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Inverse limits and openness of the induced mappings

Janusz J. Charatonik [a,b,*], Włodzimierz J. Charatonik [a,c]

[a] *Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland*
[b] *Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F., Mexico*
[c] *Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409-0020, USA*

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Abstract

The pointed versions of exactness of commutative diagrams and of exactness and limit exactness of mappings between inverse systems are introduced. These concepts are used to investigate interiority of a limit mapping between inverse limits of topological spaces. The obtained results are applied to show openness of some induced mappings between hyperspaces. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

To get nice properties of limit mappings between inverse systems of spaces commutativity of corresponding diagrams is not enough. Some stronger properties, namely exactness and limit exactness of mappings between inverse systems were considered in the literature (see, e.g., [6, p. 19] and [8, p. 58]) and have been shown to be useful tools to investigate openness of the limit mapping [8, Theorem 4, p. 61]. In the present paper we introduce pointed versions of these notions, viz. exactness of diagrams and exactness and limit exactness of mappings between two inverse systems at a point and on a subset of either the domain or the range space. The introduced concepts are used to generalize several results on limit mappings from global to pointed versions, in particular for induced mappings between hyperspaces.

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[*] Corresponding author.

E-mail addresses: jjc@hera.math.uni.wroc.pl; jjc@gauss.matem.unam.mx (J.J. Charatonik), wjcharat@hera.math.uni.wroc.pl; wjcharat@umr.edu (W.J. Charatonik).

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The paper consists of five sections. After the introduction, exactness of diagrams is considered in an auxiliary, second section. Inverse limits are studied in Section 3. We consider exactness of mappings between inverse systems (Section 3.1) and openness of the limit mapping (Section 3.2). The fourth section is devoted to induced mappings between hyperspaces of compact subsets and of subcontinua of the considered topological spaces. Exactness of the induced diagrams and of the induced mappings are studied in Sections 4.1 and 4.2, respectively. Sections 4.3 contains results related to openness of the induced limit mapping. The last chapter contains an example showing an application of introduced concepts and obtained results.

We do not collect definitions, notions and symbols used in the paper in a separate chapter as preliminaries. The needed concepts are recalled in their proper places, where they are used. However, we fix now that all considered spaces are assumed to be topological Hausdorff spaces, and all mappings are continuous. Furthermore, the following standard notation will be used. The abbreviations cl and int mean the closure and the interior respectively of a subset of a space. The composition of two mappings $f : X \to Y$ and $g : Y \to Z$ is denoted by $g \circ f$. As usual, $\mathbb{N}$ stands for the set of all positive integers.

2. Exactness of diagrams

Recall that a diagram

(2.1)
$$\begin{tikzcd}
X' & X \arrow[l, "f"'] \arrow[d, "h"] \\
Y' \arrow[from=u, "h'"'] & Y \arrow[l, "g"]
\end{tikzcd}$$

is said to be *exact* (or *bi-commutative* in [6, §3, IV, p. 19]) if it *commutes*, i.e., $h' \circ f = g \circ h$, and if the condition $h'(x') = g(y)$ implies $h^{-1}(y) \cap f^{-1}(x') \neq \emptyset$ for every $x' \in X'$ and $y \in Y$. It is known [6, §3, IV, p. 19] that

(2.2) the diagram is exact if and only if either of the following condition holds:

$$f\bigl(h^{-1}(B)\bigr) = (h')^{-1}\bigl(g(B)\bigr) \quad \text{for each } B \subset Y,$$

$$h\bigl(f^{-1}(A)\bigr) = g^{-1}\bigl(h'(A)\bigr) \quad \text{for each } A \subset X'.$$

We will define pointed versions of the above concepts as follows.

Definition 2.3. Diagram (2.1) is said to be *exact at a point* $x' \in X'$ (or *at a point* $y \in Y$) provided that $h(f^{-1}(x')) = g^{-1}(h'(x'))$ (or $f(h^{-1}(y)) = (h')^{-1}(g(y))$, respectively).

Proposition 2.4. *If diagram* (2.1) *is exact at a point* $x' \in X'$*, then*

(2.5) *for each point* $y \in Y$ *such that* $g(y) = h'(x')$ *we have* $h^{-1}(y) \cap f^{-1}(x') \neq \emptyset$.

Moreover, if diagram (2.1) *is commutative, condition* (2.5) *implies that it is exact at* x'.

Proof. Let a point $y \in Y$ be such that $g(y) = h'(x')$, i.e., $y \in g^{-1}(h'(x'))$. According to Definition 2.3 we have $g^{-1}(h'(x')) = h(f^{-1}(x'))$, whence $y \in h(f^{-1}(x'))$. Thus there is a point $x \in f^{-1}(x') \subset X$ such that $y = h(x)$. Then $x \in h^{-1}(y) \cap f^{-1}(x') \neq \emptyset$.

The inclusion $h(f^{-1}(x')) \subset g^{-1}(h'(x'))$ is a consequence of the commutativity of diagram (2.1), see [6, §3, IV, Theorem 1, p. 18]. The inclusion $g^{-1}(h'(x')) \subset h(f^{-1}(x'))$ follows from (2.5). Indeed, let $y \in g^{-1}(h'(x'))$, whence $g(y) = h'(x')$. By (2.5) there exists a point $x \in h^{-1}(y) \cap f^{-1}(x')$. Thus $h(x) = y \in h(f^{-1}(x'))$. □

Definition 2.6. Diagram (2.1) is said to be *exact on a set* $A \subset X'$ (*on a set* $B \subset Y$) provided that it is exact at each point of A (at each point of B, respectively).

The proposition below presents a pointed version of the above mentioned results of [6, §3, IV, p. 19].

Proposition 2.7. *The following two conditions are equivalent*:

(a) *diagram* (2.1) *is exact on a set* $B \subset Y$;

(b) *for each subset* $B' \subset B$ *we have*

$$f\big(h^{-1}(B')\big) = (h')^{-1}\big(g(B')\big). \tag{2.8}$$

Proof. To see that (a) implies (b) observe the following sequence of equalities.

$$\begin{aligned} f\big(h^{-1}(B')\big) &= f\Big(\bigcup\big\{h^{-1}(y)\colon\ y \in B'\big\}\Big) = \bigcup\big\{f(h^{-1}(y))\colon\ y \in B'\big\} \\ &= \bigcup\big\{(h')^{-1}(g(y))\colon\ y \in B'\big\} = (h')^{-1}\big(g(B')\big). \end{aligned}$$

To show the converse implication put $B' = \{y\}$. □

By symmetry of the assumptions the next result follows.

Proposition 2.9. *The following conditions are equivalent*:

(a) *diagram* (2.1) *is exact on a set* $A \subset X'$;

(b) *for each subset* $A' \subset A$ *we have*

$$h\big(f^{-1}(A')\big) = g^{-1}\big(h'(A')\big). \tag{2.10}$$

Putting $B = Y$ in Proposition 2.7 we get, using (2.2), the following corollary.

Corollary 2.11. *Diagram* (2.1) *is exact if and only if it is exact at each point of* X' (*equivalently, at each point of* Y).

3. Inverse limits

Suppose that for every $\lambda \in \Lambda$, where Λ is a set directed by a relation $\leqslant$, we have a topological space X_λ, and for every $\lambda, \mu \in \Lambda$ with $\lambda \leqslant \mu$, a mapping $f_\lambda^\mu : X_\mu \to X_\lambda$ is defined such that the following two conditions are satisfied:

- $f_\lambda^\mu \circ f_\mu^\nu = f_\lambda^\nu$ for any $\lambda, \mu, \nu \in \Lambda$ satisfying $\lambda \leqslant \mu \leqslant \nu$,
- f_λ^λ is the identity on X_λ for each $\lambda \in \Lambda$.

Then the family $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ is called the *inverse system of spaces* X_λ with *bonding mappings* f_λ^μ. An inverse system $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$, where $\mathbb{N}$ is the set of all positive integers directed by its natural order, is called the *inverse sequence*.

Let $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be an inverse system. An element $p = \langle p_\lambda \rangle$ of the Cartesian product $\prod\{X_\lambda\colon \lambda \in \Lambda\}$ such that $f_\lambda^\mu(p_\mu) = p_\lambda$ for any $\lambda, \mu \in \Lambda$ with $\lambda \leqslant \mu$ is called a *thread* of $\boldsymbol{S}$, and the subspace of $\prod\{X_\lambda\colon \lambda \in \Lambda\}$ consisting of all threads of $\boldsymbol{S}$ is called the *limit* of the inverse system $\boldsymbol{S}$, and is denoted by $X = \varprojlim\{X_\lambda, f_\lambda^\mu, \Lambda\}$. Further, we denote by $f_\lambda : X \to X_\lambda$ the projection from the inverse limit space into the λth factor space. Then $p_\lambda = f_\lambda(p) \in X_\lambda$ for each $\lambda \in \Lambda$. Besides, we denote by x^λ a point of X_λ, not necessary being the λth coordinate of a thread; similarly, we will use $A_\lambda \subset X_\lambda$ to denote a set of the form $f_\lambda(A)$ for some $A \subset X$, while $A^\lambda \subset X_\lambda$ need not be of this form.

The sets of the form $f_\lambda^{-1}(U^\lambda)$, where U^λ is an open subset of X_λ, called *basic open sets*, constitute a base in X. The reader is referred to Engelking's monograph [2] for more information on inverse systems.

Let two inverse systems $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ and $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ be given. By a mapping $\boldsymbol{h}$ of $\boldsymbol{S}$ to $\boldsymbol{T}$ we mean a family $\{\phi, h^\sigma\}$ consisting of a nondecreasing function $\phi : \Sigma \to \Lambda$ such that the set $\phi(\Sigma)$ is cofinal in Λ, and of mappings $h^\sigma : X_{\phi(\sigma)} \to Y_\sigma$ defined for all $\sigma \in \Sigma$ and such that $g_\sigma^\tau \circ h^\tau = h^\sigma \circ f_{\phi(\sigma)}^{\phi(\tau)}$, i.e., such that the diagram

$$(3.1\colon \sigma, \tau) \qquad \begin{tikzcd} X_{\phi(\sigma)} \arrow[d, "h^\sigma"'] & X_{\phi(\tau)} \arrow[l, "f_{\phi(\sigma)}^{\phi(\tau)}"'] \arrow[d, "h^\tau"] \\ Y_\sigma & Y_\tau \arrow[l, "g_\sigma^\tau"] \end{tikzcd}$$

is commutative for any $\sigma, \tau \in \Sigma$ satisfying $\sigma \leqslant \tau$. Any mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ induces a (continuous) mapping of $X = \varprojlim \boldsymbol{S}$ to $Y = \varprojlim \boldsymbol{T}$, called the *limit mapping induced by* $\{\phi, h^\sigma\}$ and denoted by $h = \varprojlim\{\phi, h^\sigma\} : X \to Y$ (see [2, Section 2.5, p. 101]).

3.1. Exactness of mappings between inverse systems

Recall the following concepts (see [8, p. 58]).

Definitions 3.2. A mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ is said to be *exact* if for every $\sigma, \tau \in \Sigma$ with $\sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact. A mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ is said to be *limit exact* if for every $\sigma \in \Sigma$ the diagram

$$(3.3\colon \sigma) \qquad \begin{tikzcd} X_{\phi(\sigma)} \arrow[d, "h^\sigma"'] & X \arrow[l, "f_{\phi(\sigma)}"'] \arrow[d, "h"] \\ Y_\sigma & Y \arrow[l, "g_\sigma"] \end{tikzcd}$$

is exact.

It is known (see [8, p. 58]) that

(3.4) for inverse sequences (i.e., if $\Lambda = \Sigma = \mathbb{N}$), exactness of diagrams (3.1: m, n) for $m, n \in \mathbb{N}$ implies exactness of diagrams (3.3: n), i.e., exactness of the mapping $\boldsymbol{h}$ implies its limit exactness.

