*

*Proof.* Take a point  $p$  in  $\beta\omega \setminus \omega$ . Put  $X = \omega \cup \{p\}$  with the topology inherited from  $\beta\omega$ . Note that  $p$  has no connected neighbourhood. Further, let  $f : X \rightarrow Y$  be an arbitrary continuous inductively open function of  $X$  into a linearly ordered space  $Y$ , and let  $X^*$  be an induced domain for  $f$ . Since the partial function  $f|X^* : X^* \rightarrow F(X^*) = f(X) \subset Y$  is open, the space  $f(X)$  has at most one nonisolated point, namely  $f(p)$ . We show that  $f(X)$  is discrete, and thereby property (II) is trivially satisfied.

So, assume on the contrary that  $f(X)$  is not discrete. Then  $f(p)$  is (the only) nonisolated point of  $f(X)$ . It follows that  $p$  is the only point of  $X^* \cap f^{-1}(f(p))$ . Really, any other point of  $X$  (thus of the intersection) is isolated, and so its image,  $f(p)$ , has to be isolated by openness of  $f|X^*$ . Decompose the set  $f(X) \setminus \{f(p)\} = f(\omega)$  into two disjoint infinite sets  $C_1$  and  $C_2$  such that  $f(p) \in \text{cl } C_1 \cap \text{cl } C_2$ . Then  $f^{-1}(C_1) \cap f^{-1}(C_2) = \emptyset$  and  $f^{-1}(C_1) \cup f^{-1}(C_2) = \omega$ , and observe that  $\text{cl } f^{-1}(C_1)$  and  $\text{cl } f^{-1}(C_2)$  are closed and open disjoint subsets of  $X$ , whose union is the whole space  $X$ . By the symmetry we may assume  $p \in \text{cl } f^{-1}(C_1)$ . Note that  $X^* \cap \text{cl } f^{-1}(C_1)$  is an open subset of  $X^*$ , but its image under  $f$  is not open in  $f(X)$  because it contains  $f(p)$  and it is disjoint with  $C_2$ , contrary to openness of  $f|X^*$ .

Now we show that the ordinal  $\chi(X)+1$  in condition (6) of Theorem 4 cannot be diminished to its predecessor  $\chi(X)$ . To see this, we show a more general Statement 7 below, from which the needed fact follows by taking  $\omega_1$  as the range space  $Y$ .

**7. STATEMENT.** *If  $X = \{\alpha + 1 : \alpha < \omega_1\} \cup \{\omega_1\}$  is equipped with the order topology, then  $\chi(X) = \omega_1$ , and each continuous inductively open function from  $X$  into a  $T_1$ -space  $Y$  with  $\chi(Y) = \omega$  has property (II), while  $X$  does not satisfy condition (4).*

*Proof.* Observe that the point  $\omega_1 \in X$  has no connected neighbourhood, so  $X$  does not satisfy condition (4). For a  $T_1$ -space  $Y$  with  $\chi(Y) = \omega$ , let a continuous function  $f: X \rightarrow Y$  be inductively open. We show that  $f(X)$  is a discrete space and therefore  $f$  has property (II).

Since  $f$  is inductively open, there is an induced domain  $X^* \subset X$  for  $f$ . Note that any subspace of  $X$  is either discrete or homeomorphic to  $X$ . If  $X^*$  is discrete, then  $f(X) = f(X^*)$  is discrete too, as the image of  $X^*$  under an open function  $f|X^*$ . In the other case we have  $\omega_1 \in X^*$ . Let  $\{B_n : n \in \omega\}$  be a local base of  $Y$  at  $f(\omega_1) \in Y$ . Since for each  $n \in \omega$  the set  $f^{-1}(B_n)$  is an open neighbourhood of  $\omega_1$  in  $X$ , its complement  $X \setminus f^{-1}(B_n)$  is countable, and therefore  $X \setminus f^{-1}(f(\omega_1)) = X \setminus f^{-1}(\cap \{B_n : n \in \omega\})$  is countable, too. Thus there is a point  $x^* \neq \omega_1$  such that  $x^* \in X^* \cap f^{-1}(f(\omega_1))$ . Therefore  $f(X^* \setminus \{\omega_1\}) = f(X)$  and it is discrete as the image of the discrete space  $X^* \setminus \{\omega_1\}$  under an open function  $f|X^*$ . This completes the proof.