Pointed versions of the above concept can be defined as follows.

Definitions 3.5. A mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ is said to be:

- *domain exact at a point* $a = \langle a_\lambda \rangle \in X$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for every $\sigma, \tau \in \Sigma$ satisfying $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact at the point $a_{\phi(\sigma)} \in X_{\phi(\sigma)}$;
- *domain exact on a set* $A \subset X$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for every $\sigma, \tau \in \Sigma$ satisfying $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact on the set $f_{\phi(\sigma)}(A) \subset X_{\phi(\sigma)}$;
- *domain limit exact at a point* $a = \langle a_\lambda \rangle \in X$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for each $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$ diagram (3.3: σ) is exact at the point $a_{\phi(\sigma)} \in X_{\phi(\sigma)}$;
- *domain limit exact on a set* $A \subset X$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for each $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$ diagram (3.3: σ) is exact on the set $f_{\phi(\sigma)}(A) \subset X_{\phi(\sigma)}$;
- *range exact at a point* $b = \langle b_\sigma \rangle \in Y$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for every $\sigma, \tau \in \Sigma$ satisfying $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact at the point $b_\tau \in Y_\tau$;
- *range exact on a set* $B \subset Y$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for every $\sigma, \tau \in \Sigma$ satisfying $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact on the set $g_\tau(B) \subset Y_\tau$;
- *range limit exact at a point* $b = \langle b_\sigma \rangle \in Y$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for each $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$ diagram (3.3: σ) is exact at the point b;
- *range limit exact on a set* $B \subset Y$ provided that there exists an index $\sigma_0 \in \Sigma$ such that for each $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$ diagram (3.3: σ) is exact on the set B.

Definitions 3.6. The index $\sigma_0 \in \Sigma$ mentioned in Definitions 3.5 will be called an *index of domain exactness of* $\boldsymbol{h}$ *at* a (*of domain exactness of* $\boldsymbol{h}$ *on* A; *of domain limit exactness of* $\boldsymbol{h}$ *at* a; *of domain limit exactness of* $\boldsymbol{h}$ *on* A; *of range exactness of* $\boldsymbol{h}$ *at* b; *of range exactness of* $\boldsymbol{h}$ *on* B; *of range limit exactness of* $\boldsymbol{h}$ *at* b; *of range limit exactness of* $\boldsymbol{h}$ *on* B, respectively).

As a consequence of Definitions 3.2 and 3.5 we have the following statement.

Statement 3.7. *Consider the following conditions for a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$:

(a) $\boldsymbol{h}$ *is* (*limit*) *exact*;

(b) $\boldsymbol{h}$ *is domain* (*limit*) *exact on* X *and every index* $\sigma \in \Sigma$ *is an index of domain* (*limit*) *exactness of* $\boldsymbol{h}$;

(c) $\boldsymbol{h}$ *is range* (*limit*) *exact on* X *and every index* $\sigma \in \Sigma$ *is an index of range* (*limit*) *exactness of* $\boldsymbol{h}$;

(d) *there is an index* $\sigma_0 \in \Sigma$ *such that if* $\Sigma' = \{\sigma \in \Sigma\colon \sigma_0 \leqslant \sigma\}$, *if* $\phi' : \Sigma' \to \Lambda$ *is defined by* $\phi' = \phi|\Sigma'$, *and if* $\boldsymbol{h}' = \{\phi', h^\sigma\} : \boldsymbol{S} \to \boldsymbol{T}' = \{Y_\sigma, g_\sigma^\tau, \Sigma'\}$, *then* $\boldsymbol{h}'$ *is* (*limit*) *exact*;

(e) $\boldsymbol{h}$ *is domain* (*limit*) *exact on* X;

(f) $\boldsymbol{h}$ *is domain* (*limit*) *exact at each point of* X;

(g) $\boldsymbol{h}$ *is range* (*limit*) *exact on* Y;

(h) $\boldsymbol{h}$ *is range* (*limit*) *exact at each point of* Y.

Then the following implications hold:

(3.7.1) (a) ⇒ (b), (d) ⇒ (e) ⇒ (f), (a) ⇒ (c), (d) ⇒ (g) ⇒ (h).

Moreover,

(3.7.2) *if, for all* $\lambda \in \Lambda$, *the projections* $f_\lambda : X \to X_\lambda$ *are surjective, then*

(b) ⇒ (a), (b) ⇒ (d), (e) ⇒ (d);

(3.7.2) *if, for all* $\sigma \in \Sigma$, *the projections* $g_\sigma : Y \to Y_\sigma$ *are surjective, then*

(c) ⇒ (a), (c) ⇒ (d), (g) ⇒ (d).

The next two theorems give some sufficient conditions for limit exactness of a mapping $\boldsymbol{h}$ between inverse systems $\boldsymbol{S}$ and $\boldsymbol{T}$.

Theorem 3.8. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with compact spaces* X_λ *for* $\lambda \in \Lambda$ *and with surjective bonding mappings* f_λ^μ, *and a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between them. Let a point* $a = \langle a_\lambda \rangle \in X$ *be given. If* $\boldsymbol{h}$ *is domain exact at* a, *then it is domain limit exact at* a, *and every index of domain exactness of* $\boldsymbol{h}$ *at* a *is an index of domain limit exactness of* $\boldsymbol{h}$ *at* a.

Proof. Let $\sigma_0 \in \Sigma$ be an index of domain exactness of $\boldsymbol{h}$ at a. Choose $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$. We have to show that diagram (3.3: σ) is exact at the point $a_{\phi(\sigma)} \in X_{\phi(\sigma)}$. To this aim take a point $y = \langle y_\sigma \rangle \in Y$ such that $y_\sigma = h^\sigma(a_{\phi(\sigma)})$. Since diagram (3.1: σ, τ) is exact at the point $a_{\phi(\sigma)}$ for each $\tau \in \Sigma$ with $\sigma \leqslant \tau$, the set $(h^\tau)^{-1}(y_\tau) \cap (f_{\phi(\sigma)}^{\phi(\tau)})^{-1}(a_{\phi(\sigma)})$ is a nonempty compact subset of $X_{\phi(\tau)}$. Since the bonding mappings f_λ^μ are surjective, and the factor spaces X_λ are compact, the projections f_λ are surjective. Thus the sets

$$P_\tau = f_{\phi(\tau)}^{-1}\bigl((h^\tau)^{-1}(y_\tau) \cap \bigl(f_{\phi(\sigma)}^{\phi(\tau)}\bigr)^{-1}(a_{\phi(\sigma)})\bigr) \tag{3.9}$$

are nonempty compact subsets of X. Note that for every $\tau, \tau' \in \Sigma$ with $\sigma \leqslant \tau \leqslant \tau'$ we have $P_{\tau'} \subset P_\tau$, and thus the family

$$\mathcal{P} = \{P_\tau\colon \tau \in \Sigma \text{ and } \sigma \leqslant \tau\}$$

is centered (i.e., it has the finite intersection property). Therefore the intersection P of all elements of $\mathcal{P}$ is nonempty [2, Theorems 3.1.1, p. 123 and 3.2.13, p. 141].

Take $b \in P \subset X$. To conclude the theorem, i.e., to show that diagram (3.3: σ) is exact at $a_{\phi(\sigma)}$ it is enough to show that $b_{\phi(\sigma)} = a_{\phi(\sigma)}$ and $h(b) = y$. Indeed, taking $\tau = \sigma$ in (3.9) we have $b \in P_\sigma \subset f_{\phi(\sigma)}^{-1}(a_{\phi(\sigma)})$, whence $b_{\phi(\sigma)} = a_{\phi(\sigma)}$. Further, again by (3.9), for each

$\tau \in \Sigma$ with $\sigma \leqslant \tau$ we have $b \in P_\tau \subset f^{-1}_{\phi(\tau)}((h^\tau)^{-1}(y_\tau))$, whence $b_{\phi(\tau)} \in (h^\tau)^{-1}(y_\tau)$, i.e., $h^\tau(b_\tau) = y_\tau$. The proof is complete. □

Theorem 3.10. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f^\mu_\lambda, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g^\tau_\sigma, \Sigma\}$ *with compact spaces* X_λ *for* $\lambda \in \Lambda$ *and with surjective bonding mappings* f^μ_λ*, and a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between them. Let a point* $b = \langle b_\sigma \rangle \in Y$ *be given. If* $\boldsymbol{h}$ *is range exact at* b*, then it is range limit exact at* b*, and every index of range exactness of* $\boldsymbol{h}$ *at* b *is an index of range limit exactness of* $\boldsymbol{h}$ *at* b*.*

Proof. Let $\sigma_0 \in \Sigma$ be an index of range exactness of $\boldsymbol{h}$ at b. Choose $\sigma \in \Sigma$ with $\sigma_0 \leqslant \sigma$. We have to show that diagram (3.3: σ) is exact at the point b. To this aim take a point $x^\sigma \in X_{\phi(\sigma)}$ such that $h^\sigma(x^\sigma) = b_\sigma = g_\sigma(b)$. Since diagram (3.1: σ, τ) is exact at the point $b_\tau \in Y_\tau$ for each $\tau \in \Sigma$ with $\sigma \leqslant \tau$, the sets

$$Q_\tau = f^{-1}_{\phi(\tau)}\bigl((h^\tau)^{-1}(b_\tau) \cap \bigl(f^{\phi(\tau)}_{\phi(\sigma)}\bigr)^{-1}(x^\sigma)\bigr) \tag{3.11}$$

are nonempty compact subsets of X. Note that for every $\tau, \tau' \in \Sigma$ with $\sigma \leqslant \tau \leqslant \tau'$ we have $Q_{\tau'} \subset Q_\tau$, and thus the family

$$\mathcal{Q} = \{Q_\tau\colon\ \tau \in \Sigma \text{ and } \sigma \leqslant \tau\}$$

is centered. Therefore the intersection Q of all elements of $\mathcal{Q}$ is nonempty, as previously.

Take a point $x \in Q \subset X$. To conclude the theorem, i.e., to show that diagram (3.3: σ) is exact at b it is enough to show the two equalities: $x_{\phi(\sigma)} (= f_{\phi(\sigma)}(x)) = x^\sigma$ and $h(x) = b$. Indeed, taking $\tau = \sigma$ in (3.11) we have $x \in Q_\sigma \subset f^{-1}_{\phi(\sigma)}(x_{\phi(\sigma)})$, whence the first equality follows. Further, again by (3.11), for each $\tau \in \Sigma$ with $\sigma \leqslant \tau$ we have $x \in Q_\tau \subset f^{-1}_{\phi(\tau)}((h^\tau)^{-1}(b_\tau))$, whence $x_{\phi(\tau)} \in (h^\tau)^{-1}(b_\tau)$, i.e., $h^\tau(x_{\phi(\tau)}) = b_\tau$. Thus the second equality holds, and the proof is complete. □

Using either Theorem 3.8 or Theorem 3.10, together with Statement 3.7 we get the following corollary.

Corollary 3.12. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f^\mu_\lambda, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g^\tau_\sigma, \Sigma\}$ *with compact spaces* X_λ *for* $\lambda \in \Lambda$ *and with surjective bonding mappings* f^μ_λ*. If a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between these systems is exact, then it is limit exact.*

Proof. By the implication (a) ⇒ (b) of Statement 3.7 the mapping $\boldsymbol{h}$ is domain exact on X, and, according to Definitions 3.6, every index $\sigma \in \Sigma$ is an index of domain exactness of $\boldsymbol{h}$. By Theorem 3.8, $\boldsymbol{h}$ is domain limit exact, and every index $\sigma \in \Sigma$ is an index of domain limit exactness of $\boldsymbol{h}$. Since, for compact spaces, surjectiveness of the bonding mappings implies surjectiveness of the projections (see [2, Corollary 3.2.15, p. 142]), we can use the implication (b) ⇒ (a) in (3.7.2) of Statement 3.7 to see that $\boldsymbol{h}$ is limit exact. □

Compactness of the factor spaces X_λ is an essential assumption in Theorem 3.10 and Corollary 3.12 because of the following example. The same example shows that countability of the index set is essential in (3.4).