**8. COROLLARY.** *Let the space  $X$  be as in Statement 7. Then each continuous inductively open real function  $f$  on  $X$  has property (II), while there is a point of  $X$  with no connected neighbourhood.*

**9. Remark.** The reader can verify that Statement 7 remains true if one replaces  $\omega$  by an arbitrary cardinal number  $\gamma$  and  $\omega_1$  by a cardinal number greater than  $\gamma$ .

In Theorem 4 and in corollaries and statements following it, spaces  $X$  were considered such that each inductively open multifunction, or continuous (real) function on  $X$  had property (II). However, it is natural to ask what can be said about spaces  $X$  with a stronger condition: that all continuous real functions on  $X$  (not only inductively open ones) have property (II). Our next results are related to this question. Remind that, since  $\omega + 1$  is embeddable into the real line  $R$ , all functions into  $\omega + 1$ , considered in Theorem 10 and Corollary 11 below, can be understood as real-valued.

**10. THEOREM.** *If each continuous function  $f: X \rightarrow \omega + 1$  has property (II), then the space  $X$  has finitely many components.*

*Proof.* Suppose the contrary. We shall construct a continuous function  $f: X \rightarrow \omega + 1$  without property (II). To this aim we define, by induction, a decreasing sequence of closed and open nonempty subsets  $E_n$  of  $X$ , all having the nonempty intersection. Put  $E_0 = X$  and let for some  $n \in \omega$  a nonempty closed and open subset  $E_n$  of  $X$  be given having infinitely many components. Thus  $E_n$  can be represented as the union of two nonempty closed and open sets. We denote by  $E_{n+1}$  this one of them which contains infinitely many components. Without loss of generality we may assume that  $\bigcap \{E_n : n \in \omega\} \neq \emptyset$ , because otherwise we can redefine the sets  $E_n$  putting  $E'_n = E_n \cup (X \setminus E_1)$  for all  $n > 0$ .

Now we are able to define the needed function  $f: X \rightarrow \omega + 1$  by  $f(x) = n$  if  $x \in E_n \setminus E_{n+1}$  for some  $n \in \omega$ , and  $f(x) = \omega$  if  $x \in E_n$  for all such  $n$ . It can be verified in a routine way that  $f$  is continuous. We show that  $f$  has not property (II) for  $y = \omega$ . Consider a connected subset  $X_0$  of  $X$ . Obviously  $X_0$  is contained in just one component of  $X$ , and therefore either  $X_0 \subset \bigcap \{E_n : n \in \omega\}$ , and then  $f(X_0) = \{\omega\}$ , so  $\omega$  is not an interior point of  $f(X_0)$  in  $f(X)$ , or  $X_0 \subset E_n \setminus E_{n+1}$  for some  $n$ , and then  $\omega$  is not in  $f(X_0)$ . Thus  $f$  does not have property (II).

**11. COROLLARY.** *If each continuous function  $f: X \rightarrow \omega + 1$  has property (II), then each point of the space  $X$  has a connected neighborhood.*

Recall that a space is said to be *Urysohn* if a Urysohn function does exist for any two distinct points of the space ([9], p. 16). Note that in [4], p. 13 this term is used in a different meaning, namely for completely Hausdorff or  $T_{2,1/2}$ -spaces. Obviously each Tychonoff space is Urysohn but not conversely: see [9], Example 91, p. 109. Our next result is a further contribution to the previously discussed question. Namely, if we assume that property (II) holds for each continuous function  $f: X \rightarrow R$  and if, moreover, the space  $X$  is Urysohn, then the conclusion of Theorem 10 can be sharpened: at most one component of  $X$  is nondegenerate.