Example 3.13. There are two inverse systems $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ and $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$, and a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between these systems, which is exact, while not limit exact.

Proof. Let $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be any inverse system with surjective bonding mappings and with the empty inverse limit $X = \varprojlim \boldsymbol{S}$ (see, e.g., [8, Example 1, p. 58]). Fix $\lambda_0 \in \Lambda$, put $\Sigma = \{\lambda \in \Lambda\colon \lambda_0 \leqslant \lambda\}$, and let $\phi : \Sigma \to \Lambda$ be the natural embedding. For $\lambda \in \Sigma$ put $Y_\lambda = X_{\lambda_0}$ and define $g_\lambda^\mu : Y_\mu \to Y_\lambda$ as the identity on X_{λ_0}. Put $\boldsymbol{T} = \{Y_\lambda, g_\lambda^\mu, \Sigma\}$. Thus $\varprojlim \boldsymbol{T}$ is homeomorphic to X_{λ_0}. Define further $h^\lambda = f_{\lambda_0}^\lambda : X_\lambda \to Y_\lambda = X_{\lambda_0}$. Thus diagram (3.1: λ, μ) for $\lambda, \mu \in \Sigma$ is exact by surjectiveness of the bonding mappings f_λ^μ. So $\boldsymbol{h}$ is exact. It is not limit exact since $X = \varprojlim \boldsymbol{S} = \emptyset$, while $\varprojlim \boldsymbol{T}$ is homeomorphic to X_{λ_0}, so it is nonempty. The proof is finished. □

The same example shows that no implication of (3.7.2) can be reversed. Really, since $X = \emptyset$, all the images $f_\lambda(X)$ are empty, so $\boldsymbol{h}$ is domain limit exact on X with each index $\sigma \in \Sigma$ as an index of domain limit exactness, while $\boldsymbol{h}$ is not limit exact.

The assumption of compactness of the factor spaces X_λ in Theorems 3.8 and 3.10 can be omitted provided that we consider inverse sequences instead of arbitrary inverse systems. The next two theorems and the corollary following them give precise formulations.

Theorem 3.14. *Consider two inverse sequences $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, and a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between these sequences. Let a point $b \in Y$ be fixed. If there is an index $j \in \mathbb{N}$ such that for each $n \geqslant j$ the diagram*

(3.1: $n, n+1$)

$$\begin{tikzcd}
X_n \arrow[d, "h^n"'] & X_{n+1} \arrow[l, "f_n^{n+1}"'] \arrow[d, "h^{n+1}"] \\
Y_n & Y_{n+1} \arrow[l, "g_n^{n+1}"]
\end{tikzcd}$$

is exact at the point $b_{n+1} \in Y_{n+1}$, then $\boldsymbol{h}$ is range limit exact at the point b, and the number j is an index of range limit exactness of $\boldsymbol{h}$ at b.

Proof. Fix an index $k \geqslant j$. We have to show that diagram (3.3: k) is exact at b, i.e., that for an arbitrary point $x^k \in X_k$ such that $h^k(x^k) = b_k$ we have $h^{-1}(b) \cap f_k^{-1}(x^k) \neq \emptyset$.

For each $n \leqslant k$ define $a_n = f_n^k(x^k)$. In particular, $a_k = x^k$. By exactness of diagram (3.1: $k, k+1$) at the point $b_{k+1} \in Y_{k+1}$ there is a point $a_{k+1} \in (h^{k+1})^{-1}(b_{k+1}) \cap (f_k^{k+1})^{-1}(a_k) \subset X_{k+1}$.

Assume that, for some $n \geqslant k$, we have defined a_m for every $m \leqslant n$ in such a way that $h^m(a_m) = b_m$ and that, for $m < n$, we have $f_m^{m+1}(a_{m+1}) = a_m$. By exactness of diagram (3.1: $n, n+1$) at b_{n+1} there is a point $a_{n+1} \in (h^{n+1})^{-1}(b_{n+1}) \cap (f_n^{n+1})^{-1}(a_n) \subset X_{n+1}$. Therefore by the inductive procedure the point $a = \langle a_1, a_2, a_3, \ldots\rangle$ has been defined, and we have $a \in h^{-1}(b) \cap f_k^{-1}(x^k)$. The proof is finished. □

Using a similar inductive procedure one can show the following theorem.

Theorem 3.15. *Consider two inverse sequences* $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ *and* $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, *and a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between these sequences. Let a point* $a \in X$ *be fixed. If there is an index* $j \in \mathbb{N}$ *such that for each* $n \geqslant j$ *the diagram*

$$(3.1\colon n, n+1) \qquad \begin{tikzcd} X_n \arrow[d, "h^n"'] & X_{n+1} \arrow[l, "f_n^{n+1}"'] \arrow[d, "h^{n+1}"] \\ Y_n & Y_{n+1} \arrow[l, "g_n^{n+1}"] \end{tikzcd}$$

is exact on the set $(f_j^n)^{-1}(f_j(a))$, *then* $\boldsymbol{h}$ *is domain limit exact at the point* a, *and the number* j *is an index of domain limit exactness of* $\boldsymbol{h}$ *at* a.

Corollary 3.16. *With the assumption of Theorem* 3.15 *we can conclude that the mapping* $\boldsymbol{h}$ *is domain limit exact at every point of the set* $f_j^{-1}(f_j(a))$.

It is not enough to assume in Theorem 3.15 that for each $n \geqslant j$ diagram $(3.1\colon n, n+1)$ is exact at the point a_n only, to conclude that $\boldsymbol{h}$ is limit exact at the point a, even if the considered spaces are compact. This is because exactness of diagrams $(3.1\colon n, n+1)$ at a_n and of $(3.1\colon n+1, n+2)$ at a_{n+1} do not imply exactness of $(3.1\colon n, n+2)$ at a_n. The next example shows this.

Example 3.17. There are two inverse sequences $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between them and a point $a \in X = \varprojlim \boldsymbol{S}$ such that for each $n \in \mathbb{N}$ diagram $(3.1\colon n, n+1)$ is exact at a_n, while diagram $(3.3\colon 1)$ is not exact at a_1.

Proof. To see this consider an inverse sequence $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ of discrete spaces $X_1 = \{0, 1\}$ and $X_n = \{0, 1, 2\}$ for $n \geqslant 2$ and bonding mappings f_n^m determined by the conditions $f_1^2(0) = 0$, $f_1^2(1) = 1$, $f_1^2(2) = 0$, and f_n^{n+1} is the identity mapping for $n \geqslant 2$. Then $X = \varprojlim \boldsymbol{S}$ is homeomorphic to $\{0, 1, 2\}$.

Define $Y_1 = \{0\}$, $Y_2 = \{0, 1\}$, and $Y_n = \{0, 1, 2\}$ for $n \geqslant 3$. Take g_1^2 as the constant mapping, $g_2^3(0) = 0$, $g_2^3(1) = 1$, $g_2^3(2) = 1$, and let g_n^{n+1} be the identity for $n \geqslant 3$. Thus all the bonding mappings g_n^m are determined, and putting $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$ we again see that $Y = \varprojlim \boldsymbol{T}$ is homeomorphic to $\{0, 1, 2\}$.

Define further a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ as follows. $h^1 : X_1 \to Y_1$ is the constant mapping. For $h^2 : X_2 \to Y_2$ put $h^2(0) = 0$, $h^2(1) = 1$, and $h^2(2) = 1$. Finally $h^n : X_n \to Y_n$ is the identity for $n \geqslant 3$.

One can verify that for each $n \in \mathbb{N}$ diagram $(3.1\colon n, n+1)$ is exact at 0, while diagram $(3.1\colon 1, 3)$ (and thus $(3.1\colon 1, n)$ for each $n \geqslant 3$) is not exact at 0, namely taking $1 \in Y_3$ we have $g_1^3(1) = 0 = h^1(0)$, while $(f_1^3)^{-1}(0) = \{0, 2\}$ and $(h^3)^{-1}(1) = \{1\}$, so they are disjoint.

To see that $\boldsymbol{h}$ is not limit exact at the thread $\langle 0, 0, \ldots \rangle \in X$ observe that diagram $(3.3\colon 1)$ is essentially the same as $(3.1\colon 1, 3)$. The proof is finished. □

The next two theorems concern the implication from either domain or range limit exactness to either domain or range exactness of a mapping $\boldsymbol{h}$ between inverse systems $\boldsymbol{S}$ and $\boldsymbol{T}$.

Theorem 3.18. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with compact spaces* Y_σ *for* $\sigma \in \Sigma$ *and with surjective bonding mappings* g_σ^τ*, and a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between them. Let a point* $a = \langle a_\lambda \rangle \in X$ *be given. If* $\boldsymbol{h}$ *is domain limit exact at* a*, then it is domain exact at* a*, and every index of domain limit exactness of* $\boldsymbol{h}$ *at* a *is an index of domain exactness of* $\boldsymbol{h}$ *at* a.

Proof. Let $\sigma_0 \in \Sigma$ be an index of domain limit exactness of $\boldsymbol{h}$ at a. We have to show that for every $\sigma, \tau \in \Sigma$ with $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact at the point $a_{\phi(\sigma)}$. Take any $c \in Y_\tau$ such that $g_\sigma^\tau(c) = h^\sigma(a_{\phi(\sigma)})$. By compactness of all factor spaces of $\boldsymbol{T}$ and surjectiveness of the bonding mappings, the projection g_τ is surjective [2, Corollary 3.2.15, p. 142]. Let a point $y \in Y$ be such that $y_\tau = g_\tau(y) = c$. By exactness of diagram (3.3: σ) at the point $a_{\phi(\sigma)}$ there is a point $x \in X$ such that $h(x) = y$ and $f_{\phi(\sigma)}(x) = a_{\phi(\sigma)}$. Then $x_{\phi(\tau)} \in (h^\tau)^{-1}(c) \cap (f_{\phi(\sigma)}^{\phi(\tau)})^{-1}(a_{\phi(\sigma)})$. The argument is complete. □

Theorem 3.19. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with compact spaces* X_λ *for* $\lambda \in \Lambda$ *and with surjective bonding mappings* f_λ^μ*, and a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *between them. Let a point* $b = \langle b_\sigma \rangle \in Y$ *be given. If* $\boldsymbol{h}$ *is range limit exact at* b*, then it is range exact at* b*, and every index of range limit exactness of* $\boldsymbol{h}$ *at* b *is an index of range exactness of* $\boldsymbol{h}$ *at* b.

Proof. Let $\sigma_0 \in \Sigma$ be an index of range limit exactness of $\boldsymbol{h}$ at b. We have to show that for every $\sigma, \tau \in \Sigma$ with $\sigma_0 \leqslant \sigma \leqslant \tau$ diagram (3.1: σ, τ) is exact at the point $b_\tau \in Y_\tau$. Take any $c \in X_{\phi(\sigma)}$ such that $b_\sigma = g_\sigma^\tau(b_\tau) = h^\sigma(c)$. By compactness of all factor spaces of $\boldsymbol{S}$ and surjectiveness of the bonding mappings, the projection $f_{\phi(\sigma)}$ is surjective [2, Corollary 3.2.15, p. 142]. Let a point $x \in X$ be such that $x_{\phi(\sigma)} = f_{\phi(\sigma)}(x) = c$. By exactness of diagram (3.3: σ) at the point b there is a point $a \in X$ such that $h(a) = b$ and $f_{\phi(\sigma)}(a) = x_{\phi(\sigma)}$. Then $a_{\phi(\tau)} \in (h^\tau)^{-1}(b_\tau) \cap (f_{\phi(\sigma)}^{\phi(\tau)})^{-1}(c)$. This completes the proof. □

As a consequence of Theorem 3.18 and the equivalence (a) ⇔ (b) of Statement 3.7 or of Theorem 3.19 and the equivalence (a) ⇔ (c) of Statement 3.7 (the projections are surjective, compare the proof of Corollary 3.12) we get the following corollary.