**12. PROPOSITION.** *Let a space  $X$  be Urysohn. If each continuous real function  $f$  on  $X$  has property (II), then  $X$  is the union of a connected space and a finite discrete space.*

*Proof.* By Theorem 10 the space  $X$  has finitely many components. Suppose on the contrary that some two of them,  $K_1$  and  $K_2$ , are nondegenerate. Take four points:  $x_1, x'_1$  in  $K_1$  and  $x_2, x'_2$  in  $K_2$ . Since  $X$  is Urysohn, there are two continuous functions,  $g, h: X \rightarrow [0, 1]$  such that  $g(x_1) = h(x_2) = 0$  and  $g(x'_1) = h(x'_2) = 1$ . Define now  $f: X \rightarrow \mathbb{R}$  putting  $f(X \setminus (K_1 \cup K_2)) \subset \{1\}$ ,  $f(x) = g(x)$  for  $x \in K_1$  and  $f(x) = -h(x)$  for  $x \in K_2$ . Then  $f(K_1) \subset [0, 1]$  and  $f(K_2) \subset [-1, 0]$ , and therefore for  $y = 0$  there is no connected subset  $X_y$  of  $X$  such that  $0$  is an interior point of  $f(X_y)$ . So  $f$  does not have property (II).

The next result is a converse to the previous one. It is also related to Theorem 4.

**13. PROPOSITION.** *If a space  $X$  is the union of a connected space  $C$  and a finite discrete space  $D$ , then each continuous function  $f: X \rightarrow \mathbb{R}$  has property (II).*

*Proof.* Note that  $f(X)$  has finitely many components, and that at most one of them,  $f(C)$ , is nondegenerate. Take  $y \in f(X)$ . If  $y$  is in  $f(C)$ , put  $X_y = C$ . Otherwise  $y$  is an isolated point of  $f(X)$ , and so we can take as  $X_y$  an arbitrary singleton  $\{x\}$  in  $D$  with  $f(x) = y$ .

As a consequence of Propositions 12 and 13 we have the following theorem.

**14. THEOREM.** *Let a space  $X$  be Urysohn. Then each continuous real function  $f$  on  $X$  has property (II) if and only if  $X$  is the union of a connected space and a finite discrete space.*

As it was said before, the assumptions 1° – 5° of Corollary 3 were used to show the implication (II)  $\Rightarrow$  (I). Having generalized the result related to the opposite implication, it is tempting to try either to drop, or at least to make weaker some of these assumptions. We shall show that it is not so easy as in the previous case. To this aim we are going now to perform a systematic review of assumptions 1° – 5° to study some possibilities of generalizing of Corollary 3. These possibilities consist in attempts to make the assumptions less restrictive concerning a) the domain space, b) the range space, c) the function between spaces.

1°) To see that local connectedness of  $X$  is essential take

$$X = \{(0, y) \in \mathbb{R}^2 : y \in [-1, 1]\} \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 : x \in ]0, 1]\}$$

(the  $\sin(1/x)$ -curve) and the projection  $f: X \rightarrow [0, 1]$  defined by  $f(x, y) = x$ .

Nervertheless one can try to generalize Corollary 3 by making weaker the considered assumption. As a possible way of such a generalization one could consider is a condition of  $X$  being an arcwise connected space. However, this is not the case, as Example 5.1 of [8] shows: there are an arcwise connected space  $X$  (a subcontinuum of the harmonic fan, thus a hereditarily arcwise connected and hereditarily unicoherent plane continuum) and a continuous real function  $f$  on  $X$  which is not inductively open, such that  $\text{int} f^{-1}(y) = \emptyset$  for each  $y \in f(X)$ .

2°) As regards the assumption that the space  $Y$  is linearly ordered (equivalently: that  $F(X)$  is embeddable into a linearly ordered space), one can try to extend Corollary 3 to range spaces  $Y$  being *generalized linear graphs* (i.e., being homeomorphic to connected subsets of linear graphs; see [3], p. 335). But again this attempt fails. To see this, recall that a generalized linear graph is homeomorphic to a subset of the real line if and only if it is both acyclic (i.e., contains no simple closed curve) and atriodic (i.e., contains no simple triod).

Put  $S^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$ , denote by  $T$  the union of three closed straight line segments in the plane emanating from  $(0, 0)$  and ending at  $(-1, 0)$ ,  $(0, 1)$  and  $(1, 0)$  respectively, and take the following two mappings  $f, g$  of intervals onto  $S^1$  and onto  $T$  correspondingly. Let  $f: [0, 2\pi[ \rightarrow S^1$  be given by  $f(t) = (\cos t, \sin t)$ , and  $g: [0, 4] \rightarrow T$  be defined by conditions:  $g(0) = (-1, 0)$ ,  $g(1) = (0, 0)$ ,  $g(2) = (0, 1)$ ,  $g(3) = (0, 0)$ ,  $g(4) = (1, 0)$ ; for  $k \in \{0, 1, 2, 3\}$  let  $g|_{[k, k+1]} \rightarrow \overline{g(k)g(k+1)}$  be linear (see Remarks 2 and 9 of [2]).

Remind that a mapping  $h$  between spaces  $X$  and  $Y$  is said to be *interior at a point*  $x \in X$  if for each open neighbourhood  $U$  of  $x$  we have  $h(x) \in \text{int}_{h(x)} h(U)$ . Now observe that  $f$  is not interior at  $0$ , the only point of  $f^{-1}((1, 0))$ , and  $g$  is not interior at any point of  $\{1, 3\} = g^{-1}((0, 0))$ . Thus both  $f$  and  $g$  are single-valued, satisfy conditions 1°, 3°, 4° and 5° of Corollary 3, do not satisfy 2°, have property (II) with  $X_y = X$  for all  $y$ , but are not inductively open.

These examples show that the range space of a mapping considered either in Corollary 3 or in Theorem 1, if assumed to be a generalized linear graph, has to be both acyclic and atriodic, i.e., has to be embeddable into the real line. Therefore Theorem 1 implies the following result (which resembles Theorem 3 of [2]).

**15. THEOREM.** *Let a generalized linear graph  $Y$  be given. Then condition (ii) implies condition (i) for each continuous locally nonconstant mapping  $f: X \rightarrow Y$  from a locally connected space  $X$  into  $Y$  if and only if  $Y$  is homeomorphic to a connected subset of the real line.*

*Proof.* If  $Y$  contains a simple triod, then the argumentation runs exactly like in the proof of Theorem 3 of [2]. So assume that  $Y$  contains a simple closed curve  $S$ . Choose a point  $p \in S$  that is not a ramification point of  $Y$ . The open arc  $S \setminus \{p\}$  is obviously homeomorphic to the open interval  $]0, 2\pi[$ . Join  $p$  to the open arc as its end points so that the half-closed arc  $A$  obtained in this way is homeomorphic to  $[0, 2\pi[$  under a homeomorphism which brings  $p$  to 0. Observe that  $A \cup Y = \text{bd } S$ , and consider a generalized linear graph  $X$  obtained from  $Y$  by replacing  $S$  by  $A$ . Denote by  $h$  an arbitrary continuous mapping from  $A$  to  $S$  which is locally nonconstant and interior at no point of  $h^{-1}(p)$  (this is possible by the example of such a mapping from  $[0, 2\pi[$  onto  $S^1$  described above). Define a continuous mapping  $f: X \rightarrow Y$  putting  $f(x) = x$  or  $f(x) = h(x)$  according as  $x \in X \setminus A$  or  $x \in A$ . Then conditions  $1^\circ$ ,  $3^\circ$ ,  $4^\circ$  and  $5^\circ$  hold, whereas  $f$  is interior at no point of  $f^{-1}(p)$ , and hence not inductively open. The proof is complete.  $\cdot$

The authors thank K. Omiljanowski for calling to their attention the two examples of mappings discussed in this section, and for making in this way Theorem 15 possible.