Corollary 3.20. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with compact factor spaces and with surjective bonding mappings. If a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *is limit exact, then it is exact.*

The next corollary is a consequence of Corollaries 3.12 and 3.20.

Corollary 3.21. *Let two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *be given with compact factor spaces and with surjective bonding mappings. Then a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *is exact if and only if it is limit exact.*

3.2. Openness of the limit mapping

Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be:

- *open*, if f maps each open set in X onto an open set in Y;
- *interior at a point* $p \in X$ provided that $f(p) \in \operatorname{int} f(U)$ for each open subset $U \subset X$ containing p.

Thus a mapping is open if and only if it is interior at each point of its domain [9, p. 149].

Given a space X and its subspaces A and B such that $B \subset A$, we will write $\operatorname{int}_A B$ to denote the relative interior, i.e., the interior of B with respect to A. Nevertheless, we will use the symbol $\operatorname{int}_X B$ in the sense of $\operatorname{int} B$, to indicate the space X with respect to which the interior of B is considered, especially in the case when several spaces are under consideration.

Theorem 3.22. *Let* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *be a mapping between inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$*, and let* $p = \langle p_\lambda \rangle$ *be a thread in* $X = \varprojlim \boldsymbol{S}$*. If*:

(1) *for each neighborhood* U *of* p *in* X *there is an index* $\sigma_1 \in \Sigma$ *such that for each* $\sigma \in \Sigma$ *with* $\sigma_1 \leqslant \sigma$ *we have*

$$h^\sigma(p_{\phi(\sigma)}) \in \operatorname{int}_{g_\sigma(Y)} h^\sigma\big(f_{\phi(\sigma)}(U)\big), \tag{3.23}$$

and

(2) *there is a neighborhood* V *of* p *in* X *such that the mapping* $\boldsymbol{h}$ *is domain limit exact on* V,

then the limit mapping $h : X \to Y$ *is interior at* p.

Proof. Observe first that $h^\sigma(f_{\phi(\sigma)}(U)) = g_\sigma(h(U)) \subset g_\sigma(Y)$, so the restriction $\operatorname{int}_{g_\sigma(Y)} h^\sigma(f_{\phi(\sigma)}(U))$ in (3.23) makes sense.

To show the conclusion, it is enough to show that for a basic open set $U \subset V$ containing the point p we have

$$h(p) \in \operatorname{int} h(U). \tag{3.24}$$

So, let $U = f_\lambda^{-1}(U^\lambda)$ for some $\lambda \in \Lambda$ and for an open set $U^\lambda \subset X_\lambda$. Let $\sigma_0 \in \Sigma$ be an index of domain limit exactness of $\boldsymbol{h}$ on V, and let $\sigma_1 \in \Sigma$ be as in assumption (1). Further, let $\sigma \in \Sigma$ be such that $\phi(\sigma)$ is greater than each of λ, $\phi(\sigma_0)$ and $\phi(\sigma_1)$. Then (3.23) holds. Put

$$U_{\phi(\sigma)} = f_{\phi(\sigma)}(U),$$

and note that $U = f_{\phi(\sigma)}^{-1}(U_{\phi(\sigma)})$. Further, let W^σ be an open subset of Y_σ such that

$$W^\sigma \cap g_\sigma(Y) = \operatorname{int}_{g_\sigma(Y)} h^\sigma\big(f_{\phi(\sigma)}(U)\big) = \operatorname{int}_{g_\sigma(Y)} h^\sigma(U_{\phi(\sigma)}).$$

By the choice of σ, diagram (3.3: σ) is exact on the set $U_{\phi(\sigma)}$, and thus using Proposition 2.9 we have

$$h(U)=h\big(f^{-1}_{\phi(\sigma)}(U_{\phi(\sigma)})\big)=g^{-1}_{\sigma}\big(h^{\sigma}(U_{\phi(\sigma)})\big)\supset g^{-1}_{\sigma}\big(W^{\sigma}\cap g_{\sigma}(Y)\big)=g^{-1}_{\sigma}(W^{\sigma}),$$

with $h(p)\in g^{-1}_{\sigma}(W^{\sigma})$ by (3.23). Thus (3.24) holds and the proof is finished. □

Corollary 3.25. *Let $\boldsymbol{h}:\boldsymbol{S}\to\boldsymbol{T}$ be a mapping between inverse systems $\boldsymbol{S}=\{X_\lambda, f^{\mu}_{\lambda}, \Lambda\}$ and $\boldsymbol{T}=\{Y_\sigma, g^{\tau}_{\sigma}, \Sigma\}$, and let $p=\langle p_\lambda\rangle$ be a thread in $X=\varprojlim\boldsymbol{S}$. If*:

1. *for each neighborhood U of p in X there is an index $\sigma_1\in\Sigma$ such that for each $\sigma\in\Sigma$ with $\sigma_1\leqslant\sigma$ the mapping h^{σ} is interior at $p_{\phi(\sigma)}$, and*
2. *there is a neighborhood V of p in X such that the mapping $\boldsymbol{h}$ is domain limit exact on V,*

then the limit mapping $h:X\to Y$ is interior at p.

Proof. Take any basic open set $U\subset X$ such that $p\in U$. Let $U=f^{-1}_{\lambda}(U^{\lambda})$ for some open set $U^{\lambda}\subset X_\lambda$, and let σ_1 be such that $\phi(\sigma_1)\geqslant\lambda$. Take $\sigma\geqslant\sigma_1$. Then $p_{\phi(\sigma)}$ is an element of the open set $U^{\phi(\sigma)}=(f^{\phi(\sigma)}_{\lambda})^{-1}(U^{\lambda})$. Note that, by interiority of h^{σ} at $p_{\phi(\sigma)}$, we have $h^{\sigma}(p_{\phi(\sigma)})\in\operatorname{int}_{Y_\sigma}h^{\sigma}(U^{\phi(\sigma)})$, and thus to prove that condition (3.23) of Theorem 3.22 is satisfied, it is enough to show that

$$h^{\sigma}(p_{\phi(\sigma)})\cap g_{\sigma}(Y)\subset h^{\sigma}\big(f_{\phi(\sigma)}(U)\big). \tag{3.26}$$

So, take $y_\sigma\in h^{\sigma}(p_{\phi(\sigma)})$, and let $x^{\phi(\sigma)}\in U^{\phi(\sigma)}$ be such that $h^{\sigma}(x^{\phi(\sigma)})=y_\sigma$. By exactness of diagram (3.3: σ) there is a point $x\in X$ such that $h(x)=y$ and $f_\sigma(x)=x^{\sigma}$. Since $f^{-1}_{\phi(\sigma)}(U^{\phi(\sigma)})=U$, we have $x\in U$. Thus $y_\sigma=h^{\sigma}(f_{\phi(\sigma)}(x))\in h^{\sigma}(f_{\phi(\sigma)}(U))$. This shows (3.26) and finishes the proof. □

Corollary 3.27. *Let $\boldsymbol{h}:\boldsymbol{S}\to\boldsymbol{T}$ be a limit exact mapping between inverse systems $\boldsymbol{S}=\{X_\lambda, f^{\mu}_{\lambda}, \Lambda\}$ and $\boldsymbol{T}=\{Y_\sigma, g^{\tau}_{\sigma}, \Sigma\}$, and let $p=\langle p_\lambda\rangle$ be a thread in $X=\varprojlim\boldsymbol{S}$. If for each neighborhood U of p in X there is an index $\sigma_1\in\Sigma$ such that for each $\sigma\in\Sigma$ with $\sigma_1\leqslant\sigma$ condition*

$$h^{\sigma}(p_{\phi(\sigma)})\in\operatorname{int}_{g_\sigma(Y)}h^{\sigma}\big(f_{\phi(\sigma)}(U)\big), \tag{3.23}$$

holds, then the limit mapping $h:X\to Y$ is interior at p.

The next corollary follows from Corollary 3.25.

Corollary 3.28. *Let $\boldsymbol{h}:\boldsymbol{S}\to\boldsymbol{T}$ be a limit exact mapping between inverse systems $\boldsymbol{S}=\{X_\lambda, f^{\mu}_{\lambda}, \Lambda\}$ and $\boldsymbol{T}=\{Y_\sigma, g^{\tau}_{\sigma}, \Sigma\}$, and let $p=\langle p_\lambda\rangle$ be a thread in $X=\varprojlim\boldsymbol{S}$. If for each $\sigma\in\Sigma$ the mapping $h^{\sigma}:X_{\phi(\sigma)}\to Y_\sigma$ is interior at the point $p_{\phi(\sigma)}$, then the limit mapping $h:X\to Y$ is interior at p.*

The above corollary generalizes the following result of Puzio [8, Theorem 4, p. 61].

Theorem 3.29. *If a mapping $\boldsymbol{h}: \boldsymbol{S} \to \boldsymbol{T}$ between inverse systems $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ and $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ is limit exact, and if all the mappings h^σ are open for $\sigma \in \Sigma$, then the limit mapping $h: X \to Y$ is open.*

Examples are known showing that limit exactness of the mapping $\boldsymbol{h}$ is essential in the above results (see, e.g., [3, Section 3, p. 57]), however the spaces used in the examples are not compact. We will construct a similar example for metric continua. By a *continuum* we mean a compact connected space.

Example 3.30. There is a mapping $\boldsymbol{h}: \boldsymbol{S} \to \boldsymbol{T}$ between inverse sequences of continua $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$ such that all mappings h^n, f_n^m, g_n^m are open, while the limit mapping h is not.

Proof. For $n \in \mathbb{N}$ let X_n be the cone over

$$B_n = \left\{-1, -\tfrac{1}{2}, \ldots, -\tfrac{1}{n}, 0, \tfrac{1}{n}, \ldots, \tfrac{1}{2}, 1\right\}$$

and Y_n be the cone over

$$B_n^+ = \left\{0, \tfrac{1}{n}, \ldots, \tfrac{1}{2}, 1\right\}.$$

Let $h_n': B_n \to B_n^+$ be defined by

$$h_n'(x) = \begin{cases} x, & \text{for } x \in B_n^+, \\ 0, & \text{for } x \in B_n \setminus B_n^+. \end{cases}$$

Similarly, if $m > n$ define $(f_n^m)': B_m \to B_n$ by

$$(f_n^m)'(x) = \begin{cases} x, & \text{for } x \in B_n, \\ 0, & \text{for } x \in B_m \setminus B_n. \end{cases}$$

Finally, let mappings $h^n: X_n \to Y_n$, $f_n^m: X_m \to X_n$ and $g_n^m: Y_m \to Y_n$ be understood as the natural extensions of h_n', $(f_n^m)'$ and $(g_n^m)'$, respectively. Note that they are open. Then $X = \varprojlim\{X_n, f_n^m, \mathbb{N}\}$ is the cone over $\{-1, -\frac{1}{2}, \ldots, 0, \ldots, \frac{1}{2}, 1\}$, and $Y = \varprojlim\{Y_n, g_n^m, \mathbb{N}\}$ is the cone over the harmonic sequence $\{0, \ldots, \frac{1}{2}, 1\}$. The mapping $h: X \to Y$ projects the left part of X onto the limit segment of Y, so it is not open. □

Remark 3.31. Note that if we assume in (2) of Theorem 3.22 that the mapping $\boldsymbol{h}$ is range limit exact on $h(V)$ instead of being domain limit exact on V, then the conclusion does not have to be true. Indeed, denote in Example 3.30 by $t = (0, 1)$ the vertex of the cone X over the set $\{-1, -\frac{1}{2}, \ldots, 0, \ldots, \frac{1}{2}, 1\}$. Take $p = (-1, 0)$ and $V = pt \setminus \{t\}$, where pt is the straight line segment from p to t. Since each h^n is open, assumption (1) of Theorem 3.22 is satisfied. One can verify that diagram (3.3: n) is exact on $h(V)$ for every $n \in \mathbb{N}$. Although, h is not interior at p.