$3^\circ$ ) The following example shows that connectivity of all values of  $F$  is necessary in Corollary 3 and in Theorem 2, as well as connectivity of the graph of  $F$  is essential in condition (2) of Theorem 9 of [7].

16. *Example.* There exists a real finite-valued multifunction  $F: [-2, 1] \rightarrow 2^{\{-2, 2\}}$  satisfying conditions  $1^\circ$ ,  $2^\circ$ ,  $4^\circ$  and  $5^\circ$  of Corollary 3, not satisfying  $3^\circ$ , with the nonconnected graph, which is not inductively open.

*Proof.* For  $x \in [-2, 1] \setminus \{-1\}$  put  $F(x) = \{x, -x\}$ ; and  $F(-1) = \{1\}$ . To see  $F$  is not inductively open note that the point 1 has to be in a possible induced domain for  $F$  because it is the only point of  $F^{-1}(1)$ , while the image of no small neighbourhood of 1 is open.

$4^\circ$ ) Lower semicontinuity of  $F$  is essential in Corollary 3 and Theorem 2, and moreover, it cannot be replaced by upper semicontinuity of  $F$  thanks to the following example.

17. *Example.* There exists a real upper semicontinuous multifunction  $F: [0, 1] \rightarrow 2^{[0, 1]}$  with closed values, satisfying conditions  $1^\circ$ ,  $2^\circ$ ,  $3^\circ$  and  $5^\circ$  of Corollary 3, not satisfying  $4^\circ$ , which is not inductively open.

*Proof.* Put  $F(x) = \{x\}$  if  $x \in [0, 1/4[ \cup ]1/2, 1]$ ,  $F(1/4) = [1/4, 3/4]$  and  $F(x) = \{1-x\}$  if  $x \in ]1/4, 1/2[$ . To see  $F$  is not inductively open observe that  $F^{-1}(3/8) = \{1/4\}$ , so  $1/4$  must be in a possible induced domain for  $F$ . But the images of small neighborhoods of  $1/4$  contain  $3/4$  as a boundary point.

$5^\circ$ ) Assumptions  $5^\circ$  of Corollary 3 and (3) of Theorem 2 are essential. This is shown by the next example.

18. *Example.* There exists a real, continuous, not single-valued multifunction  $F: [0, 1] \rightarrow 2^{[0, 1]}$  with closed values, satisfying conditions  $1^\circ - 4^\circ$  of Corollary 3, not satisfying  $5^\circ$  (even not satisfying condition (3) of Theorem 2), which is not inductively open.

*Proof.* Put  $F(x) = \{x\}$  if  $x \in [0, 1/2]$  and  $F(x) = [1/2, x]$  if  $x \in ]1/2, 1]$ . To see  $F$  is not inductively open observe that  $F^{-1}(1) = \{1\}$ , so 1 must be in a possible induced domain for  $F$ . But the images of small neighborhoods of 1 contain  $1/2$  as a boundary point.

Condition  $5^\circ$  is also essential in Theorem 1 (i.e., in the single-valued version of Corollary 3) thanks to an obvious example of a function  $f: [0, 1] \rightarrow [0, 1]$  which increases from 0 to  $1/2$  on  $[0, 1/3]$ , is constant on  $[1/3, 2/3]$ , and again increases, from  $1/2$  to 1, on  $[2/3, 1]$ .