4. Exactness and openness of the induced mappings

In the present section we will consider exactness of induced diagrams and openness of induced mappings between hyperspaces. Some definitions are in order first.

Given a Hausdorff space X, we let 2^X denote the hyperspace of all nonempty compact subsets of X equipped with the Vietoris topology (see [7, (0.12), p. 10]). The basis of the Vietoris topology in 2^X consists of sets of the form

$$\langle U_1, \ldots, U_n \rangle = \big\{ A \in 2^X\colon\ A \subset U_1 \cup \cdots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, n\} \big\},$$

where each U_i is open in X (see [7, (0.10), p. 9]). If X is a metric space with a metric d, then the topology on 2^X coincides with the one generated by the Hausdorff metric H defined by

$$H(A, B) = \max\big\{ \sup\{d(a, B)\colon\ a \in A\},\ \sup\{d(b, A)\colon\ b \in B\} \big\}$$

(see, e.g., [7, (0.1), p. 1 and (0.13), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. The reader is referred to Nadler's book [7] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between Hausdorff spaces X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \quad \text{for every } A \in 2^X$$

and

$$C(f)(A) = f(A) \quad \text{for every } A \in C(X).$$

The following results concerning induced mappings for the class of open mappings are known (see [5, Theorem 4.3]; compare also [4, Theorem 3.2]).

Statement 4.1. *Let a surjective mapping $f : X \to Y$ between continua X and Y be given. Consider the following conditions*:

(a) $f : X \to Y$ *is open*;

(b) $C(f) : C(X) \to C(Y)$ *is open*;

(c) $2^f : 2^X \to 2^Y$ *is open.*

Then (a) *and* (c) *are equivalent, and each of them is implied by* (b).

An example is known [5, Example, p. 244] of an open surjective mapping $f : X \to Y$ between locally connected continua X and Y such that the induced mapping $C(f) : C(X) \to C(Y)$ is not open.

Pointed versions of the implications in Statement 4.1 are presented below.

Theorem 4.2. *Let $f : X \to Y$ be a mapping between Hausdorff spaces. If 2^f (if $C(f)$) is interior at $\{a\} \in 2^X$ (at $\{a\} \in C(X)$, respectively), then f is interior at a.*

Proof. We will argue for $C(f)$; the argument for 2^f is the same. Assume that $C(f)$ is interior at $\{a\} \in C(f)$. Let U be an open set in X containing the point a. By interiority of $C(f)$ at $\{a\}$ we have $\{f(a)\} \in \operatorname{int} C(f)(C(X) \cap \langle U \rangle)$. Thus there are open sets $V_1, \dots, V_n$ in Y such that $\{f(a)\} \in \langle V_1, \dots, V_n \rangle \subset C(f)(C(X) \cap \langle U \rangle)$. Put $V = V_1 \cap \cdots \cap V_n$. Then $\{f(a)\} \in \langle V \rangle \subset \langle V_1, \dots, V_n \rangle$. To finish the proof it is enough to show that $V \subset f(U)$. Take $y \in V$; then $\{y\} \in \langle V \rangle \subset C(f)(C(X) \cap \langle U \rangle)$, whence there is $A \in C(X) \cap \langle U \rangle$ such that $C(f)(A) = \{y\}$. Take $x \in A$. Then $x \in U$ and $f(x) = y$. The argument is complete. □

Theorem 4.3. *Let $f : X \to Y$ be a mapping of a compact Hausdorff space X into a Hausdorff space Y, and let $A \in 2^X$. If f is interior at each point of A, then 2^f is interior at A.*

Proof. Let $A \in \langle U_1, \dots, U_n \rangle$. By regularity of 2^X (see [2, Theorem 3.1.9, p. 125, 3.12.27 (b), p. 244] and apply [2, Proposition 1.5.5, p. 38]) there are open sets $V_1, \dots, V_m$ in X such that $A \in \langle V_1, \dots, V_m \rangle \subset \langle \operatorname{cl} V_1, \dots, \operatorname{cl} V_m \rangle \subset \langle U_1, \dots, U_n \rangle$. For each $j \in \{1, \dots, m\}$ put $W_j = \operatorname{int} f(V_j)$, and note that by the assumption of interiority of f at each point of A we have $W_j \neq \emptyset$. It is enough to show that $f(A) \in \langle W_1, \dots, W_m \rangle \subset 2^f(\langle U_1, \dots, U_n \rangle)$. The first part is a consequence of the definitions. To verify the second one take $B \in \langle W_1, \dots, W_m \rangle$ and put $C = (\operatorname{cl} V_1 \cup \cdots \cup \operatorname{cl} V_m) \cap f^{-1}(B)$. Thus C is compact by the compactness of X. Then $C \in \langle \operatorname{cl} V_1, \dots, \operatorname{cl} V_m \rangle \subset \langle U_1, \dots, U_n \rangle$ and $f(C) = B$. Thus $B \in 2^f(\langle U_1, \dots, U_n \rangle)$. The proof is finished. □

Theorem 4.3 can be generalized from compact to locally compact spaces. Namely we have the following corollary.

Corollary 4.4. *Let $f : X \to Y$ be a surjective mapping between Hausdorff spaces, with the domain X being locally compact, and let $A \in 2^X$. If f is interior at each point of A, then 2^f is interior at A.*

Proof. For each point $x \in A$ let $U(x)$ be a compact neighborhood of x. By compactness of A there are points $x_1, \dots, x_n \in A$ such that $A \subset U = U(x_1) \cup \cdots \cup U(x_n)$. Then $U \in 2^X$ and $A \subset \operatorname{int} U$. Let $\mathcal{V} = \langle V_1, \dots, V_m \rangle$ be an open neighborhood of A such that $V_1 \cup \cdots \cup V_m \subset U$. Then $2^f(\mathcal{V}) = 2^{f|U}(\mathcal{V})$ is a neighborhood of $f(A)$ because of Theorem 4.3. So, the conclusion holds. □

Answering a question of the authors, Professor Alejandro Illanes has constructed the following example which shows that the local compactness of X an essential assumption in Corollary 4.4.

Example 4.5. There is a metric, not locally compact space X, and an open surjective mapping $f : X \to [0, 1]$ such that $2^f : 2^X \to 2^{[0,1]}$ is not open.

Proof. The construction will be performed in the Euclidean plane $\mathbb{R}^2$. Denote by Q the set of all rationals in $[0, 1]$. Let A and B be two subsets of Q such that $Q = A \cup B$,

$A \cap B = \emptyset$, and both A and B are dense in $[0,1]$. Put $X = (Q \times A) \cup (([0,1] \setminus Q) \times B)$, and let $f : X \to [0,1]$ be the natural projection on the first factor.

To see that f is open we verify its interiority at each point $p \in X$. To this aim let S be a square with center p whose sides are parallel to the coordinate axes. Then $S \cap X$ is a neighborhood of p in X. Since the sets A and B are dense in $[0,1]$, the neighborhood $S \cap X$ projects onto an interval $f(S \cap X) \subset [0,1]$ with $p \in \operatorname{int} f(S \cap X)$. Thus f is open.

Now we will prove that $2^f : 2^X \to 2^{[0,1]}$ is not open. Let $a \in A$. Then $(0,a) \in X$. Let $\mathcal{U}$ be any open neighborhood of $\{(0,a)\}$ in 2^X. Suppose $2^f(\mathcal{U})$ is open. Since $\{0\} \in 2^f(\mathcal{U})$, there is $\varepsilon > 0$ such that $[0,\varepsilon] \in 2^f(\mathcal{U})$. Thus there is a compact set $K \in \mathcal{U}$ for which $f(K) = [0,\varepsilon]$. Define $K(x) = K \cap ([0,1] \times \{x\})$ for $x \in Q$. Thus $K = \bigcup\{K(x)\colon x \in Q\}$, whence $[0,\varepsilon] = f(K) = \bigcup\{f(K(x))\colon x \in Q\}$. By the Baire category theorem there exists $x_0 \in Q$ such that $\operatorname{int} f(K(x_0)) \neq \emptyset$. Note that if $x_0 \in A$, then $f(K(x_0)) \subset Q$, and if $x_0 \in B$, then $f(K(x_0)) \subset [0,1] \setminus Q$. So, in any case $\operatorname{int} f(K(x_0)) = \emptyset$, a contradiction that finishes the proof. $\square$

The inverse implication to that of Theorem 4.3 is not true. The next example shows this.

Example 4.6. There is a mapping $f : [0,1] \to [0,1]$ such that the induced mapping 2^f is interior at $[0,1]$, while f is not interior at a point $a \in [0,1]$.

Proof. The mapping f defined by

$$f(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{3}\right], \\ 1 - x & \text{for } x \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ 2x - 1 & \text{for } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

has the needed properties for $a = \frac{2}{3}$. $\square$

Remark 4.7. An analogous result to Theorem 4.3 for the induced mapping $C(f)$ is not true: so-called tent mapping $f : [0,1] \to [0,1]$ with $f(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and $f(x) = 2 - 2x$ for $x \in (\frac{1}{2}, 1]$ is open, thus interior at each point of $[0,1]$, while $C(f)$ is not interior at $X = [0,1]$.

If spaces and mappings are given as in diagram (2.1), then one considers the diagrams

$$\begin{tikzcd} 2^{X'} \arrow[d, "2^{h'}"'] & 2^X \arrow[l, "2^f"'] \arrow[d, "2^h"] \\ 2^{Y'} & 2^Y \arrow[l, "2^g"] \end{tikzcd} \tag{4.8}$$

and

$$\begin{tikzcd} C(X') \arrow[d, "C(h')"'] & C(X) \arrow[l, "C(f)"'] \arrow[d, "C(h)"] \\ C(Y') & C(Y) \arrow[l, "C(g)"] \end{tikzcd} \tag{4.9}$$

(called the diagrams *induced* by diagram (2.1)).

4.1. Exactness of the induced diagrams

It is evident that if diagram (2.1) is commutative, then diagrams (4.8) and (4.9) are commutative, too, and conversely. Concerning exactness of these diagrams we have the following results.

Theorem 4.10. *Let spaces X, X', Y and Y' be given. If, for some point $a \in X'$ (for some point $b \in Y$) the induced diagram* (4.8)*, or the induced diagram* (4.9)*, is exact at $\{a\}$ (at $\{b\}$), then the diagram* (2.1) *is exact at a (at b, respectively).*

Proof. We will argue for the point a and for the diagram (4.8). The argument for the other three cases is the same. Let (4.8) be exact at $\{a\} \in 2^{X'}$. Take $b \in Y$ such that $h'(a) = g(b)$. By the assumption there exists $P \in (2^f)^{-1}(\{a\}) \cap (2^h)^{-1}(\{b\}) \neq \emptyset$. Thus $2^f(P) = \{a\}$ and $2^h(P) = \{b\}$, whence it follows that if $p \in P$ then $f(p) = a$ and $h(p) = b$, so $p \in f^{-1}(a) \cap h^{-1}(b)$, and therefore the diagram (2.1) is exact at a. The proof is complete. □

Corollary 4.11. *If, for some spaces X, X', Y and Y' the induced diagram* (4.9) *is exact, then diagram* (2.1) *is exact.*

The inverse implication to that of Corollary 4.11 does not hold. The next example shows this. We denote by $\mathbb{C}$ the set of all complex numbers.

Example 4.12. If $X = X' = Y = Y'$ is the unit circle $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$, the mappings $f : X \to X'$ and $g : Y \to Y'$ are defined by $f(z) = g(z) = z^3$, and mappings $h : X \to Y$ and $h' : X' \to Y'$ are defined by $h(z) = h'(z) = z^2$, then diagram (2.1) is exact, while the induced diagram (4.9) is not.

Proof. Take $z_1 \in X'$ and $z_2 \in Y$ such that $a = z_1^2 = z_2^3$. We have to find such a number $z \in X$ that $z^3 = z_1$ and $z^2 = z_2$. It follows that $z^6 = a$, so these numbers z cut the unit circle S^1 into six equal parts. We label them $c_0, \ldots, c_5$ assuming that they are ordered cyclicly on S^1. Those of them which satisfy the equation $z^3 = z_1$ are each second, while those with $z^2 = z_2$ are each third. Thus there is exactly one c_j for some $j \in \{0, \ldots, 5\}$ such that $c_j^3 = z_1$ and $c_j^2 = z_2$. Thus diagram (2.1) is exact.

To show that diagram (4.9) is not exact take $A = X'$ and let $B \in C(Y)$ be an arc of the length $2\pi/3$. Then $C(h')(A) = S^1 = C(g)(B)$. Each of the two elements of $(C(h))^{-1}(B)$ is a subarc of S^1 of length $\pi/3$, so its image under $C(f)$ is an arc of length π, and therefore $(C(f))^{-1}(A) \cap (C(h))^{-1}(B) = \emptyset$. The argument is complete. □

Remarks 4.13.

(a) Note that all mappings in diagram (2.1) of Example 4.12 are open. Thus openness of the mappings in diagram (2.1) and its exactness do not suffice for exactness of the induced diagram (4.9).

(b) The exponents 2 and 3 in the definitions of mappings f, g, h and h' of Example 4.12 can be replaced by any pair of relatively prime positive integers.

Note that, in the next results, according to Definition 2.5, if $A \in 2^{X'}$ then A is a subset of X' in (2.1), and is a point of $2^{X'}$ in (4.8) (a point of $C(X')$ in (4.9)).

Theorem 4.14. *Let spaces X, X', Y and Y' be given, and let $A \in 2^{X'}$ (let $B \in 2^{Y}$). If the diagram* (2.1) *is exact on A (on B), then diagram* (4.8) *is exact at A (at B, respectively).*

Proof. We will argue for exactness on/at A. The other case is symmetric. Take $B \in 2^Y$ such that $h'(A) = g(B)$. Put $P = f^{-1}(A) \cap h^{-1}(B)$. We will show that $P \in (2^f)^{-1}(A) \cap (2^h)^{-1}(B)$, i.e., that $f(P) = A$ and $h(P) = B$. The inclusions $f(P) \subset A$ and $h(P) \subset B$ are consequences of the definitions, and because of the symmetry it is enough to prove that $A \subset f(P)$. So, take a point $a \in A$ and choose a point $b \in B$ such that $h'(a) = g(b)$. Then $f^{-1}(a) \cap h^{-1}(b) \neq \emptyset$ by the exactness of diagram (2.1). Let $p \in f^{-1}(a) \cap h^{-1}(b)$. Then $p \in P$ and $f(p) = a$, whence $a \in f(P)$. The proof is finished. □

The following is a consequence of Theorems 4.10 and 4.14.

Corollary 4.15. *Let spaces X, X', Y and Y' be given, and let $A \in 2^{X'}$ (let $B \in 2^{Y}$). Then the induced diagram* (4.8) *is exact at every compact subset of A (of B) if and only if diagram* (2.1) *is exact on A (on B, respectively).*

Corollary 4.16. *Let spaces X, X', Y and Y' be given. Then the induced diagram* (4.8) *is exact if and only if diagram* (2.1) *is exact.*

The inverse implication to that of Theorem 4.10 in case of diagram (4.8) is a consequence of a stronger result, namely of Theorem 4.14. We will show that it is not true in case of diagram (4.9). Precisely, we have the following example.

Example 4.17. There are metric continua X, X', Y and Y', a point $a \in X'$, and mappings f, g, h and h' as in diagram (2.1) such that diagram (2.1) is exact, while the induced diagram (4.9) is not exact at $\{a\}$.

Proof. Let $\mathfrak{C} \subset [0,1]$ be the standard ternary Cantor set. In the plane $\mathbb{R}^2$ (equipped with the Cartesian rectangular coordinates (x, y) of points) let X be the cone with the vertex $v = (1/2, 1)$ over the Cantor set of points $\{(c, 0)\colon c \in \mathfrak{C}\}$ located in the closed unit interval of the x-axis. For each element $c \in \mathfrak{C}$ let L_c stands for the straight line segment joining v and $(c, 0)$. Thus

$$X = \bigcup\{L_c\colon c \in \mathfrak{C}\}.$$

Put $X' = [0,1]$ and let $f : X \to X'$ be the projection defined by $f((x, y)) = y$ for each point $(x, y) \in X$. Further, let $\varphi : \mathfrak{C} \to [0,1]$ be the well known Cantor–Lebesgue step function that maps $\mathfrak{C}$ onto $[0,1]$ (see, e.g., [6, §16, II, (8), p. 150]; compare [9, Chapter II,

§4, p. 35]). Taking Y as the triangle with vertices $v = (1/2, 1)$, $(0, 0)$ and $(1, 0)$, we define a mapping $h : X \to Y$ such that $h(v) = v$ and for each $c \in \mathfrak{C}$ the restriction $h|L_c$ maps L_c linearly onto the straight line segment from v to the point $(\varphi(c), 0)$. Defining $h' : X' \to Y' = [0, 1]$ as the identity, and $g : Y \to Y'$ as the projection defined by $g((x, y)) = y$ for each $(x, y) \in Y$, we see that diagram (2.1) commutes just by the definitions, because if y means the second coordinate of a point $p \in X$, then we have $h'(f(p)) = y = g(h(p))$. Since the function $\varphi : \mathfrak{C} \to [0, 1]$ is a surjection, for each point (x, y) of the triangle Y there is a number $c \in \mathfrak{C}$ with $\varphi(c) = x$. Then $g((x, y)) = y = (h')^{-1}(y)$, whence $(c, y) \in f^{-1}(y) \cap h^{-1}((x, y))$. This means that diagram (2.1) is exact.

Put $a = 0 \in X'$. To see that the induced diagram (4.9) is not exact at $\{a\}$ it is enough to note that there is no subcontinuum of X that is mapped onto $[0, 1] \times \{0\}$ under h. □

The next example shows that the converse to Theorem 4.14 is not true. In other words, it shows that the phrase “on every compact subset of A" cannot be replaced by “at A" in Corollary 4.15.

Example 4.18. There is a space X and a self mapping $f : X \to X$ such that putting $X = X' = Y = Y'$ and $f = g = h = h'$ in diagram (2.1), the induced diagram (4.8) is exact at X' and at Y, while diagram (2.1) is not exact.

Proof. Consider the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane $\mathbb{C}$. Put $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$ and $R = \{\infty\} \cup \{(1 + 1/t)\exp(\mathrm{i}t)\colon t \in (0, \infty)\}$, and let $X = S^1 \cup R$. Define $f : X \to X$ by $f(z) = z^2$ for $z \in S^1$, $f(\infty) = \infty$, and $f((1 + 1/t)\exp(\mathrm{i}t)) = (1 + 1/(2t))\exp(2\mathrm{i}t)$ for $t \in (0, \infty)$. Then $f(S^1) = S^1$ and $f(R) = R$.

Observe that f is one-to-one on R, whence $(2^f)^{-1}(X) = \{X\}$. This implies that diagram (4.8) is exact at X' and at Y. To see that diagram (2.1) is not exact it is enough to take $1 \in X'$ and $-1 \in Y$. Then $g(-1) = 1 = h'(1)$, while $f^{-1}(1) \cap h^{-1}(-1) = \emptyset$. The proof is complete. □

4.2. Exactness of the induced mappings

Let, as previously, $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ be an inverse system. We denote by $\boldsymbol{2^S}$ the inverse system $\{2^{X_\lambda}, 2^{f_\lambda^\mu}, \Lambda\}$, and by $\boldsymbol{C}(\boldsymbol{S})$ the inverse system $\{C(X_\lambda), C(f_\lambda^\mu), \Lambda\}$. It follows from [7, Theorem (1.169), p. 171 and Remark (1.170), p. 174] that if $X = \varprojlim \boldsymbol{S}$, then 2^X is homeomorphic to the inverse limit $\varprojlim \boldsymbol{2^S}$ and $C(X)$ is homeomorphic to the inverse limit $\varprojlim \boldsymbol{C}(\boldsymbol{S})$ (see also [2, 3.12.27 (f), p. 245 and 6.3.22 (f), p. 380]).

Given a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between inverse systems, we define the *induced mappings* $\boldsymbol{2^h} : \boldsymbol{2^S} \to \boldsymbol{2^T}$ and $\boldsymbol{C}(\boldsymbol{h}) : \boldsymbol{C}(\boldsymbol{S}) \to \boldsymbol{C}(\boldsymbol{T})$ as systems of induced mappings

$$2^{h^\sigma} : 2^{X_{\phi(\sigma)}} \to 2^{Y_\sigma} \quad \text{and} \quad C(h^\sigma) : C(X_{\phi(\sigma)}) \to C(Y_\sigma),$$

correspondingly. Recall that, for every $\sigma, \tau \in \Sigma$ with $\sigma \leqslant \tau$ we have

$$2^{h^\sigma} \circ 2^{f_{\phi(\sigma)}^{\phi(\tau)}} = 2^{g_\sigma^\tau} \circ 2^{h^\tau} \quad \text{and} \quad C(h^\sigma) \circ C\bigl(f_{\phi(\sigma)}^{\phi(\tau)}\bigr) = C(g_\sigma^\tau) \circ C(h^\tau),$$

respectively, and that, by a result of Segal, see [7, Theorem 1.169, p. 171, and Remark 1.170, p. 174], we have

$$2^h = \varprojlim \mathbf{2}^{\boldsymbol{h}} \quad \text{and} \quad C(h) = \varprojlim \boldsymbol{C}(\boldsymbol{h}).$$

Theorem 4.19. *Let $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ be a mapping between inverse systems. Then the following implications hold*:

(4.20) *if $\mathbf{2}^{\boldsymbol{h}}$ is domain* (*limit*) *exact at $\{a\}$, then $\boldsymbol{h}$ is domain* (*limit*) *exact at a*;
(4.21) *if $\boldsymbol{C}(\boldsymbol{h})$ is domain* (*limit*) *exact at $\{a\}$, then $\boldsymbol{h}$ is domain* (*limit*) *exact at a*;
(4.22) *if $\boldsymbol{h}$ is domain* (*limit*) *exact on $A \subset X$, then $\mathbf{2}^{\boldsymbol{h}}$ is domain* (*limit*) *exact at A*;
(4.23) *if $\mathbf{2}^{\boldsymbol{h}}$ is range* (*limit*) *exact at $\{b\}$, then $\boldsymbol{h}$ is range* (*limit*) *exact at b*;
(4.24) *if $\boldsymbol{C}(\boldsymbol{h})$ is range* (*limit*) *exact at $\{b\}$, then $\boldsymbol{h}$ is range* (*limit*) *exact at b*;
(4.25) *if $\boldsymbol{h}$ is range* (*limit*) *exact on $B \subset Y$, then $\mathbf{2}^{\boldsymbol{h}}$ is range* (*limit*) *exact at B*;
(4.26) *$\boldsymbol{h}$ is* (*limit*) *exact if and only if $\mathbf{2}^{\boldsymbol{h}}$ is* (*limit*) *exact*;
(4.27) *if $\boldsymbol{C}(\boldsymbol{h})$ is* (*limit*) *exact, then $\boldsymbol{h}$ is* (*limit*) *exact.*

Proof. Implications (4.20) and (4.21), as well as (4.23) and (4.24) are consequences of Theorem 4.10. Conditions (4.22) and (4.25) follow from Theorem 4.14. Finally (4.26) and (4.27) are implied by Corollaries 4.15 and 4.11, correspondingly. □

Now we will consider the inverse implications to ones discussed in Theorem 4.19. The inverse implication to (4.20) holds even in a stronger form, which is (4.22). Similarly, the inverse implication to (4.23) holds in a stronger form, which is (4.25). To see that the inverse implication to (4.21) is not true the following two examples are presented.