Condition  $5^\circ$  is, in case of single-valued real functions, equivalent to the following:

(9) for each nonempty open set  $U$  in  $X$  there are points

$$x_1, x_2 \in U \text{ such that } x_1 \neq x_2 \text{ and } F(x_1) \neq F(x_2).$$

The equivalence is no longer true if multifunctions  $F$  are considered. Evidently  $5^\circ$  implies (9), but not conversely, as Example 18 shows. It follows from the same example that  $5^\circ$  cannot be weakened to (9) in Theorem 2, and that a version of Corollary 3 for continuous real multifunctions with closed and connected values is not true in case when condition  $5^\circ$  is replaced by (9).

We close the paper showing that continuity of the function  $f$  in Theorem 1, even in the case of real functions, cannot be replaced by either lower or upper semicontinuity understood in the following, well known, sense. A real function  $f: X \rightarrow R$  is *lower (upper) semicontinuous* provided that for each point  $x_0 \in X$  and for each real number  $r > 0$  there exists a neighborhood  $U$  of  $x_0$  such that the inequality  $f(x) > f(x_0) - r$  ( $f(x) < f(x_0) + r$ ) holds for each point  $x \in U$ . Really, define two functions  $g, h: [0, 1] \rightarrow R$  by  $g(0) = 0$  and  $g(x) = 1 - x$  if  $x \in ]0, 1[$ , and by  $h(0) = 1$  and  $h(x) = x$  if  $x \in ]0, 1[$ . Then  $g$  is lower while  $h$  is upper semicontinuous, both are locally nonconstant and not open, hence not inductively open (being one-to-one).

#### REFERENCES:

- [1] A. V. Arhangel'skii, Open and near open mappings. Connections between spaces. Trudy Moskov. Mat. Obšč. 15 (1966), 181–223 (Trans. Moscow Math. Soc. 15 (1966), 204–250).
- [2] J. J. Charatonik, Openness properties of mappings on some connected spaces, Rend. Mat. (to appear).
- [3] J. J. Charatonik and S. Miklos, Generalized graphs and their open mappings, Rend. Mat. (7) 2 (1982), 335–354.

- [4] *R. Engelking*, General topology, PWN Warszawa 1977.  
 [5] *P. J. Nyikos*, Order-theoretic base axioms, in: Surveys in general topology, edited by G. M. Reed, Academic Press 1980; pp. 367–398.  
 [6] *B. Ricceri*, On inductively open real functions, Proc. Amer. Math. Soc. **90** (1984), 485–487.  
 [7] ———, On multifunctions of one real variable, J. Math. Anal. Appl. **124** (1987), 225–236.  
 [8] *B. Ricceri* and *A. Villani*, Openness properties of continuous real functions on connected spaces, Rend. Mat. (7) **2** (1982), 679–687.  
 [9] *L. A. Section* and *J. A. Seebach, Jr.*, Counterexamples in topology, Second edition, Springer Verlag 1978.

(Received September 29, 1987)

*Institute of Mathematics,  
 University of Wrocław  
 pl. Grunwaldzki 2/4,  
 50–384 Wrocław, Poland*

*Department of Mathematics,  
 University of Catania  
 Città Universitaria, Viale A. Doria 6,  
 90125 Catania, Italy*

## INDUKTIVNO OTVORENA PRESLIKAVANJA I PROSTORI S OTVORENIM KOMPONENTAMA

*J. J. Charatonik, W. J. Charatonik, Wrocław, Poljska i  
 B. Ricceri, Catania, Italija*

### Sadržaj

Ovdje se dalje istražuje rezultat iz [6] gdje je dokazana ekvivalencija između induktivne otvorenosti neprekidne realne funkcije i uvjeta koji kaže da je svaka točka kodomene nutarnja s obzirom na sliku povezanog podskupa domene. Dokazano je da se jedan smjer može proširiti na višeznačne funkcije, dok su pokušaji generalizacije drugog smjera bezuspješni. Karakterizirani su prostori na kojima svaka (induktivno otvorena) neprekidna realna funkcija zadovoljava gornji uvjet i dobiveni neki daljnji srodni rezultati.