Example 4.28. There are two inverse sequences of metric continua $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, and a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between them such that it is exact (and thus limit exact), while $\boldsymbol{C}(\boldsymbol{h})$ is neither domain exact nor domain limit exact at some singleton.

Proof. As in Example 4.17 let $\mathfrak{C} \subset [0,1]$ be the standard ternary Cantor set. For each $n \in \mathbb{N}$ let X_n be the cone over $\mathfrak{C}^n$ and Y_n be the cone over $[0,1]^n$. Therefore points of X_n can be written in the form $(c_1, \dots, c_n, t)$ with $c_j \in \mathfrak{C}$ for $j \in \{1, \dots, n\}$ and $t \in [0,1]$, as well as points of Y_n can be written in the form $(x_1, \dots, x_n, t)$ with $x_j, t \in [0,1]$ for $j \in \{1, \dots, n\}$. We assume that the value $t = 1$ corresponds to the vertices of the cones, i.e., that we have $(c_1, \dots, c_n, 1) = (c'_1, \dots, c'_n, 1)$ in X_n, and $(x_1, \dots, x_n, 1) = (x'_1, \dots, x'_n, 1)$ in Y_n. Let, as in Example 4.17, the mapping $\varphi : \mathfrak{C} \to [0,1]$ be the Cantor–Lebesgue step function. For each $n \in \mathbb{N}$ define $f_n^{n+1} : X_{n+1} \to X_n$, $g_n^{n+1} : Y_{n+1} \to Y_n$ and $h^n : X_n \to Y_n$ by the conditions

$$f_n^{n+1}\big((c_1, \dots, c_n, c_{n+1}, t)\big) = (c_1, \dots, c_n, t),$$

$$g_n^{n+1}\big((x_1, \dots, x_n, x_{n+1}, t)\big) = (x_1, \dots, x_n, t),$$

and

$$h^n\big((c_1, \dots, c_n, t)\big) = \big(\varphi(c_1), \dots, \varphi(c_n), t\big),$$

respectively. We will show that $\boldsymbol{h}:\boldsymbol{S}\to\boldsymbol{T}$ is exact. In diagram (3.1: $n, n+1$) take two points $p=(c_1,\dots,c_n,t)\in X_n$ and $q=(x_1,\dots,x_n,x_{n+1},t')\in Y_{n+1}$ with $h^n(p)=g_n^{n+1}(q)$. Then $\varphi(c_j)=x_j$ for $j\in\{1,\dots,n\}$ and $t=t'$, or $t=t'=1$. Take $c_{n+1}\in\varphi^{-1}(x_{n+1})\subset\mathfrak{C}$. Therefore $(c_1,\dots,c_n,c_{n+1},t)\in(f_n^{n+1})^{-1}(p)\cap(h^{n+1})^{-1}(q)$. So, $\boldsymbol{h}$ is exact, thus limit exact (see, e.g., [8, p. 58] or Corollary 3.12).

Let $X=\varprojlim\boldsymbol{S}$, and take a thread $a=\langle a_1,a_2,a_3,\dots\rangle\in X$ with $a_n=(0,\dots,0)$ (i.e., the sequence of $n+1$ zeros). We will show that the induced diagram

$$\begin{tikzcd}
C(X_n) \arrow[d, "C(h^n)"'] & C(X_{n+1}) \arrow[l, "C(f_n^{n+1})"'] \arrow[d, "C(h^{n+1})"] \\
C(Y_n) & C(Y_{n+1}) \arrow[l, "C(g_n^{n+1})"]
\end{tikzcd} \tag{4.29}$$

is not exact at $\{a_n\}$. To this aim for each $i\in\mathbb{N}$ let s_i be a sequence of i zeros, and put $B_{i+1}=\{s_i\}\times[0,1]\times\{0\}\subset Y_{i+1}$. Note that $g_n^{n+1}(B_{n+1})=\{h^n(a_n)\}$. The set $(h^{n+1})^{-1}(B_{n+1})$ is of the form $\{s_n\}\times\mathfrak{C}\times\{0\}$, and all its subcontinua are singletons, so $(C(h^{n+1}))^{-1}(B_{n+1})=\emptyset$, and therefore the induced diagram (4.29) is not exact at $\{a_n\}$. It is not limit exact at $\{a_n\}$ by Theorem 3.18. The proof is finished. □

As it has been seen in the very final part of the previous proof, diagram (4.29) is not exact because $C(h^{n+1})$ is not surjective, i.e., because h^{n+1} is not weakly confluent [7, Theorem (0.49.1), p. 24]. However, one can have a similar example with all mappings h^n being weakly confluent.

Example 4.30. There are two inverse sequences of metric continua $\boldsymbol{S}'=\{X'_n,\ (f')_n^m,\mathbb{N}\}$ and $\boldsymbol{T}'=\{Y'_n,(g')_n^m,\mathbb{N}\}$, and a mapping $\boldsymbol{h}':\boldsymbol{S}'\to\boldsymbol{T}'$ between them such that it is exact (and thus limit exact), all induced mappings $C((h')^n)$ are surjective, while $\boldsymbol{C}(\boldsymbol{h}')$ is neither domain exact nor domain limit exact at some singleton.

Proof. Consider the two inverse sequences $\boldsymbol{S}$ and $\boldsymbol{T}$ of Example 4.28. For each $n\in\mathbb{N}$ let X'_n be the one-point union of X_n and Y_n with the vertices identified. Define $(f')_n^{n+1}:X'_{n+1}\to X'_n$ by

$$(f')_n^{n+1}|X_{n+1}=f_n^{n+1}\quad\text{and}\quad(f')_n^{n+1}|Y_{n+1}=g_n^{n+1}.$$

Then $\boldsymbol{S}'$ is defined. Then put $\boldsymbol{T}'=\boldsymbol{T}$, and for each $n\in\mathbb{N}$ define $(h')^n:X'_n\to Y'_n=Y_n$ by

$$(h')^n|X_n=h^n\quad\text{and}\quad(h')^n|Y_n\ \text{is the identity.}$$

Since $(h')^n$ is a retraction, the mapping $C((h')^n)$ is a surjection. The argument used in the proof of Example 4.28 shows that $\boldsymbol{C}(\boldsymbol{h}')$ is not domain exact. The reader can verify that $\boldsymbol{h}'$ is exact. □

Examples 4.28 and 4.30 show that the converse to implication (4.27) of Theorem 4.19 is not true.

The converse implication to that of (4.22), i.e., a stronger form of (4.20), in which the singleton $\{a\}$ is replaced by any set $A \in 2^X$, and to that of (4.25), i.e., a stronger version of (4.23), in which the singleton $\{b\}$ is replaced by any set $B \in 2^Y$, are not true by the next example.

Example 4.31. There are two inverse sequences of metric continua $\boldsymbol{S} = \{X_n,\ f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, and a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between them such that the induced mapping $\mathbf{2}^{\boldsymbol{h}} : \mathbf{2}^{\boldsymbol{S}} \to \mathbf{2}^{\boldsymbol{T}}$ is domain exact and domain limit exact at $X_\infty = \varprojlim \boldsymbol{S}$, and range exact and range limit exact at $Y_\infty = \varprojlim \boldsymbol{T}$, while h is neither domain exact nor domain limit exact on X_∞, and it is neither range exact nor range limit exact at Y_∞.

Proof. As the reader has observed, we have changed our usual notation for $\varprojlim \boldsymbol{S}$ from X into X_∞. This is because we will use the symbol X to denote the continuum of Example 4.18. The same change from Y to Y_∞ is just to keep the symmetry of notations. Further, let $f : X \to X$ be the self mapping defined there. With this meaning of X and f for each $n \in \mathbb{N}$ put $X_n = Y_n = X$ and $f_n^{n+1} = g_n^{n+1} = h^n = f$. Thus the needed inverse sequences $\boldsymbol{S}$, $\boldsymbol{T}$ and the mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ are defined. We will show that $\mathbf{2}^{\boldsymbol{h}}$ is domain exact at $X_\infty \in 2^{X_\infty}$. According to Definition 3.5 we have to verify that the (induced) diagram

$$\begin{tikzcd}
2^{X_n} \arrow[d, "2^{h_n}"'] & 2^{X_m} \arrow[l, "2^{f_n^m}"'] \arrow[d, "2^{h_m}"] \\
2^{Y_n} & 2^{Y_m} \arrow[l, "2^{g_n^m}"]
\end{tikzcd}$$

in which $n \leqslant m$, is exact at $X_n \in 2^{X_n}$. By Example 4.18 the only point P of 2^{Y_m} satisfying $2^{g_n}(P) = 2^{h_n}(X_n) = Y_n$ is $P = Y_m$, and thereby we are done.

Note that $\boldsymbol{h}$ is not domain exact at the thread $p = \langle 1, 1, 1, \ldots\rangle \in X_\infty$, and it is not range exact at $h(p) = \langle 1, 1, 1, \ldots\rangle \in Y_\infty$. Further, $\mathbf{2}^{\boldsymbol{h}}$ is domain limit exact at X_∞ by Theorem 3.8, and it is range limit exact at Y_∞ by Theorem 3.10. Note that $\boldsymbol{h}$ is neither domain limit exact at p by Theorem 3.18, nor it is range limit exact by Theorem 3.19. The argument is complete. □

The next example shows that the converse implication to (4.24) is not true.

Example 4.32. There are two inverse sequences of metric continua $\boldsymbol{S} = \{X_n,\ f_n^m, \mathbb{N}\}$ and $\boldsymbol{T} = \{Y_n, g_n^m, \mathbb{N}\}$, and a mapping $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ between them such that it is exact (and thus limit exact), while $\boldsymbol{C}(\boldsymbol{h})$ is neither range exact nor range limit exact at some singleton.

Proof. As previously defined, let $\mathfrak{C} \subset [0, 1]$ be the Cantor set. For each $n \in \mathbb{N}$ let X_n be the cone over $[0, 1] \times \mathfrak{C} \times \mathfrak{C}$, and $Y_n = [0, 1]$. Thus points of X_n can be written in the form (s, c_1, c_2, t) with $s, t \in [0, 1]$ and $c_1, c_2 \in \mathfrak{C}$.

As in Example 4.28, we assume that the value $t = 1$ corresponds to the vertices of the cones. Let again the mapping $\varphi : \mathfrak{C} \to [0, 1]$ be the Cantor–Lebesgue step function, and let $\psi : \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ be a homeomorphism. For each $n \in \mathbb{N}$ define

$$f_n^{n+1}(s, c_1, c_2, t) = \bigl(\varphi(c_1), \psi(c_2), t\bigr);$$

$$h^n(s, c_1, c_2, t) = t \quad \text{and} \quad g_n^{n+1}(t) = t.$$

To show exactness and limit exactness of $\boldsymbol{h}$ it is enough, by equivalence (a) and (c) of Statement 3.7 and by Theorems 3.14 and 3.19, to prove that diagrams (3.1: $n, n + 1$) are exact for each $n \in \mathbb{N}$. So, take $x^n \in X_n$ and $y^{n+1} \in Y_{n+1}$ such that $g_n^{n+1}(y^{n+1}) = h^n(x^n)$. Then $x^n = (s, c_1, c_2, t)$ and $y^{n+1} = t$ for some $s, t \in [0, 1]$ and $c_1, c_2 \in \mathfrak{C}$. Take $c \in \mathfrak{C}$ such that $\varphi(c) = s$. Then

$$\bigl(0, c, \psi^{-1}(c_1, c_2), t\bigr) \in \bigl(h^{n+1}\bigr)^{-1}\bigl(y^{n+1}\bigr) \cap \bigl(f_n^{n+1}\bigr)^{-1}\bigl(x^n\bigr).$$

This finishes the proof of exactness of $\boldsymbol{h}$.

Take a thread $q = \langle 0, 0, \ldots\rangle \in Y = \varprojlim \boldsymbol{T}$. We will show that $\boldsymbol{C}(\boldsymbol{h})$ is is neither range exact nor range limit exact at $\{q\}$. Again by Theorem 3.19 it is enough to show that for each $n \in \mathbb{N}$ diagram (4.29) is not exact at $\{0\} \in C(Y_{n+1})$. Put $B = [0, 1] \times \{0, 0, 0\}$. Then $h^n(B) = 0 = g_n^{n+1}(0)$ and $(f_n^{n+1})^{-1}(B) = [0, 1] \times \mathfrak{C} \times \{\psi^{-1}(0, 0)\} \times \{0\}$. Note that the image under f_n^{n+1} of any subcontinuum of $(f_n^{n+1})^{-1}(B)$ is a singleton, whence $(C(f_n^{n+1}))^{-1}(B) = \emptyset$. Consequently, diagram (4.29) is not exact at $\{0\} \in C(Y_{n+1})$. The proof is complete. □

4.3. Openness of the induced limit mapping

We prove the following result.

Theorem 4.33. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with locally compact factor spaces* X_λ. *Let* $X = \varprojlim \boldsymbol{S}$ *and* $Y = \varprojlim \boldsymbol{T}$, *and let a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *be domain limit exact on some compact neighborhood* U *of* $A \in 2^X$. *If there exists a* $\sigma_1 \in \Sigma$ *such that for each* $\sigma \in \Sigma$ *with* $\sigma_1 \leqslant \sigma$ *the mapping* $h^\sigma : X_{\phi(\sigma)} \to Y_\sigma$ *is interior at every point of the set* $A_{\phi(\sigma)} = f_{\phi(\sigma)}(A)$, *then the induced mapping* 2^h *is interior at* A.

Proof. By (4.22) of Theorem 4.19 the induced mapping 2^h is domain limit exact at each $P \in 2^U$, i.e., it is domain limit exact on 2^U. On the other hand, Corollary 4.4 implies that 2^{h^σ} is interior at $A_{\phi(\sigma)}$ for each $\sigma \in \Sigma$ with $\sigma_1 \leqslant \sigma$. Thus Corollary 3.25 can be applied with $\boldsymbol{2^S}$, $\boldsymbol{2^T}$ and $\boldsymbol{2^h}$ in place of $\boldsymbol{S}$, $\boldsymbol{T}$ and $\boldsymbol{h}$, respectively, and $A \in 2^U \subset 2^X$ in place of P, to get the conclusion. □

Corollary 4.34. *Consider two inverse systems* $\boldsymbol{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\boldsymbol{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *with locally compact factor spaces* X_λ. *Let* $X = \varprojlim \boldsymbol{S}$ *and* $Y = \varprojlim \boldsymbol{T}$, *and let a mapping* $\boldsymbol{h} : \boldsymbol{S} \to \boldsymbol{T}$ *be limit exact and such that the mappings* $h^\sigma : X_{\phi(\sigma)} \to Y_\sigma$ *are open for each* $\sigma \in \Sigma$. *Let* $h : X \to Y$ *be the limit mapping. Then the induced mapping* $2^h : 2^X \to 2^Y$ *is open.*

5. Applications

To illustrate how the theorems considered in the previous sections work we will show that the induced mapping $C(h)$ is open for some mappings h from a solenoid into itself. We start with the necessary definitions. For a given sequence $\xi = \{k_1, k_2, \ldots\}$ of positive integers, define the following inverse sequence. For each $n \in \mathbb{N}$ let X_n be the unit circle $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$, and define $f_n^{n+1}(z) = z^{k_n}$. Put $\boldsymbol{S} = \{X_n, f_n^m, \mathbb{N}\}$. Then the inverse limit $X = \varprojlim \boldsymbol{S}$ is called a *solenoid* determined by the sequence ξ. In particular, if the sequence ξ is constant from some place on, with $k_i = k$ for almost all $i \in \mathbb{N}$, then the solenoid is called *k-adic*. It is known that if all terms k_i for $i \in \mathbb{N}$ are bigger than 1, then the solenoid X is a homogeneous indecomposable continuum. The reader is referred, e.g., to [1, pp. 222 and 223] for more information on other definitions and characterizations of solenoids known from the literature.

Example 5.1. For each $n \in \mathbb{N}$ put $X_n = Y_n = S^1$, $f_n^{n+1}(z) = g_n^{n+1}(z) = z^3$, and $h^n(z) = z^2$. Let $h: X \to Y$ be the limit mapping between the inverse limits (being the same triadic solenoid). Then for each $n \in \mathbb{N}$ the induced mappings $C(h^n)$ are not open, while $C(h)$ is.

Proof. To see that $C(h^n)$ is not open for any n consider the family $\mathcal{U}$ of subarcs of X_n with length greater than π. Then $\mathcal{U}$ is an open subset of $C(X_n)$. The image $C(h^n)(\mathcal{U}) = \{Y_n\}$ is not open in $C(Y_n)$.

To show that $C(h)$ is open we will prove its interiority at each element $A \in C(X)$. We consider the cases $A \neq X$ and $A = X$ separately. In the case $A \neq X$ we will apply Theorem 3.22. Put $A_n = f_n(A)$. The whole proof will be divided into four claims.

Claim 1. *For each $A \in C(X) \setminus \{X\}$ and for each neighborhood $\mathcal{U}$ of A in $C(X)$ there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have $h^n(A_n) \in \operatorname{int} C(h^n)(C(f_n)(\mathcal{U}))$.*

Proof. To show Claim 1 let $n_1 \in \mathbb{N}$ be such that $f_{n_1}(A)$ is a proper subset of X_{n_1}. Let V^{n_1} be a subarc of X_{n_1} satisfying $A_{n_1} = f_{n_1}(A) \subset \operatorname{int} V^{n_1}$. For every $n > n_1$ define V^n as the component of $(f_{n_1}^n)^{-1}(V^{n_1})$ that contains $A_n = f_n(A)$. Then $A_n \subset \operatorname{int} V^n$ and $h^n|V^n$ is a homeomorphism, so $C(h^n)|C(V^n)$ is also a homeomorphism, and the image $C(h^n)(C(V^n))$ is a neighborhood of $h^n(A_n)$ in $C(Y_n)$.

Let $\mathcal{U}$ be a neighborhood of A in $C(X)$. Then there is an index $n_2 \in \mathbb{N}$ and an open set U^{n_2} in X_{n_2} such that $f_{n_2}^{-1}(U^{n_2}) \subset \mathcal{U}$. Let $n_0 = \max\{n_1 + 1, n_2\}$. For each $n \geqslant n_0$ put $W^n = (f_{n_2}^n)^{-1}(U^{n_2}) \cap V^n$. Then W^n is an open subset of X_n and $A_n \subset W^n$. Moreover, $C(h^n)|C(W^n)$ is a homeomorphism with $C(h^n)(C(W^n))$ open in Y_n. Therefore $h^n(A_n) \in C(h^n)(C(W^n)) \subset \operatorname{int} C(h^n)(C(f_n)(\mathcal{U}))$. This finishes the proof of Claim 1. □

Claim 2. *For each $A \in C(X) \setminus \{X\}$ there exists a compact neighborhood $\mathcal{V}$ of A such that $\boldsymbol{C(h)}$ is domain exact on $\mathcal{V}$.*

Proof. To show Claim 2 let $n_0 \in \mathbb{N}$ be such that $f_{n_0}(A)$ is a proper subset of X_{n_0}. Let V^{n_0} be a subarc of X_{n_0} satisfying $A_{n_0} = f_{n_0}(A) \subset \operatorname{int} V^{n_0} \neq X_{n_0}$, and denote by V^{n_0+1}

the component of $(f_{n_0}^{n_0+1})^{-1}(V^{n_0})$ that contains A_{n_0+1}. Then V^{n_0+1} is an arc in X_{n_0+1} of length less than $2\pi/3$. Put $\mathcal{V} = \{P \in C(X)\colon\ f_{n_0+1}(P) \subset V^{n_0+1}\}$. Then $\mathcal{V}$ is a compact neighborhood of A in $C(X)$. We will show that $\boldsymbol{C(h)}$ is domain exact on $\mathcal{V}$, and $n_0 + 1$ is an index of domain exactness of $\boldsymbol{C(h)}$ on $\mathcal{V}$. To this aim consider the induced diagram

$$\begin{tikzcd}
C(X_n) \arrow[d, "C(h^n)"'] & C(X_m) \arrow[l, "C(f_n^m)"'] \arrow[d, "C(h^m)"] \\
C(Y_n) & C(Y_m) \arrow[l, "C(g_n^m)"]
\end{tikzcd}$$

for $n_0 + 1 \leqslant n \leqslant m$, and a continuum $V \in \mathcal{V}$. Let W^m be a subcontinuum of Y_m satisfying $g_n^m(W^m) = h^n(V_n)$, where $V_n = f_n(V)$. By the choice of n_0 the continuum $h^n(V_n)$ is a proper subcontinuum of Y_n, and therefore W^m is a component of $(g_n^m)^{-1}(h^n(V_n))$. Choose a thread $v = \langle v_1, v_2, \ldots\rangle \in V$ and $w^m \in W^m$ such that $h^n(v_n) = g_n^m(w^m)$. By Remark 4.13(b) diagram (3.1: n, m) is exact, and thus there is a point $v^m \in (f_n^m)^{-1}(v_n) \cap (h^m)^{-1}(w^m)$. Let V^m be the component of $(h^m)^{-1}(W^m)$ that contains the point v^m. Then $V^m \in (C(f_n^m))^{-1}(V_n) \cap (C(h^m))^{-1}(W^m)$. This finishes the proof of Claim 2. □

Claim 3. *For each $A \in C(X) \setminus \{X\}$ the induced mapping $C(h)$ is interior at A.*

Indeed, it follows from Claims 1 and 2, using Theorem 3.22.

Claim 4. *The induced mapping $C(h)$ is interior at X.*

Proof. To show Claim 4 for each $n \in \mathbb{N}$ put $\mathcal{U}_n = \{A \in C(X)\colon\ f_n(A) = X_n\}$. We will prove that the family $\{\mathcal{U}_n\colon\ n \in \mathbb{N}\}$ is a local base of $C(X)$ at its element X. To this aim define

$$U = \bigl\{\exp(\mathrm{i}t)\colon\ t \in (0, 3\pi/2)\bigr\} \subset X_{n+1}$$

and

$$V = \bigl\{\exp(\mathrm{i}t)\colon\ t \in (-\pi, \pi/2)\bigr\} \subset X_{n+1},$$

and note that $\mathcal{U}_n$ contains the basic open set $\langle f_{n+1}^{-1}(U), f_{n+1}^{-1}(V)\rangle$. The property $\bigcap\{\mathcal{U}_n\colon\ n \in \mathbb{N}\} = \{X\}$ is obvious. To complete the argument it is enough to observe that $\mathcal{U}_{n+1} \subset C(h)(\mathcal{U}_n)$. Thus Claim 4 is shown. □

By Claims 3 and 4 the induced mapping $C(h)$ is interior at each element A of $C(X)$, whence it is open, as needed. The proof is complete. □

Remark 5.2. Observe, similarly as in Remark 4.13(b), that the exponents 2 and 3 in the definitions of h^n, f_n^{n+1} and g_n^{n+1} of Example 5.1 can be replaced by any pair of relatively prime positive integers.

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